CHAPTER 2

Stiffness Matrices, Spring and Bar Elements

2.1 INTRODUCTION

The primary characteristics of a finite element are embodied in the element *stiffness matrix*. For a structural finite element, the stiffness matrix contains the geometric and material behavior information that indicates the resistance of the element to deformation when subjected to loading. Such deformation may include axial, bending, shear, and torsional effects. For finite elements used in nonstructural analyses, such as fluid flow and heat transfer, the term *stiffness matrix* is also used, since the matrix represents the resistance of the element to change when subjected to external influences.

This chapter develops the finite element characteristics of two relatively simple, one-dimensional structural elements, a linearly elastic spring and an elastic tension-compression member. These are selected as introductory elements because the behavior of each is relatively well-known from the commonly studied engineering subjects of statics and strength of materials. Thus, the "bridge" to the finite element method is not obscured by theories new to the engineering student. Rather, we build on known engineering principles to introduce finite element concepts. The linear spring and the tension-compression member (hereafter referred to as a *bar* element and also known in the finite element literature as a *spar, link,* or *truss* element) are also used to introduce the concept of *interpolation functions.* As mentioned briefly in Chapter 1, the basic premise of the finite element method is to describe the continuous variation of the field variable (in this chapter, physical displacement) in terms of discrete values at the finite element nodes. In the interior of a finite element, as well as along the boundaries (applicable to two- and three-dimensional problems), the field variable is described via interpolation functions (Chapter 6) that must satisfy prescribed conditions.

Finite element analysis is based, dependent on the type of problem, on several mathematic/physical principles. In the present introduction to the method, we present several such principles applicable to finite element analysis. First, and foremost, for spring and bar systems, we utilize the principle of static equilibrium but—and this is essential—we include *deformation* in the development; that is, we are not dealing with rigid body mechanics. For extension of the finite element method to more complicated elastic structural systems, we also state and apply the first theorem of Castigliano [1] and the more widely used principle of minimum potential energy [2]. Castigliano's first theorem, in the form presented, may be new to the reader. The first theorem is the counterpart of Castigliano's second theorem, which is more often encountered in the study of elementary strength of materials [3]. Both theorems relate displacements and applied forces to the equilibrium conditions of a mechanical system in terms of mechanical energy. The use here of Castigliano's first theorem is for the distinct purpose of introducing the concept of minimum potential energy without resort to the higher mathematic principles of the calculus of variations, which is beyond the mathematical level intended for this text.

2.2 LINEAR SPRING AS A FINITE ELEMENT

A linear elastic spring is a mechanical device capable of supporting axial loading only and constructed such that, over a reasonable operating range (meaning extension or compression beyond undeformed length), the elongation or contraction of the spring is directly proportional to the applied axial load. The constant of proportionality between deformation and load is referred to as the *spring constant, spring rate,* or *spring stiffness* [4], generally denoted as k, and has units of force per unit length. Formulation of the linear spring as a finite element is accomplished with reference to Figure 2.1a. As an elastic spring supports axial loading only, we select an *element coordinate system* (also known as a *local* coordinate system) as an x axis oriented along the length of the spring, as shown. The element coordinate system is embedded in the element and chosen, by geometric convenience, for simplicity in describing element behavior. The element

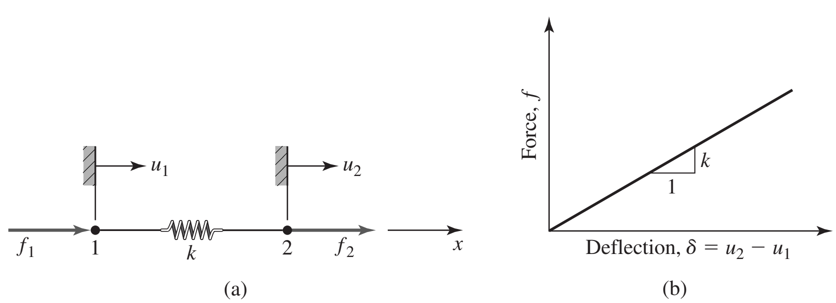

Figure 2.1
(a) Linear spring element with nodes, nodal displacements, and nodal forces.
(b) Load-deflection curve.

or local coordinate system is contrasted with the *global* coordinate system. The global coordinate system is that system in which the behavior of a complete structure is to be described. By *complete structure* is meant the assembly of many finite elements (at this point, several springs) for which we desire to compute response to loading conditions. In this chapter, we deal with cases in which the local and global coordinate systems are essentially the same except for translation of origin. In two- and three-dimensional cases, however, the distinctions are quite different and require mathematical rectification of element coordinate systems to a common basis. The common basis is the global coordinate system.

Returning attention to Figure 2.1a, the ends of the spring are the *nodes* and the nodal displacements are denoted by u_1 and u_2 and are shown in the positive sense. If these nodal displacements are known, the total elongation or contraction of the spring is known as is the *net force* in the spring. At this point in our development, we require that forces be applied to the element only at the nodes (distributed forces are accommodated for other element types later), and these are denoted as f_1 and f_2 and are also shown in the positive sense.

Assuming that both the nodal displacements are zero when the spring is undeformed, the net spring deformation is given by

$$\delta = u_2 - u_1 \tag{2.1}$$

and the resultant axial force in the spring is

$$f = k\delta = k(u_2 - u_1) \tag{2.2}$$

as is depicted in Figure 2.1b.

For equilibrium, $f_1 + f_2 = 0$ or $f_1 = -f_2$, and we can rewrite Equation 2.2 in terms of the applied nodal forces as

$$f_1 = -k(u_2 - u_1) \tag{2.3a}$$

$$f_2 = k(u_2 - u_1) \tag{2.3b}$$

which can be expressed in matrix form (see Appendix A for a review of matrix algebra) as

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{2.4}$$

or

$$[k_e]\{u\} = \{f\} \tag{2.5}$$

where

$$[k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{2.6}$$

is defined as the element stiffness matrix in the element coordinate system (or local system), $\{u\}$ is the column matrix (vector) of nodal displacements, and $\{f\}$ is the column matrix (vector) of element nodal forces. (In subsequent chapters, the matrix notation is used extensively. A general matrix is designated by brackets [ ] and a column matrix (vector) by braces { }.)

Equation 2.6 shows that the element stiffness matrix for the linear spring element is a 2×2 matrix. This corresponds to the fact that the element exhibits two nodal displacements (or degrees of freedom) and that the two displacements are not independent (that is, the body is continuous and elastic). Furthermore, the matrix is symmetric. A symmetric matrix has off-diagonal terms such that $k_{ij} = k_{ji}$. Symmetry of the stiffness matrix is indicative of the fact that the body is linearly elastic and each displacement is related to the other by the same physical phenomenon. For example, if a force F (positive, tensile) is applied at node 2 with node 1 held fixed, the *relative* displacement of the two nodes is the same as if the force is applied *symmetrically* (negative, tensile) at node 1 with node 2 fixed. (Counterexamples to symmetry are seen in heat transfer and fluid flow analyses in Chapters 7 and 8.) As will be seen as more complicated structural elements are developed, this is a general result: An element exhibiting N degrees of freedom has a corresponding $N \times N$, symmetric stiffness matrix.

Next consider solution of the system of equations represented by Equation 2.4. In general, the nodal forces are prescribed and the objective is to solve for the unknown nodal displacements. Formally, the solution is represented by

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [k_e]^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{2.7}$$

where $[k_e]^{-1}$ is the inverse of the element stiffness matrix. However, this inverse matrix does not exist, since the determinant of the element stiffness matrix is identically zero. Therefore, the element stiffness matrix is *singular,* and this also proves to be a general result in most cases. The physical significance of the singular nature of the element stiffness matrix is found by reexamination of Figure 2.1a, which shows that no displacement constraint whatever has been imposed on motion of the spring element; that is, the spring is not connected to any physical object that would prevent or limit motion of either node. With no constraint, it is not possible to solve for the nodal displacements individually. Instead, only the *difference* in nodal displacements can be determined, as this difference represents the elongation or contraction of the spring element owing to elastic effects. As discussed in more detail in the general formulation of interpolation functions (Chapter 6) and structural dynamics (Chapter 10), a properly formulated finite element must allow for constant value of the field variable. In the example at hand, this means rigid body motion. We can see the rigid body motion capability in terms of a single spring (element) and in the context of several connected elements. For a single, unconstrained element, if arbitrary forces are applied at each node, the spring not only deforms axially but also undergoes acceleration according to Newton's second law. Hence, there exists not only deformation but overall motion. If, in a connected system of spring elements, the overall system response is such that nodes 1 and 2 of a particular element displace the same amount, there is no elastic deformation of the spring and therefore no elastic force in the spring. This physical situation must be included in the element formulation. The capability is indicated mathematically by singularity of the element stiffness matrix. As the stiffness matrix is formulated on the basis of *deformation* of the element, we cannot expect to compute nodal displacements if there is no deformation of the element.

Equation 2.7 indicates the mathematical operation of inverting the stiffness matrix to obtain solutions. In the context of an individual element, the singular nature of an element stiffness matrix precludes this operation, as the inverse of a singular matrix does not exist. As is illustrated profusely in the remainder of the text, the general solution of a finite element problem, in a global, as opposed to element, context, involves the solution of equations of the form of Equation 2.5. For realistic finite element models, which are of huge dimension in terms of the matrix order ($N \times N$) involved, computing the inverse of the stiffness matrix is a very inefficient, time-consuming operation, which should not be undertaken except for the very simplest of systems. Other, more-efficient solution techniques are available, and these are discussed subsequently. (Many of the end-of-chapter problems included in this text are of small order and can be efficiently solved via matrix inversion using "spreadsheet" software functions or software such as MATLAB.)

2.2.1 System Assembly in Global Coordinates

Derivation of the element stiffness matrix for a spring element was based on equilibrium conditions. The same procedure can be applied to a connected system of spring elements by writing the equilibrium equation for each node. However, rather than drawing free-body diagrams of each node and formally writing the equilibrium equations, the nodal equilibrium equations can be obtained more efficiently by considering the effect of each element separately and adding the element force contribution to each nodal equation. The process is described as "assembly," as we take individual stiffness components and "put them together" to obtain the system equations. To illustrate, via a simple example, the assembly of element characteristics into *global* (or *system*) equations, we next consider the system of two linear spring elements connected as shown in Figure 2.2.

For generality, it is assumed that the springs have different spring constants k_1 and k_2. The nodes are numbered 1, 2, and 3 as shown, with the springs sharing node 2 as the physical connection. Note that these are *global* node numbers. The *global* nodal displacements are identified as U_1, U_2, and U_3, where the upper case is used to indicate that the quantities represented are *global* or *system* displacements as opposed to individual element displacements. Similarly, applied nodal

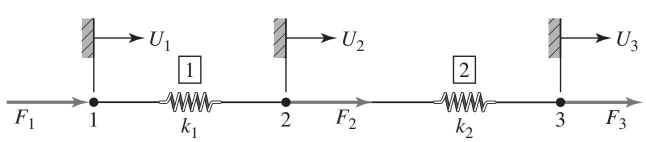

Figure 2.2 System of two springs with node numbers, element numbers, nodal displacements, and nodal forces.

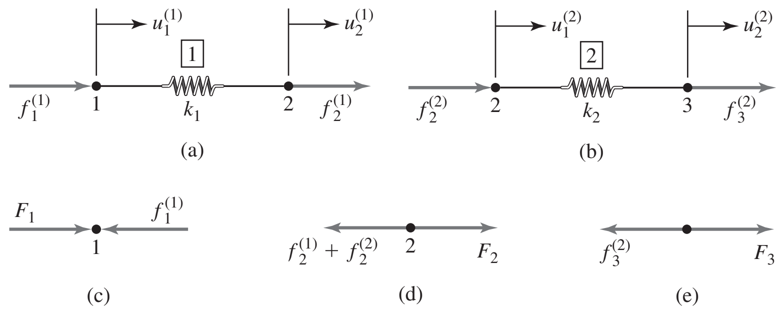

Figure 2.3 Free-body diagrams of elements and nodes for the two-element system of Figure 2.2.

forces are F_1, F_2, and F_3. Assuming the system of two spring elements to be in equilibrium, we examine free-body diagrams of the springs individually (Figure 2.3a and 2.3b) and express the equilibrium conditions for each spring, using Equation 2.4, as

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \tag{2.8a}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} \tag{2.8b}$$

where the superscript is element number.

To begin "assembling" the equilibrium equations describing the behavior of the system of two springs, the displacement *compatibility conditions,* which relate element displacements to system displacements, are written as

$$u_1^{(1)} = U_1 \qquad u_2^{(1)} = U_2 \qquad u_1^{(2)} = U_2 \qquad u_2^{(2)} = U_3 \tag{2.9}$$

The compatibility conditions state the physical fact that the springs are connected at node 2, remain connected at node 2 after deformation, and hence, must have the same nodal displacement at node 2. Thus, element-to-element displacement continuity is enforced at nodal connections. Substituting Equations 2.9 into Equations 2.8, we obtain

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \tag{2.10a}$$

and

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} \tag{2.10b}$$

Here, we use the notation $f_i^{(j)}$ to represent the force exerted on element j at node i.

Equation 2.10 is the equilibrium equations for each spring element expressed in terms of the specified global displacements. In this form, the equations clearly show that the elements are physically connected at node 2 and have the same displacement U_2 at that node. These equations are not yet amenable to direct combination, as the displacement vectors are not the same. We expand both matrix equations to 3×3 as follows (while formally expressing the facts that element 1 is not connected to node 3 and element 2 is not connected to node 1):

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \end{Bmatrix} \tag{2.11}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} \tag{2.12}$$

The addition of Equations 2.11 and 2.12 yields

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} \tag{2.13}$$

Next, we refer to the free-body diagrams of each of the three nodes depicted in Figure 2.3c, 2.3d, and 2.3e. The equilibrium conditions for nodes 1, 2, and 3 show that

$$f_1^{(1)} = F_1 \qquad f_2^{(1)} + f_2^{(2)} = F_2 \qquad f_3^{(2)} = F_3 \tag{2.14}$$

respectively. Substituting into Equation 2.13, we obtain the final result:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{2.15}$$

which is of the form $[K]\{U\} = \{F\}$, similar to Equation 2.5. However, Equation 2.15 represents the equations governing the *system* composed of two connected spring elements. By direct consideration of the equilibrium conditions, we obtain the system stiffness matrix $[K]$ (note use of upper case) as

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \tag{2.16}$$

Note that the system stiffness matrix is (1) symmetric, as is the case with all linear systems referred to orthogonal coordinate systems; (2) singular, since no constraints are applied to prevent rigid body motion of the system; and (3) the system matrix is simply a superposition of the individual element stiffness matrices with proper assignment of element nodal displacements and associated stiffness coefficients to system nodal displacements. The superposition procedure is formalized in the context of frame structures in the following paragraphs.

EXAMPLE 2.1

Consider the two element system depicted in Figure 2.2 given that

Node 1 is attached to a fixed support, yielding the displacement constraint $U_1 = 0$.
$k_1 = 50$ lb./in., $k_2 = 75$ lb./in., $F_2 = F_3 = 75$ lb.

for these conditions determine nodal displacements U_2 and U_3.

■ **Solution**

Substituting the specified values into Equation 2.15 yields

$$\begin{bmatrix} 50 & -50 & 0 \\ -50 & 125 & -75 \\ 0 & -75 & 75 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 75 \\ 75 \end{Bmatrix}$$

and we note that, owing to the constraint of zero displacement at node 1, nodal force F_1 becomes an unknown reaction force. Formally, the first algebraic equation represented in this matrix equation becomes

$$-50U_2 = F_1$$

and this is known as a *constraint equation,* as it represents the equilibrium condition of a node at which the displacement is constrained. The second and third equations become

$$\begin{bmatrix} 125 & -75 \\ -75 & 75 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 75 \\ 75 \end{Bmatrix}$$

which can be solved to obtain $U_2 = 3$ in. and $U_3 = 4$ in. Note that the matrix equations governing the unknown displacements are obtained by simply striking out the first row and column of the 3×3 matrix system, since the constrained displacement is zero. Hence, the constraint does not affect the values of the *active* displacements (we use the term *active* to refer to displacements that are unknown and must be computed). Substitution of the calculated values of U_2 and U_3 into the constraint equation yields the value $F_1 = -150$ lb., which value is clearly in equilibrium with the applied nodal forces of 75 lb. each. We also illustrate element equilibrium by writing the equations for each element as

$$\begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix} \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} -150 \\ 150 \end{Bmatrix} \text{lb.} \qquad \text{for element 1}$$

$$\begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} -75 \\ 75 \end{Bmatrix} \text{lb.} \qquad \text{for element 2}$$

Example 2.1 illustrates the general procedure for solution of finite element models: Formulate the system equilibrium equations, apply the specified constraint conditions, solve the reduced set of equations for the "active" displacements, and substitute the computed displacements into the constraint equations to obtain the unknown reactions. While not directly applicable for the spring element, for more general finite element formulations, the computed displacements are also substituted into the strain relations to obtain element strains, and the strains are, in turn, substituted into the applicable stress-strain equations to obtain element stress values.

EXAMPLE 2.2

Figure 2.4a depicts a system of three linearly elastic springs supporting three equal weights W suspended in a vertical plane. Treating the springs as finite elements, determine the vertical displacement of each weight.

■ Solution

To treat this as a finite element problem, we assign node and element numbers as shown in Figure 2.4b and ignore, for the moment, that displacement U_1 is known to be zero by the fixed support constraint. Per Equation 2.6, the stiffness matrix of each element is (preprocessing)

$$\left[k^{(1)}\right] = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}$$

$$\left[k^{(2)}\right] = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix}$$

$$\left[k^{(3)}\right] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

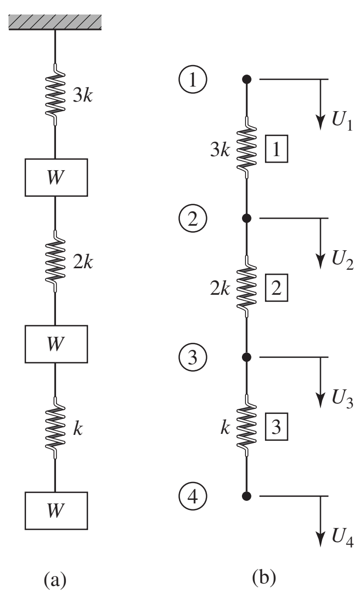

Figure 2.4 Example 2.2: elastic spring supporting weights.

The element-to-global displacement relations are

$$u_1^{(1)} = U_1 \qquad u_2^{(1)} = u_1^{(2)} = U_2 \qquad u_2^{(2)} = u_1^{(3)} = U_3 \qquad u_2^{(3)} = U_4$$

Proceeding as in the previous example, we then write the individual element equations as

$$\begin{bmatrix} 3k & -3k & 0 & 0 \\ -3k & 3k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} \tag{1}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2k & -2k & 0 \\ 0 & -2k & 2k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^{(2)} \\ f_2^{(2)} \\ 0 \end{Bmatrix} \tag{2}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_1^{(3)} \\ f_2^{(3)} \end{Bmatrix} \tag{3}$$

Adding Equations 1–3, we obtain

$$k \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ W \\ W \\ W \end{Bmatrix} \tag{4}$$

where we utilize the fact that the sum of the element forces at each node must equal the applied force at that node and, at node 1, the force is an unknown reaction.

Applying the displacement constraint $U_1 = 0$ (*this is also preprocessing*), we obtain

$$-3kU_2 = F_1 \tag{5}$$

as the constraint equation and the matrix equation

$$k \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} W \\ W \\ W \end{Bmatrix} \tag{6}$$

for the active displacements. Again note that Equation 6 is obtained by eliminating the constraint equation from 4 corresponding to the prescribed zero displacement.

Simultaneous solution (*the solution step*) of the algebraic equations represented by Equation 6 yields the displacements as

$$U_2 = \frac{W}{k} \qquad U_3 = \frac{2W}{k} \qquad U_4 = \frac{3W}{k}$$

and Equation 5 gives the reaction force as

$$F_1 = -3W$$

(This is *postprocessing*.)

Note that the solution is exactly that which would be obtained by the usual statics equations. Also note the general procedure as follows:

Formulate the individual element stiffness matrices.

Write the element to global displacement relations.

Assemble the global equilibrium equation in matrix form.

Reduce the matrix equations according to specified constraints.

Solve the system of equations for the unknown nodal displacements (primary variables).

Solve for the reaction forces (secondary variable) by back-substitution.

EXAMPLE 2.3

Figure 2.5 depicts a system of three linear spring elements connected as shown. The node and element numbers are as indicated. Node 1 is fixed to prevent motion, and node 3 is given a specified displacement δ as shown. Forces $F_2 = -F$ and $F_4 = 2F$ are applied at nodes 2 and 4. Determine the displacement of each node and the force required at node 3 for the specified conditions.

■ Solution

This example includes a *nonhomogeneous* boundary condition. In previous examples, the boundary conditions were represented by zero displacements. In this example, we have both a zero (homogeneous) and a specified nonzero (nonhomogeneous) displacement condition. The algebraic treatment must be different as follows. The system equilibrium equations are expressed in matrix form (Problem 2.6) as

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 4k & -3k & 0 \\ 0 & -3k & 5k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ -F \\ F_3 \\ 2F \end{Bmatrix}$$

Substituting the specified conditions $U_1 = 0$ and $U_3 = \delta$ results in the system of equations

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 4k & -3k & 0 \\ 0 & -3k & 5k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ \delta \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ -F \\ F_3 \\ 2F \end{Bmatrix}$$

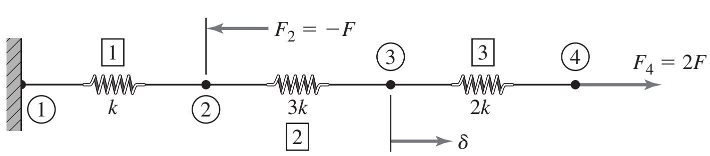

Figure 2.5 Example 2.3: Three-element system with specified nonzero displacement at node 3.

Since $U_1 = 0$, we remove the first row and column to obtain

$$\begin{bmatrix} 4k & -3k & 0 \\ -3k & 5k & -2k \\ 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} U_2 \\ \delta \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -F \\ F_3 \\ 2F \end{Bmatrix}$$

as the system of equations governing displacements U_2 and U_4 and the unknown nodal force F_3. This last set of equations clearly shows that we cannot simply strike out the row and column corresponding to the *nonzero* specified displacement δ because it appears in the equations governing the active displacements. To illustrate a general procedure, we rewrite the last matrix equation as

$$\begin{bmatrix} 5k & -3k & -2k \\ -3k & 4k & 0 \\ -2k & 0 & 2k \end{bmatrix} \begin{Bmatrix} \delta \\ U_2 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ -F \\ 2F \end{Bmatrix}$$

Next, we formally partition the stiffness matrix and write

$$\begin{bmatrix} 5k & -3k & -2k \\ -3k & 4k & 0 \\ -2k & 0 & 2k \end{bmatrix} \begin{Bmatrix} \delta \\ U_2 \\ U_4 \end{Bmatrix} = \begin{bmatrix} [K_{\delta\delta}] & [K_{\delta U}] \\ [K_{U\delta}] & [K_{UU}] \end{bmatrix} \begin{Bmatrix} \{\delta\} \\ \{U\} \end{Bmatrix} = \begin{Bmatrix} \{F_\delta\} \\ \{F_U\} \end{Bmatrix}$$

with

$$[K_{\delta\delta}] = [5k]$$

$$[K_{\delta U}] = [-3k \quad -2k]$$

$$[K_{U\delta}] = [K_{\delta U}]^T = \begin{bmatrix} -3k \\ -2k \end{bmatrix}$$

$$[K_{UU}] = \begin{bmatrix} 4k & 0 \\ 0 & 2k \end{bmatrix}$$

$$\{\delta\} = \{\delta\}$$

$$\{U\} = \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}$$

$$\{F_\delta\} = \{F_3\}$$

$$\{F_U\} = \begin{Bmatrix} -F \\ 2F \end{Bmatrix}$$

From the second "row" of the partitioned matrix equations, we have

$$[K_{U\delta}]\{\delta\} + [K_{UU}]\{U\} = \{F_U\}$$

and this can be solved for the unknown displacements to obtain

$$\{U\} = [K_{UU}]^{-1}(\{F\} - [K_{U\delta}]\{\delta\})$$

provided that $[K_{UU}]^{-1}$ exists. Since the constraints have been applied correctly, this inverse does exist and is given by

$$[K_{UU}]^{-1} = \begin{bmatrix} \dfrac{1}{4k} & 0 \\ 0 & \dfrac{1}{2k} \end{bmatrix}$$

Substituting, we obtain the unknown displacements as

$$\{U\} = \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{4k} & 0 \\ 0 & \dfrac{1}{2k} \end{bmatrix} \begin{Bmatrix} -F + 3k\delta \\ 2F + 2k\delta \end{Bmatrix} = \begin{Bmatrix} -\dfrac{F}{4k} + \dfrac{3\delta}{4} \\ \dfrac{F}{k} + \delta \end{Bmatrix}$$

The required force at node 3 is obtained by substitution of the displacement into the upper partition to obtain

$$F_3 = -\frac{5}{4}F + \frac{3}{4}k\delta$$

Finally, the reaction force at node 1 is

$$F_1 = -kU_2 = \frac{F}{4} - \frac{3}{4}k\delta$$

As a check on the results, we substitute the computed and prescribed displacements into the individual element equations to insure that equilibrium is satisfied.

Element 1

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} -kU_2 \\ kU_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

which shows that the nodal forces on element 1 are equal and opposite as required for equilibrium.

Element 2

$$\begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix} \begin{Bmatrix} -\dfrac{F}{4k} + \dfrac{3}{4}\delta \\ \delta \end{Bmatrix}$$

$$= \begin{Bmatrix} -\dfrac{3F}{4k} - \dfrac{3}{4}k\delta \\ \dfrac{3F}{4k} + \dfrac{3}{4}k\delta \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

which also verifies equilibrium.

Element 3

$$\begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} \delta \\ \dfrac{F}{k} + \delta \end{Bmatrix} = \begin{Bmatrix} -2F \\ 2F \end{Bmatrix} = \begin{Bmatrix} f_3^{(3)} \\ f_4^{(3)} \end{Bmatrix}$$

Therefore element 3 is in equilibrium as well.

2.3 ELASTIC BAR, SPAR/LINK/TRUSS ELEMENT

While the linear elastic spring serves to introduce the concept of the stiffness matrix, the usefulness of such an element in finite element analysis is rather limited. Certainly, springs are used in machinery in many cases and the availability of a finite element representation of a linear spring is quite useful in such cases. The spring element is also often used to represent the elastic nature of supports for more complicated systems. A more generally applicable, yet similar, element is an elastic bar subjected to axial forces only. This element, which we simply call a *bar element,* is particularly useful in the analysis of both two- and three-dimensional frame or truss structures. Formulation of the finite element characteristics of an elastic bar element is based on the following assumptions:

1. The bar is geometrically straight.
2. The material obeys Hooke's law.
3. Forces are applied only at the ends of the bar.
4. The bar supports axial loading only; bending, torsion, and shear are not transmitted to the element via the nature of its connections to other elements.

The last assumption, while quite restrictive, is not impractical; this condition is satisfied if the bar is connected to other structural members via pins (2-D) or ball-and-socket joints (3-D). Assumptions 1 and 4, in combination, show that this is inherently a one-dimensional element, meaning that the elastic displacement of any point along the bar can be expressed in terms of a single independent variable. As will be seen, however, the bar element can be used in modeling both two- and three-dimensional structures. The reader will recognize this element as the familiar two-force member of elementary statics, meaning, for equilibrium, the forces exerted on the ends of the element must be colinear, equal in magnitude, and opposite in sense.

Figure 2.6 depicts an elastic bar of length L to which is affixed a uniaxial coordinate system x with its origin arbitrarily placed at the left end. This is the *element* coordinate system or reference frame. Denoting axial displacement at any position along the length of the bar as $u(x)$, we define nodes 1 and 2 at each end as shown and introduce the nodal displacements $u_1 = u(x = 0)$ and $u_2 = u(x = L)$. Thus, we have the continuous field variable $u(x)$, which is to be expressed (approximately) in terms of two nodal variables u_1 and u_2. To accomplish this discretization, we assume the existence of *interpolation* functions $N_1(x)$ and $N_2(x)$ (also known as *shape* or *blending* functions) such that

$$u(x) = N_1(x)u_1 + N_2(x)u_2 \tag{2.17}$$

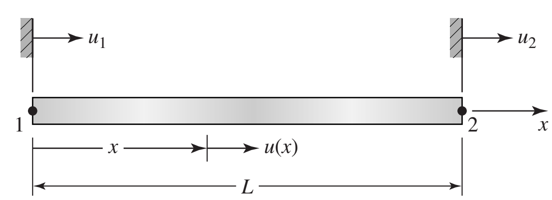

Figure 2.6 A bar (or truss) element with element coordinate system and nodal displacement notation.

(It must be emphasized that, although an equality is indicated by Equation 2.17, the relation, for finite elements in general, is an approximation. For the bar element, the relation, in fact, is exact.) To determine the interpolation functions, we require that the boundary values of $u(x)$ (the nodal displacements) be identically satisfied by the discretization such that

$$u(x = 0) = u_1 \qquad u(x = L) = u_2 \tag{2.18}$$

Equations 2.17 and 2.18 lead to the following boundary (nodal) conditions:

$$N_1(0) = 1 \qquad N_2(0) = 0 \tag{2.19}$$

$$N_1(L) = 0 \qquad N_2(L) = 1 \tag{2.20}$$

which must be satisfied by the interpolation functions. It is required that the displacement expression, Equation 2.17, satisfy the end (nodal) conditions identically, since the nodes will be the connection points between elements and the displacement continuity conditions are enforced at those connections. As we have two conditions that must be satisfied by each of two one-dimensional functions, the simplest forms for the interpolation functions are polynomial forms:

$$N_1(x) = a_0 + a_1 x \tag{2.21}$$

$$N_2(x) = b_0 + b_1 x \tag{2.22}$$

where the polynomial coefficients are to be determined via satisfaction of the boundary (nodal) conditions. We note here that any number of mathematical forms of the interpolation functions could be assumed while satisfying the required conditions. The reasons for the linear form is explained in detail in Chapter 6.

Application of conditions represented by Equation 2.19 yields $a_0 = 1$, $b_0 = 0$ while Equation 2.20 results in $a_1 = -(1/L)$ and $b_1 = x/L$. Therefore, the interpolation functions are

$$N_1(x) = 1 - x/L \tag{2.23}$$

$$N_2(x) = x/L \tag{2.24}$$

and the continuous displacement function is represented by the discretization

$$u(x) = (1 - x/L)u_1 + (x/L)u_2 \tag{2.25}$$

As will be found most convenient subsequently, Equation 2.25 can be expressed in matrix form as

$$u(x) = [N_1(x) \quad N_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{u\} \tag{2.26}$$

where $[N]$ is the row matrix of interpolation functions and $\{u\}$ is the column matrix (vector) of nodal displacements.

Having expressed the displacement field in terms of the nodal variables, the task remains to determine the relation between the nodal displacements and applied forces to obtain the stiffness matrix for the bar element. Recall from elementary strength of materials that the deflection δ of an elastic bar of length L and uniform cross-sectional area A when subjected to axial load P is given by

$$\delta = \frac{PL}{AE} \tag{2.27}$$

where E is the modulus of elasticity of the material. Using Equation 2.27, we obtain the equivalent spring constant of an elastic bar as

$$k = \frac{P}{\delta} = \frac{AE}{L} \tag{2.28}$$

and could, by analogy with the linear elastic spring, immediately write the stiffness matrix as Equation 2.6. While the result is exactly correct, we take a more general approach to illustrate the procedures to be used with more complicated element formulations.

Ultimately, we wish to compute the nodal displacements given some loading condition on the element. To obtain the necessary equilibrium equations relating the displacements to applied forces, we proceed from displacement to strain, strain to stress, and stress to loading, as follows. In uniaxial loading, as in the bar element, we need consider only the normal strain component, defined as

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} \tag{2.29}$$

which, when applied to Equation 2.25, gives

$$\varepsilon_x = \frac{u_2 - u_1}{L} \tag{2.30}$$

which shows that the spar element is a constant strain element. This is in accord with strength of materials theory: The element has constant cross-sectional area and is subjected to constant forces at the end points, so the strain does not vary along the length. The axial stress, by Hooke's law, is then

$$\sigma_x = E\varepsilon_x = E\frac{u_2 - u_1}{L} \tag{2.31}$$

and the associated axial force is

$$P = \sigma_x A = \frac{AE}{L}(u_2 - u_1) \tag{2.32}$$

Taking care to observe the correct algebraic sign convention, Equation 2.32 is now used to relate the applied nodal forces f_1 and f_2 to the nodal displacements u_1 and u_2. Observing that, if Equation 2.32 has a positive sign, the element is in tension and nodal force f_2 must be in the positive coordinate direction while nodal force f_1 must be equal and opposite for equilibrium; therefore,

$$f_1 = -\frac{AE}{L}(u_2 - u_1) \tag{2.33}$$

$$f_2 = \frac{AE}{L}(u_2 - u_1) \tag{2.34}$$

Equations 2.33 and 2.34 are expressed in matrix form as

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{2.35}$$

Comparison of Equation 2.35 to Equation 2.4 shows that the stiffness matrix for the bar element is given by

$$[k_e] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.36}$$

As is the case with the linear spring, we observe that the element stiffness matrix for the bar element is symmetric, singular, and of order 2×2 in correspondence with two nodal displacements or *degrees of freedom*. It must be emphasized that the stiffness matrix given by Equation 2.36 is expressed in the *element coordinate system,* which in this case is one-dimensional. Application of this element formulation to analysis of two- and three-dimensional structures is considered in the next chapter.

EXAMPLE 2.4

Figure 2.7a depicts a tapered elastic bar subjected to an applied tensile load P at one end and attached to a fixed support at the other end. The cross-sectional area varies linearly from A_0 at the fixed support at $x = 0$ to $A_0/2$ at $x = L$. Calculate the displacement of the end of the bar (a) by modeling the bar as a single element having cross-sectional area equal to the area of the actual bar at its midpoint along the length, (b) using two bar elements of equal length and similarly evaluating the area at the midpoint of each, and (c) using integration to obtain the exact solution.

■ **Solution**

(a) For a single element, the cross-sectional area is $3A_0/4$ and the element "spring constant" is

$$k = \frac{AE}{L} = \frac{3A_0E}{4L}$$

and the element equations are

$$\frac{3A_0E}{4L}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \end{Bmatrix}$$

The element and nodal displacements are as shown in Figure 2.7b. Applying the constraint condition $U_1 = 0$, we find

$$U_2 = \frac{4PL}{3A_0E} = 1.333\frac{PL}{A_0E}$$

as the displacement at $x = L$.

(b) Two elements of equal length $L/2$ with associated nodal displacements are depicted in Figure 2.7c. For element 1, $A_1 = 7A_0/8$ so

$$k_1 = \frac{A_1E}{L_1} = \frac{7A_0E}{8(L/2)} = \frac{7A_0E}{4L}$$

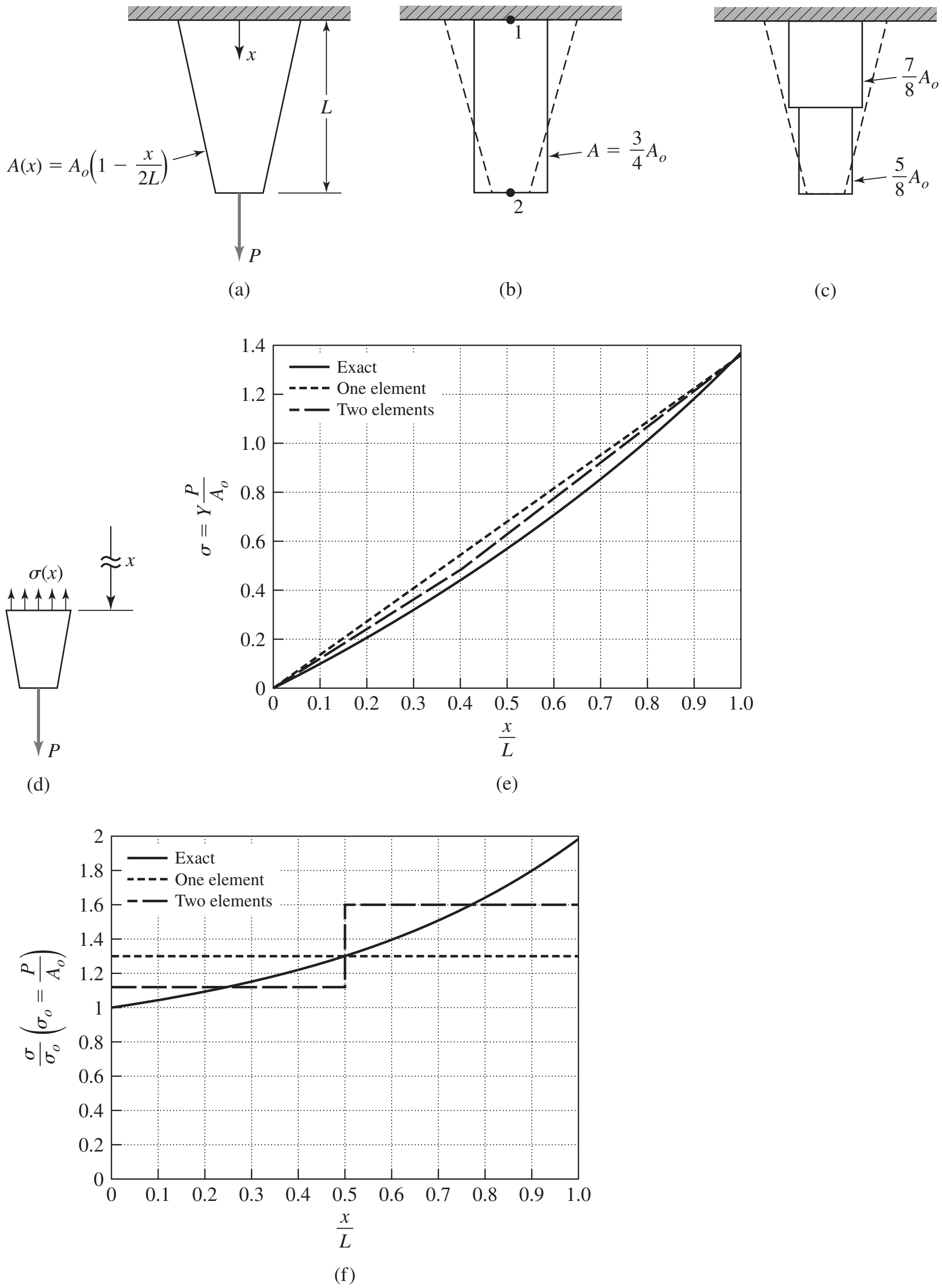

Figure 2.7
(a) Tapered axial bar, (b) one-element model, (c) two-element model, (d) free-body diagram for an exact solution, (e) displacement solutions, (f) stress solutions.

while for element 2, we have

$$A_1 = \frac{5A_0}{8} \quad \text{and} \quad k_2 = \frac{A_2 E}{L_2} = \frac{5A_0 E}{8(L/2)} = \frac{5A_0 E}{4L}$$

Since no load is applied at the center of the bar, the equilibrium equations for the system of two elements is

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \end{Bmatrix}$$

Applying the constraint condition $U_1 = 0$ results in

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Adding the two equations gives

$$U_2 = \frac{P}{k_1} = \frac{4PL}{7A_0 E}$$

and substituting this result into the first equation results in

$$U_3 = \frac{k_1 + k_2}{k_2} = \frac{48PL}{35A_0 E} = 1.371\frac{PL}{A_0 E}$$

(c) To obtain the exact solution, we refer to Figure 2.7d, which is a free-body diagram of a section of the bar between an arbitrary position x and the end $x = L$. For equilibrium,

$$\sigma_x A = P \quad \text{and since} \quad A = A(x) = A_0\left(1 - \frac{x}{2L}\right)$$

the axial stress variation along the length of the bar is described by

$$\sigma_x = \frac{P}{A_0\left(1 - \dfrac{x}{2L}\right)}$$

Therefore, the axial strain is

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P}{EA_0\left(1 - \dfrac{x}{2L}\right)}$$

Since the bar is fixed at $x = 0$, the displacement at $x = L$ is given by

$$\delta = \int_0^L \varepsilon_x \,\mathrm{d}x = \frac{P}{EA_0}\int_0^L \frac{\mathrm{d}x}{\left(1 - \dfrac{x}{2L}\right)}$$

$$= \frac{2PL}{EA_0}[-\ln(2L - x)]\Big|_0^L = \frac{2PL}{EA_0}[\ln(2L) - \ln L] = \frac{2PL}{EA_0}\ln 2 = 1.386\frac{PL}{A_0 E}$$

Comparison of the results of parts b and c reveals that the two element solution exhibits an error of only about 1 percent in comparison to the exact solution from strength of materials theory. Figure 2.7e shows the displacement variation along the length for the three solutions. It is extremely important to note, however, that the computed axial stress for the finite element solutions varies significantly from that of the exact solution. The axial stress for the two-element solution is shown in Figure 2.7f, along with the calculated stress from the exact solution. Note particularly the discontinuity of calculated stress values for the two elements at the connecting node. This is typical of the derived, or secondary, variables, such as stress and strain, as computed in the finite element method. As more and more smaller elements are used in the model, the values of such discontinuities decrease, indicating solution convergence. In structural analyses, the finite element user is most often more interested in stresses than displacements, hence it is essential that convergence of the secondary variables be monitored.

2.4 STRAIN ENERGY, CASTIGLIANO'S FIRST THEOREM

When external forces are applied to a body, the mechanical work done by those forces is converted, in general, into a combination of kinetic and potential energies. In the case of an elastic body constrained to prevent motion, all the work is stored in the body as elastic potential energy, which is also commonly referred to as *strain energy*. Here, strain energy is denoted U_e and mechanical work W. From elementary statics, the mechanical work performed by a force $\vec{F}$ as its point of application moves along a path from position 1 to position 2 is defined as

$$W = \int_1^2 \vec{F} \cdot \mathrm{d}\vec{r} \tag{2.37}$$

where

$$\mathrm{d}\vec{r} = \mathrm{d}x\vec{i} + \mathrm{d}y\vec{j} + \mathrm{d}z\vec{k} \tag{2.38}$$

is a differential vector along the path of motion. In Cartesian coordinates, work is given by

$$W = \int_{x_1}^{x_2} F_x\,\mathrm{d}x + \int_{y_1}^{y_2} F_y\,\mathrm{d}y + \int_{z_1}^{z_2} F_z\,\mathrm{d}z \tag{2.39}$$

where F_x, F_y, and F_z are the Cartesian components of the force vector.

For linearly elastic deformations, deflection is directly proportional to applied force as, for example, depicted in Figure 2.8 for a linear spring. The slope of the force-deflection line is the spring constant such that $F = k\delta$. Therefore, the work required to deform such a spring by an arbitrary amount δ_0 from its

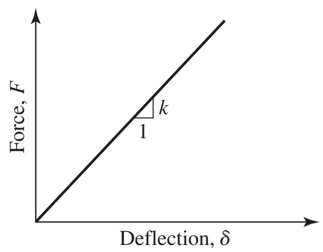

Figure 2.8 Force-deflection relation for a linear elastic spring.

free length is

$$W = \int_0^{\delta_0} F\, d\delta = \int_0^{\delta_0} k\delta\, d\delta = \frac{1}{2}k\delta_0^2 = U_e \tag{2.40}$$

and we observe that the work and resulting elastic potential energy are quadratic functions of displacement and have the units of force-length. This is a general result for linearly elastic systems, as will be seen in many examples throughout this text.

Utilizing Equation 2.28, the strain energy for an axially loaded elastic bar fixed at one end can immediately be written as

$$U_e = \frac{1}{2}k\delta^2 = \frac{1}{2}\frac{AE}{L}\delta^2 \tag{2.41}$$

However, for a more general purpose, this result is converted to a different form (applicable to a bar element only) as follows:

$$U_e = \frac{1}{2}k\delta^2 = \frac{1}{2}\frac{AE}{L}\left(\frac{PL}{AE}\right)^2 = \frac{1}{2}\left(\frac{P}{A}\right)\left(\frac{P}{AE}\right)AL = \frac{1}{2}\sigma\varepsilon V \tag{2.42}$$

where V is the total volume of deformed material and the quantity $\frac{1}{2}\sigma\varepsilon$ is *strain energy per unit volume,* also known as *strain energy density*. In Equation 2.42, stress and strain values are those corresponding to the *final* value of applied force. The factor $\frac{1}{2}$ arises from the linear relation between stress and strain as the load is applied from zero to the final value P. In general, for uniaxial loading, the strain energy per unit volume u_e is defined by

$$u_e = \int_0^{\varepsilon} \sigma\, d\varepsilon \tag{2.43}$$

which is extended to more general states of stress in subsequent chapters. We note that Equation 2.43 represents the area under the elastic stress-strain diagram.

Presently, we will use the work-strain energy relation to obtain the governing equations for the bar element using the following theorem.

Castigliano's First Theorem [1]

For an elastic system in equilibrium, the partial derivative of total strain energy with respect to deflection at a point is equal to the applied force in the direction of the deflection at that point.

Consider an elastic body subjected to N forces F_j for which the total strain energy is expressed as

$$U_e = W = \sum_{j=1}^{N} \int_0^{\delta_j} F_j \, \mathrm{d}\delta_j \tag{2.44}$$

where δ_j is the deflection at the point of application of force F_j in the direction of the line of action of the force. If all points of load application are fixed except one, say, i, and that point is made to deflect an infinitesimal amount $\Delta\delta_i$ by an incremental infinitesimal force ΔF_i, the change in strain energy is

$$\Delta U_e = \Delta W = F_i \Delta\delta_i + \int_0^{\Delta\delta_i} \Delta F_i \, \mathrm{d}\delta_i \tag{2.45}$$

where it is assumed that the original force F_i is constant during the infinitesimal change. The integral term in Equation 2.45 involves the product of infinitesimal quantities and can be neglected to obtain

$$\frac{\Delta U_e}{\Delta\delta_i} = F_i \tag{2.46}$$

which in the limit as $\Delta\delta_i$ approaches zero becomes

$$\frac{\partial U}{\partial\delta_i} = F_i \tag{2.47}$$

The first theorem of Castigliano is a powerful tool for finite element formulation, as is now illustrated for the bar element. Combining Equations 2.30, 2.31, and 2.43, total strain energy for the bar element is given by

$$U_e = \frac{1}{2}\sigma_x \varepsilon_x V = \frac{1}{2} E \left(\frac{u_2 - u_1}{L} \right)^2 AL \tag{2.48}$$

Applying Castigliano's theorem with respect to each displacement yields

$$\frac{\partial U_e}{\partial u_1} = \frac{AE}{L}(u_1 - u_2) = f_1 \tag{2.49}$$

$$\frac{\partial U_e}{\partial u_2} = \frac{AE}{L}(u_2 - u_1) = f_2 \tag{2.50}$$

which are observed to be identical to Equations 2.33 and 2.34.

The first theorem of Castigliano is also applicable to rotational displacements. In the case of rotation, the partial derivative of strain energy with respect to a rotational displacement is equal to the moment/torque applied at the point of concern in the sense of the rotation. The following example illustrates the application in terms of a simple torsional member.

EXAMPLE 2.5

A solid circular shaft of radius R and length L is subjected to constant torque T. The shaft is fixed at one end, as shown in Figure 2.9. Formulate the elastic strain energy in terms of the angle of twist θ at $x = L$ and show that Castigliano's first theorem gives the correct expression for the applied torque.

■ Solution

From strength of materials theory, the shear stress at any cross section along the length of the member is given by

$$\tau = \frac{Tr}{J}$$

where r is radial distance from the axis of the member and J is polar moment of inertia of the cross section. For elastic behavior, we have

$$\gamma = \frac{\tau}{G} = \frac{Tr}{JG}$$

where G is the shear modulus of the material, and the strain energy is then

$$U_e = \frac{1}{2}\int_V \tau\gamma \,\mathrm{d}V = \frac{1}{2}\int_0^L \left[\int_A \left(\frac{Tr}{J}\right)\left(\frac{Tr}{JG}\right)\mathrm{d}A\right]\mathrm{d}x$$

$$= \frac{T^2}{2J^2G}\int_0^L\int_A r^2\,\mathrm{d}A\,\mathrm{d}x = \frac{T^2L}{2JG}$$

where we have used the definition of the polar moment of inertia

$$J = \int_A r^2\,\mathrm{d}A$$

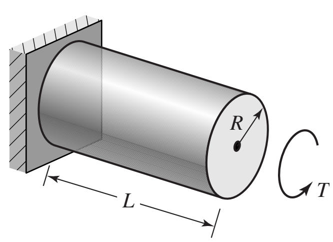

Figure 2.9 Example 2.5: Circular cylinder subjected to torsion.

Again invoking the strength of materials results, the angle of twist at the end of the member is known to be

$$\theta = \frac{TL}{JG}$$

so the strain energy can be written as

$$U_e = \frac{1}{2}\frac{L}{JG}\left(\frac{JG\theta}{L}\right)^2 = \frac{JG}{2L}\theta^2$$

Per Castangliano's first theorem,

$$\frac{\partial U_e}{\partial \theta} = T = \frac{JG\theta}{L}$$

which is exactly the relation shown by strength of materials theory. The reader may think that we used circular reasoning in this example, since we utilized many previously known results. However, the formulation of strain energy must be based on known stress and strain relationships, and the application of Castigliano's theorem is, indeed, a different concept.

For linearly elastic systems, formulation of the strain energy function in terms of displacements is relatively straightforward. As stated previously, the strain energy for an elastic system is a quadratic function of displacements. The quadratic nature is simplistically explained by the facts that, in elastic deformation, stress is proportional to force (or moment or torque), stress is proportional to strain, and strain is proportional to displacement (or rotation). And, since the elastic strain energy is equal to the mechanical work expended, a quadratic function results. Therefore, application of Castigliano's first theorem results in linear algebraic equations that relate displacements to applied forces. This statement follows from the fact that a derivative of a quadratic term is linear. The coefficients of the displacements in the resulting equations are the components of the stiffness matrix of the system for which the strain energy function is written. Such an energy-based approach is the simplest, most-straightforward method for establishing the stiffness matrix of many structural finite elements.

EXAMPLE 2.6

(a) Apply Castigliano's first theorem to the system of four spring elements depicted in Figure 2.10 to obtain the system stiffness matrix. The vertical members at nodes 2 and 3 are to be considered rigid.
(b) Solve for the displacements and the reaction force at node 1 if

$$k_1 = 4\text{ N/mm} \qquad k_2 = 6\text{ N/mm} \qquad k_3 = 3\text{ N/mm}$$

$$F_2 = -30\text{ N} \qquad F_3 = 0 \qquad F_4 = 50\text{ N}$$

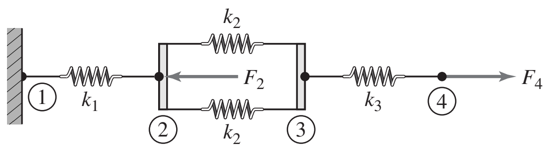

Figure 2.10 Example 2.6: Four spring elements.

■ Solution

(a) The total strain energy of the system of four springs is expressed in terms of the nodal displacements and spring constants as

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2$$

Applying Castigliano's theorem, using each nodal displacement in turn,

$$\frac{\partial U_e}{\partial U_1} = F_1 = k_1(U_2 - U_1)(-1) = k_1(U_1 - U_2)$$

$$\frac{\partial U_e}{\partial U_2} = F_2 = k_1(U_2 - U_1) + 2k_2(U_3 - U_2)(-1) = -k_1U_1 + (k_1 + 2k_2)U_2 - 2k_2U_3$$

$$\frac{\partial U_e}{\partial U_3} = F_3 = 2k_2(U_3 - U_2) + k_3(U_4 - U_3)(-1) = -2k_2U_2 + (2k_2 + k_3)U_3 - k_3U_4$$

$$\frac{\partial U_e}{\partial U_4} = F_4 = k_3(U_4 - U_3) = -k_3U_3 + k_3U_4$$

which can be written in matrix form as

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

and the system stiffness matrix is thus obtained via Castigliano's theorem.

(b) Substituting the specified numerical values, the system equations become

$$\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 16 & -12 & 0 \\ 0 & -12 & 15 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ -30 \\ 0 \\ 50 \end{Bmatrix}$$

Eliminating the constraint equation, the active displacements are governed by

$$\begin{bmatrix} 16 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ 0 \\ 50 \end{Bmatrix}$$

which we solve by manipulating the equations to convert the coefficient matrix (the stiffness matrix) to upper-triangular form; that is, all terms below the main diagonal become zero.

Step 1. Multiply the first equation (row) by 12, multiply the second equation (row) by 16, add the two and replace the second equation with the resulting equation to obtain

$$\begin{bmatrix} 16 & -12 & 0 \\ 0 & 96 & -48 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -360 \\ 50 \end{Bmatrix}$$

Step 2. Multiply the third equation by 32, add it to the second equation, and replace the third equation with the result. This gives the triangularized form desired:

$$\begin{bmatrix} 16 & -12 & 0 \\ 0 & 96 & -48 \\ 0 & 0 & 48 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -360 \\ 1240 \end{Bmatrix}$$

In this form, the equations can now be solved from the "bottom to the top," and it will be found that, at each step, there is only one unknown. In this case, the sequence is

$$U_4 = \frac{1240}{48} = 25.83 \text{ mm}$$

$$U_3 = \frac{1}{96}[-360 + 48(25.83)] = 9.17 \text{ mm}$$

$$U_2 = \frac{1}{16}[-30 + 12(9.17)] = 5.0 \text{ mm}$$

The reaction force at node 1 is obtained from the constraint equation

$$F_1 = -4U_2 = -4(5.0) = -20 \text{ N}$$

and we observe system equilibrium since the external forces sum to zero as required.

2.5 MINIMUM POTENTIAL ENERGY

The first theorem of Castigliano is but a forerunner to the general principle of *minimum potential energy*. There are many ways to state this principle, and it has been proven rigorously [2]. Here, we state the principle without proof but expect the reader to compare the results with the first theorem of Castigliano. The principle of minimum potential energy is stated as follows:

Of all displacement states of a body or structure, subjected to external loading, that satisfy the geometric boundary conditions (imposed displacements), the displacement state that also satisfies the equilibrium equations is such that the total *potential energy is a minimum for stable equilibrium.*

We emphasize that the *total* potential energy must be considered in application of this principle. The total potential energy includes the stored elastic potential energy (the strain energy) as well as the potential energy of applied loads. As is customary, we use the symbol Π for total potential energy and divide the total potential energy into two parts, that portion associated with strain energy U_e and the portion associated with external forces U_F. The total potential energy is

$$\Pi = U_e + U_F \tag{2.51}$$

where it is to be noted that the term external *forces* also includes moments and torques.

In this text, we will deal only with elastic systems subjected to *conservative* forces. A *conservative force* is defined as one that does mechanical work independent of the path of motion and such that the work is reversible or recoverable. The most common example of a *nonconservative* force is the force of sliding friction. As the friction force always acts to oppose motion, the work done by friction forces is always negative and results in energy loss. This loss shows itself physically as generated heat. On the other hand, the mechanical work done by a conservative force, Equation 2.37, is reversed, and therefore recovered, if the force is released. Therefore, the mechanical work of a conservative force is considered to be a loss in potential energy; that is,

$$U_F = -W \tag{2.52}$$

where W is the mechanical work defined by the scalar product integral of Equation 2.37. The total potential energy is then given by

$$\Pi = U_e - W \tag{2.53}$$

As we show in the following examples and applications to solid mechanics in Chapter 9, the strain energy term U_e is a quadratic function of system displacements and the work term W is a linear function of displacements. Rigorously, the minimization of total potential energy is a problem in the *calculus of variations* [5]. We do not suppose that the intended audience of this text is familiar with the calculus of variations. Rather, we simply impose the minimization principle of calculus of multiple variable functions. If we have a total potential energy expression that is a function of, say, N displacements U_i, $i = 1, \dots, N$; that is,

$$\Pi = \Pi(U_1, U_2, \dots, U_N) \tag{2.54}$$

then the total potential energy will be minimized if

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad i = 1, \dots, N \tag{2.55}$$

Equation 2.55 will be shown to represent N algebraic equations, which form the finite element approximation to the solution of the differential equation(s) governing the response of a structural system.

EXAMPLE 2.7

Repeat the solution to Example 2.6 using the principle of minimum potential energy.

■ **Solution**

Per the previous example solution, the elastic strain energy is

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2$$

and the potential energy of applied forces is

$$U_F = -W = -F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

Hence, the total potential energy is expressed as

$$\Pi = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2 - F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

In this example, the principle of minimum potential energy requires that

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad i = 1, 4$$

giving in sequence $i = 1, 4$, the algebraic equations

$$\frac{\partial \Pi}{\partial U_1} = k_1(U_2 - U_1)(-1) - F_1 = k_1(U_1 - U_2) - F_1 = 0$$

$$\frac{\partial \Pi}{\partial U_2} = k_1(U_2 - U_1) + 2k_2(U_3 - U_2)(-1) - F_2$$

$$= -k_1U_1 + (k_1 + 2k_2)U_2 - 2k_2U_3 - F_2 = 0$$

$$\frac{\partial \Pi}{\partial U_3} = 2k_2(U_3 - U_2) + k_3(U_4 - U_3)(-1) - F_3$$

$$= -2k_2U_2 + (2k_2 + k_3)U_3 - k_3U_4 - F_3 = 0$$

$$\frac{\partial \Pi}{\partial U_4} = k_3(U_4 - U_3) - F_4 = -k_3U_3 + k_3U_4 - F_4 = 0$$

which, when written in matrix form, are

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

and can be seen to be identical to the previous result. Consequently, we do not resolve the system numerically, as the results are known.

We now reexamine the energy equation of the Example 2.7 to develop a more-general form, which will be of significant value in more complicated systems to be discussed in later chapters. The system or global displacement vector is

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} \tag{2.56}$$

and, as derived, the global stiffness matrix is

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \tag{2.57}$$

If we form the matrix triple product

$$\frac{1}{2}\{U\}^T[K]\{U\} = \frac{1}{2}[\,U_1 \quad U_2 \quad U_3 \quad U_4\,] \times \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} \tag{2.58}$$

and carry out the matrix operations, we find that the expression is identical to the strain energy of the system. As will be shown, the matrix triple product of Equation 2.58 represents the strain energy of any elastic system. If the strain energy can be expressed in the form of this triple product, the stiffness matrix will have been obtained, since the displacements are readily identifiable.

2.6 SUMMARY

Two linear mechanical elements, the idealized elastic spring and an elastic tension-compression member (bar) have been used to introduce the basic concepts involved in formulating the equations governing a finite element. The element equations are obtained by both a straightforward equilibrium approach and a strain energy method using the first theorem of Castigliano. The principle of minimum potential also is introduced. The next chapter shows how the one-dimensional bar element can be used to demonstrate the finite element model assembly procedures in the context of some simple two- and three-dimensional structures.

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1. Budynas, R. *Advanced Strength and Applied Stress Analysis.* 2d ed. New York: McGraw-Hill, 1998.
2. Love, A. E. H. *A Treatise on the Mathematical Theory of Elasticity.* New York: Dover Publications, 1944.

3. Beer, F. P., E. R. Johnston, and J. T. DeWolf. *Mechanics of Materials.* 3d ed. New York: McGraw-Hill, 2002.
4. Shigley, J., and R. Mischke. *Mechanical Engineering Design*. New York: McGraw-Hill, 2001.
5. Forray, M. J. *Variational Calculus in Science and Engineering*. New York: McGraw-Hill, 1968.

PROBLEMS

2.1–2.3 For each assembly of springs shown in the accompanying figures (Figures P2.1–P2.3), determine the global stiffness matrix using the system assembly procedure of Section 2.2.

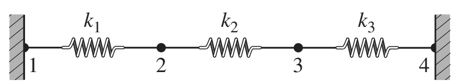

Figure P2.1

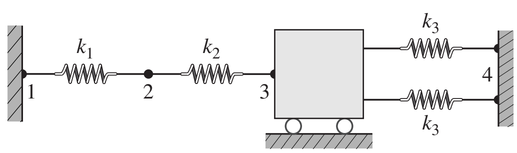

Figure P2.2

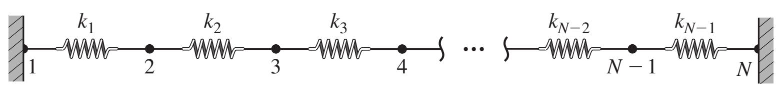

Figure P2.3

2.4 For the spring assembly of Figure P2.4, determine force F_3 required to displace node 2 an amount $\delta = 0.75$ in. to the right. Also compute displacement of node 3. Given

$$k_1 = 50 \text{ lb./in.} \quad \text{and} \quad k_2 = 25 \text{ lb./in.}$$

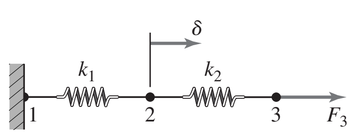

Figure P2.4

2.5 In the spring assembly of Figure P2.5, forces F_2 and F_4 are to be applied such that the resultant force in element 2 is zero and node 4 displaces an amount

$\delta = 1$ in. Determine (a) the required values of forces F_2 and F_4, (b) displacement of node 2, and (c) the reaction force at node 1.

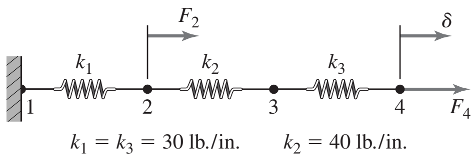

Figure P2.5

2.6 Verify the global stiffness matrix of Example 2.3 using (a) direct assembly and (b) Castigliano's first theorem.

2.7 Two trolleys are connected by the arrangement of springs shown in Figure P2.7. (a) Determine the complete set of equilibrium equations for the system in the form $[K]\{U\} = \{F\}$. (b) If $k = 50$ lb./in., $F_1 = 20$ lb., and $F_2 = 15$ lb., compute the displacement of each trolley and the force in each spring.

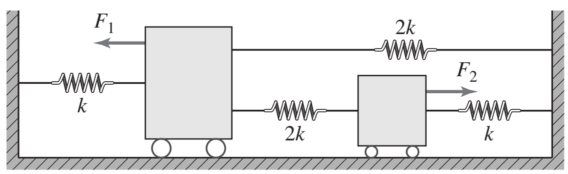

Figure P2.7

2.8 Use Castigliano's first theorem to obtain the matrix equilibrium equations for the system of springs shown in Figure P2.8.

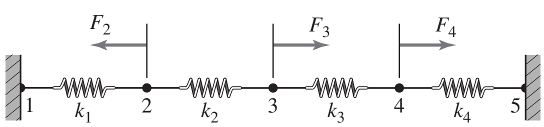

Figure P2.8

2.9 In Problem 2.8, let $k_1 = k_2 = k_3 = k_4 = 10$ N/mm, $F_2 = 20$ N, $F_3 = 25$ N, $F_4 = 40$ N and solve for (a) the nodal displacements, (b) the reaction forces at nodes 1 and 5, and (c) the force in each spring.

2.10 A steel rod subjected to compression is modeled by two bar elements, as shown in Figure P2.10. Determine the nodal displacements and the axial stress in each element. What other concerns should be examined?

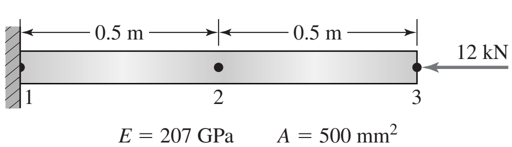

Figure P2.10

2.11 Figure P2.11 depicts an assembly of two bar elements made of different materials. Determine the nodal displacements, element stresses, and the reaction force.

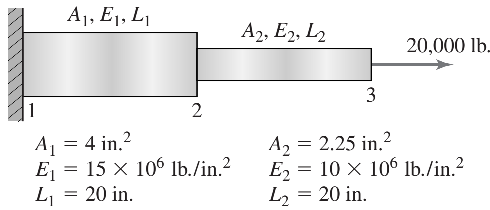

Figure P2.11

2.12 Obtain a four-element solution for the tapered bar of Example 2.4. Plot element stresses versus the exact solution. Use the following numerical values:

$$E = 10 \times 10^6 \text{ lb./in.}^2 \qquad A_0 = 4 \text{ in.}^2 \qquad L = 20 \text{ in.} \qquad P = 4000 \text{ lb.}$$

2.13 A weight W is suspended in a vertical plane by a linear spring having spring constant k. Show that the equilibrium position corresponds to minimum total potential energy.

2.14 For a bar element, it is proposed to discretize the displacement function as

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

with interpolation functions

$$N_1(x) = \cos\frac{\pi x}{2L}$$

$$N_2(x) = \sin\frac{\pi x}{2L}$$

Are these valid interpolation functions? (Hint: Consider strain and stress variations.)

2.15 The torsional element shown in Figure P2.15 has a solid circular cross section and behaves elastically. The nodal displacements are rotations θ_1 and θ_2 and the associated nodal loads are applied torques T_1 and T_2. Use the potential energy principle to derive the element equations in matrix form.

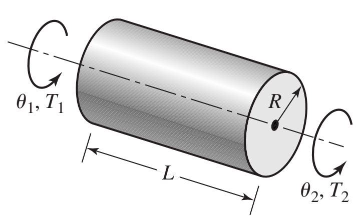

Figure P2.15