EXOTIC OPTIONS

2nd Edition

A Guide to Second Generation

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To my father Zuheng Zhang

PREFACE TO THE SECOND EDITION

Nearly one year has elapsed since the first edition of this book came into existence in early 1997. Within the past year, two significant events occurred which are directly related to the derivatives profession. The first was that on October 14, 1997, the Nobel committee gave the 1997 Nobel Prize in Economic Sciences to Professor Robert Merton of Harvard University and Professor Myron Scholes of Stanford University for their work on the development of option pricing theory. The Nobel committee made it clear that had he lived, Fischer Black would have shared the prize. As described in the first edition of this book, most of the models and pricing formulas in this book have been within a Black-Scholes-Merton world which has been central to the development of financial engineering as both a discipline and profession.

The other is the still-going-on financial crisis spreading from East Asia to around the globe. This crisis started with the rapid devaluation of Thailand's bhat early in July, spreaded to the neighboring Southeast Asian countries of Indonesia, Malaysia, Singapore, and Philippines. Because of similar economics structures and foreign exchange rate policies, these countries began their competitive devaluation of their currencies. Within months, the crisis moved North to Hong Kong, Taiwan, and then South Korea. After defending the New Taiwan dollar for one week, the Taiwan central bank also followed the Southeast countries on October 16, by letting its currency float against the US dollar. The devaluation of the New Taiwanese dollar pressured Hong Kong Hang Seng Stock Index down for four consecutively days from October 17 to October 20 with an accumulated 3175 points, or nearly 15%. The tremendous fall of Hong Kong Stock market pushed the US stock markets (the Dow Jones Index dived 554 points on October 27, the largest one-day drop since the Black Monday in October 1987) and stock markets around the world down significantly. Volatilities in both currency and equity markets have increased significantly during the crisis. Derivatives should have good potentiality for wider use, especially in East Asia, as there still shows no signs that the crisis will tranquilize in the near future as evidenced from the second-round of crisis starting from Indonesia on January 9, 1998.

As stated in the first edition of this book, the innovation process for newer products has slowed down because the concept of vanilla options has been extended in almost every aspect. With only a few new types of exotic options such as "pure vega digital" options (it could be classified as one type of correction option with the measurement asset specified as the implied volatility of another option, see Chapter 15 of this edition for more details), the market has been learning and familiarizing with the existing products as evidenced in the foreign exchange options markets with various types of barrier options, knockouts, range binaries, one-touch bets, and etc. covered in Chapters 10, and 11, and 15, and particularly the use of average rate barriers (these barrier options are special cases of outside Asian-barrier options classified in Section 11.9) [see Nusbaum (1997) for more detailed descriptions of recent uses of exotic options]. At the same time, traders have been improving the ways to hedge various types of exotic options and researchers incorporating volatility surface into pricing many types of exotic options.

Also as stated in the first edition of this book, exotic option have been applied to many exotic underlying markets, especially in the fast growing credit derivatives market. As credit derivatives expand investor universe, popular with different investor classes including insurance companies, mutual funds, pension funds, banks hedge funds, and corporations, they have expanded exponentially in the past few years. Trigger options are among the most popular types of credit options amounted nearly 20% of all the credit derivatives market size or over 40% of all types of credit options (Financial Times, November 22, 1996). The release of a draft of standardized term sheet for credit swaps defining specifically default-event and other important legal terms by ISDA (International Swaps and Derivatives Association) and the launch of JP Morgan's CreditMetrics package in the first half of 1997 will help this growing market to develop more smoothly with less legal confusion and boost credit derivatives including various types of credit exotic options.

There are some changes in this edition. First of all, a lot of types have been detected and corrected in this edition. Secondly, quite a few errors have been found and corrected in this edition. Thirdly, new materials such as limiting cases of the Black-Scholes model, "pure vega" digital options, outside double-barrier options, trigger compound options, joint quanto options, and some other materials are included in this edition.

Since the publication of the first edition in early 1997, I have received many letters, calls, and e-mails both directly and through the publisher. I have also received many messages through many of my friends and colleagues. All these letters, calls, and messages either pointed out some typos, errors, questions or gave me good suggestions.

I want to take this opportunity to thank all these who have helped me to improve the quality and contents of the book. It would be too long to list the names of all these people. Still, I particularly want to thank Keiji Ohmori, Andrew H. Chen, Lixin Wang, Steven Allan, Bay Way Wee, I. R. Low, A. A. Kotze, Svein Stokke, John Murray, Peng Wei, Cindy Wong, Tim Owens, Gaile Gong, Garry de Jagger, James Xu, Yuko Kawai, Michel Kurek, and many others.

Peter G. Zhang
January 10, 1998 in Tokyo, Japan

PREFACE TO THE FIRST EDITION

In memory of Fisher Black, without whose tremendous contribution to both theory and practice, derivatives research and industry would not have reached the current stage, and certainly, this book which concentrates on a Black-Scholes environment would not have been started.

> *With derivatives you can have almost any payoff pattern you want. If you can draw it on paper, or describe it in words, someone can design a derivative that gives you that payoff.*
>
> *Fisher Black* (1995)

These days we often come across such terms as exotic options in newspapers, journals, magazines, and many other financial reports. You may wonder, as I did two years ago, what exactly they are. At that time I had just started to work in the financial industry and was much puzzled by the phrase "exotic options". Such puzzlement left me feeling uneasy as I had previously spent a few years at school studying option pricing theory. I tried to find some systematic sources to reeducate myself, and it turned out to be nearly impossible as there was no systematic source on this subject. Two years later, although the situation has changed somewhat, exotic options still remain mysterious to many people. I have tried to keep a systematic record on this subject, though initially it was not my intention to write a book on exotic options.

My first research paper on this subject was on spread options nearly two years ago. Since then, many other papers have followed. The writing process has been so wonderful that I would never have learned so fast and thoroughly had I not written this book. This book records the accumulation of my knowledge on exotic options, and I would like to share my learning curve with all of you.

To some degree, exotic options are as old as vanilla options. The earliest article on exotic options can be traced to an article titled "Alternative Forms of Options" by Snyder, published in the *Financial Analysts*

Journal. It was in 1969, four years earlier than the establishment of the Chicago Board of Options Exchange (CBOE), the first organized options exchange in the world, and four years earlier than the birth of the seminal work of Black and Scholes, who made the path-breaking contribution to derivatives industry. However, exotic options became somewhat popular only from the late 1970s and have experienced significant growth in the past decade or so. The primary motivations driving the recent innovations of derivatives are cost-reduction and special customer needs such as off-balance sheet opportunities, tax considerations, and so on.

The study of exotic options is indispensable not only for their own active and important trading but also because they provide easy and efficient building blocks for other more complicated financial derivatives. In order not to inundate many readers, I try to spend a significant amount of time in almost every chapter on how to use the pricing formulas and how to apply them in practice.

A series of events in the derivatives industry since 1994, Orange County, Kidder Peabody, Procter & Gamble, Gibson Greetings, Askin Capital, and so on, has attracted a lot of attention in the financial industry as well as among the general public. These events created calls for transparency of special-purpose derivatives activities. One of the objects of this book is to provide a convenient source of information for exotic options and thus to improve the transparency of the market. I try to provide as complete a source as possible on this subject. In each chapter I try to introduce one type of exotic options, what it can achieve, and how to price and use it.

We will concentrate on a Black-Scholes environment throughout this book for the purpose of transparency and easy comparisons with vanilla options, because the Black-Scholes model is best known. In Section 4.3 of Chapter 4, we provide a derivation of the Black-Scholes formula using the method to solve the related partial differential equation. And in Section 4.4 of Chapter 4, we provide an intuitive and concise method to derive the Black-Scholes formula. The method shown in Section 4.4 to derive the Black-Scholes formula is the same method we use to price essentially all exotic options in this book.

It is well-known in physics that energy can neither be created nor destroyed, it can only be transferred from one form to another. Risk, or more specifically, financial risk, is such a complicated subject that many researchers and financial institutions have been struggling to find ways to measure it. Yet intuition suggests that risk, like energy, can neither be created nor destroyed: it is inherent within the financial system. If we consider standard options as vehicles to transfer risk between two parties (writers and buyers), then exotic options are vehicles tailored or specially-designed to transfer risk between them. These tailored vehicles are more effective in risk-transferring and at the same time they require more training to drive. Therefore, exotic options are also called risk management products.

Although there has been continuing concern in recent years about the future of the derivatives industry — especially in 1994 as a result of the losses in the market — the general trend is still promising. This is because risk will be better understood with new technology and new studies. Improved understanding will lead investors and institutions to decide what kind of risk they have to tolerate and what they will have to eliminate. This trend will keep institutions not only using most of their existing vehicles but also creating more to meet their increasingly specific needs.

This book is organized as follows. Part I includes two chapters. Chapter 1 gives a bird's eye view of exotic options, and Chapter 2 reviews option pricing theory — the arbitrage-free principle, which will be used to price all options. Part II reviews vanilla options in two chapters. Chapter 3 reviews various aspects of vanilla options, the extensions of the Black-Scholes option pricing formula, modern Greeks, implied volatilities and so on. Chapter 4 reviews the methods to price American options. The review is necessary because most of the terminology used in describing exotic options are from vanilla options. Comparisons between each type of exotic option and its corresponding vanilla option are helpful for us to grasp the characteristics of exotic options. Those with good understanding of vanilla options may skip Chapters 2 to 4 without losing any integrity.

Besides a few chapters which review theories on vanilla options (Chapters 3 and 4), introduce an approximation method (Chapter 6), or point out some limitations of existing methods in pricing correlation options (Chapter 28), each of the other chapters is designed to cover one kind of exotic options. Part III introduces and prices one of the most popular exotic options — path-dependent options — in eight chapters. Chapter 5 introduces geometric Asian options and finds closed-form solutions for them. Chapter 6 illustrates how to approximate arithmetic Asian options with their corresponding geometric Asian options. Using the general mean which includes all existing averages as special cases, we find a linear approximation for any arithmetic average with its corresponding geometric average. This result is used to approximate many other kinds of exotic options in this book. Chapter 7 extends standard Asian options with equal weights to all observations to flexible Asian options which allow uneven weights to different observations in the average. Chapter 8 introduces forward-start options, and Chapter 9 analyzes one-clique options.

Chapter 10 studies vanilla or standard barrier options. We first describe the difficulty in pricing barrier options and then derive the conditional density functions necessary to price all eight types of barrier options. Closed-form solutions are obtained for all eight types of vanilla barrier options including knockout barrier options with time-dependent, or deferrable, rebates. Because of the great variety of barrier options, we study nonstandard or exotic barrier options in Chapter 11. We study time-dependent barrier options, forward-start barrier options, earlier-ending barrier options, window-barrier options, outside barrier options, Asian barrier options, dual-barrier options, and so on. The most interesting feature of our analysis in this chapter is that we provide unified closed-form pricing formulas for all eight types of earlier-ending as well as outside barrier options. These unified pricing formulas include all the pricing formulas of vanilla barrier options as special cases. The unified pricing formulas make it much easier to analyse Greeks. Chapter 12 studies lookback options including floating strike, fixed strike, lookback options, and partial lookback options.

Part IV covers fifteen popular correlation options in sixteen chapters. The order of the chapters largely follows the complexity of the products. Chapter 13 introduces and prices exchange options. Chapter 14 discusses options paying the best/worst of two risky assets and cash. Chapter 15 reviews standard digital options and introduces and prices correlation digital options which include standard digital options as special cases. Chapter 16 studies quotient options or options written on the ratio of two asset prices. Chapter 17 covers product options and prices foreign equity options with domestic strikes using the product option pricing formula. Chapter 18 introduces foreign equity options. Chapter 19 studies equity-linked foreign exchange options. Chapter 20 prices quanto options. Chapter 21 prices rainbow options on two or more than two underlying assets. Chapter 22 discusses both simple spread options and multiple spread options. Chapter 23 covers options written on the spread between the two rainbows or between the maximum and minimum of two asset prices. Chapter 24 discusses dual-strike options or options with two underlying assets and two different strike prices. Chapter 27 prices basket options and discusses basket digital options. Chapter 28 points out the limitations of using constant correlation coefficients in pricing all correlation options in Part IV and tries to estimate the errors of constant correlation coefficients, when they are non-deterministic.

Part V covers exotic options not covered in Part III and Part IV. Chapter 29 describes package options or portfolios of vanilla options, their underlying assets, and cash. Chapter 30 studies nonlinear payoff options.

Chapter 31 prices compound options and discusses how to use the compound option pricing formula to price American options. Chapter 32 studies chooser's options or "as-you-like-options". Chapter 33 describes contingent premium options or pay-later options. Chapter 34 describes many other kinds of exotic options such as Bermuda options, installment options, and so on.

Part VI covers the hedging of exotic options and their further development. Chapter 35 briefly describes the popular methods in hedging most kinds of exotic options and the difficulties. Chapter 36 discusses possible directions in the future development of exotic options and concludes the book.

We have designed questions and exercises at the end of each chapter. The questions are designed for readers to review the important concepts in each chapter, and the exercises are designed to provide some hands-on experiences on how to calculate option prices and other related measures such as sensitivities. Most of the exercises are straightforward applications of the contents in the corresponding chapters. There are, however, a few exercises in each chapter marked with "*" which require more mathematical training. These exercises are mainly designed for analysts or Ph.D. students.

Peter G. Zhang
August 31, 1995 in New York

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PART I: INTRODUCTION TO EXOTIC OPTIONS AND OPTION PRICING METHODOLOGY

We give a bird's eye view of the world of exotic options in Chapter 1. Particularly, we describe briefly the basic characteristics of major exotic options and their historical development. After a brief review of standard options, we classify exotic options into three major groups: path-dependent options, correlation options, and other exotic options. We emphasize how each kind of exotic options is different from standard options or plain vanilla options.

After describing the principle in pricing all kinds of derivatives including options — the no-arbitrage or the "no-free-lunch" argument — in Chapter 2, we show various methods in pricing options. We provide two methods to derive the Black-Scholes pricing formula, one by solving the option partial differential equation and the other using the risk-neutral evaluation relationship. We also describe the popular methods of Monte Carlo simulations and the recombining trees to price various kinds of options.

Chapter 1

FROM VANILLA OPTIONS TO EXOTIC OPTIONS

One day, on my way back home on the PATH train from New York to New Jersey, I was working on the preliminary version of this book when a lady sitting beside me glanced at my work and asked: "*What are exotic options? Are they similar to exotic dancing?*"

"*Sort of,*" I replied even though I knew little about exotic dancing, "*they share the same exoticness.*"

I believe there are many more people who know exotic dancing than those who know exotic options. After finishing this book, I would like to find someone to teach me exotic dancing.

Exotic options are not new to the financial markets. Some came into existence several years before the birth of the Chicago Board of Options Exchange (CBOE) — the first organized options exchange in the world, established in 1973 [Snyder (1969)]. Trading volume for standard options was rather thin in the pre-CBOE period, and for nonstandard options, it was even thinner. A few years after the establishment of the CBOE, a slow and inconspicuous revolution in option concepts and trading started to take place. Towards the end of the 1970's and the beginning of the 1980's, when standard options trading at exchanges became better understood and their trading volume exploded, financial institutions began to search for alternative forms of options to meet their particular needs and increase their business. All these alternative options are called exotic options. In the late 1980's and early 1990's, exotic options became more visible in daily presses and more popular among financial communities. Their trading became more active in the over-the-counter (OTC) marketplace, and their users were big corporations, financial institutions, fund managers, and recently private bankers.

Most of these exotic options are traded at the OTC, although a few have been listed in exchanges recently. The American Stock Exchange, for example, trades quanto options, while the New York Mercantile Exchange trades spread options. Trading of these options in exchanges represents only a small percentage of all exotic options volume. Because of the opaque nature of the OTC marketplace, exotic options still remain exotic to many investors, professionals, and even many of those who have good knowledge of standard options. These exotic products have been incorporated into general books to such an extent that many financial institutions now feel that they can neither live with nor without them.

If we call standard options first-generation options, nonstandard options can be called second-generation options. Second-generation options are exotic options, which are also called special-purpose options or customer-tailored options, implying that each type of exotic options can somehow serve a special purpose which standard options cannot do conveniently or cheaply. These names somehow explain why exotic options came into existence, and why they have grown significantly in varieties and volumes. Although there are many different kinds of exotic options, all of them are, in one way or another, either direct or indirect extensions of standard options.

Exotic options differ from standard options in at least one aspect. For example, a deferred option or forward-start option is an option whose effective starting time is sometime in the future after the contract is signed rather than in the present. A compound option is an option written on a standard option rather than on an underlying asset directly. A spread option is an option written on the difference between two prices or indices, rather than on one single price or index as in the case of standard options, and so on. In general, exotic options are almost exclusively traded in the OTC marketplace rather than in the exchanges. It is difficult to find a source which classifies these products into a small number of groups. This chapter will provide a bird's eye view of exotic options.

Before we enter into exotic options, Section 1.1 reviews some concepts of standard, or plain vanilla, options. Section 1.2 describes the most popular group of exotic options — path-dependent options, which include Asian or average-price options, barrier options, lookback options, and forward-start options. Section 1.3 introduces another large group of exotic options — correlation options which include spread options, out-performance options, two-color rainbow options, quanto options, exchange options, basket options, and others. Section 1.4 introduces other popular exotic options such as chooser options or as-you-like options, power options, binary or digital options, and so on. Section 1.5 concludes the chapter.

1.1. PLAIN VANILLA OPTIONS

Although options have existed for about a century, it was the establishment of the Chicago Board of Options Exchange (CBOE) in October 1973 — the first options exchange in the world — that brought legitimacy to option trading and made option trading more attractive to hedgers as well as speculators through reducing counter-party risks by the Exchange. The earliest options trading at the CBOE were call options on 16 US stocks, and put options on stocks were introduced to the Exchange a few years later. Options on many other underlying assets such as bonds and currencies were later introduced to the CBOE and other exchanges. Options trade nowadays on stocks and stock indexes, bonds and bond indexes, currencies and currency indexes, many commodities and commodity indexes, futures, and other indexes or underlying instruments.

Before we start to describe exotic options, it is necessary for us to review some basic terms of standard or plain vanilla options. A standard option is a financial contract which gives its holder the right to buy or sell the underlying asset at a prespecified price (called exercise or strike price) within a prespecified time (called time to maturity or time to expiration of the option). If the holder can exercise his/her right to buy or sell the underlying asset only at maturity, the option is called a European option. If the holder can exercise his/her right to buy or sell the underlying asset any time at or before the option's maturity, the option is called an American option. As American options permit their holders to exercise any time before the expiration of the options, they are at least as expensive as the corresponding European options. The amount of money the holder has to pay the seller or the writer of an option is called the premium of the option.

There are many ways to classify standard options. The most popular one divides options into two groups: call options and put options. A call (put) option gives its holder the right to buy (sell) the corresponding underlying asset at a prespecified strike price. If the strike price of a call option is lower (higher) than the spot price of the underlying asset, it is called an in-the-money or ITM call option (out-of-the-money or OTM call option). If the strike price of a call option is equal to the spot price of the underlying asset, it is called an at-the-money or ATM call option. The word "moneyness" is often used to represent whether an option is ITM, ATM, or OTM.

Black and Scholes (1973) pioneered the modern option pricing theory. Interestingly enough, this model was published in the same year as the establishment of the CBOE in 1973. The most important feature of the Black-Scholes model is that option prices are determined by observable factors which can be found readily. This theoretical breakthrough in pricing, to a large degree, has helped not only the development of option industry but also the entire derivative industry.

In general, the premium of a vanilla option is affected by a number of factors such as volatility, strike price, time to maturity, and so on. In the well-known Black-Scholes option pricing theory, the following five factors affect the option prices: (1) current price of the underlying asset; (2) exercise or strike price of the option; (3) time to maturity; (4) volatility of the underlying asset; and (5) interest rate. The Black-Scholes pricing theory is easily extended to include the payout rate of the underlying asset as the sixth factor. The payout rate of the underlying asset is a foreign interest rate in the case of currency options, and dividend yield in the case of domestic equity options. European options can be conveniently priced using the Black-Scholes formula with which the premiums of European options can be calculated directly by substituting values of the above-mentioned parameters into the pricing expressions. Unfortunately, compact formulas are not available for American options, and we can only use approximation formulas or numerical methods to obtain the premiums of American options.

All vanilla options share a few common characteristics: one underlying asset; the effective starting time is present; only the price of the underlying asset at the option's maturity affects the payoff of the option; whether an option is a call or a put is known when sold; the payoff is always the difference between the underlying asset price and the strike price, and so on. Vanilla options have many limitations resulting from their lack of flexibility. Each kind of exotic options, to some degree, overcomes one particular limitation of vanilla options. This will be clearly seen in the following sections.

Table 1.1 lists the total option trading volumes in the six options exchanges in the US (CBOE, American Stock Exchange, Philadelphia Stock Exchange, Pacific Stock Exchange, Midwest Stock Exchange, and the New York Stock Exchange) and the annual growth rates as percentages over the years from 1973 to 1991. We can observe that the total trading volume increased more than one hundred times in the first ten years from 1973 to 1982. The US equity options markets matured in the early 1980's and trading volumes have remained above one hundred million contracts since 1983. Among many reasons that could explain the maturity, the obvious one is that equity index options and futures attracted substantial volumes away from the individual equity options markets.

Following the CBOT, CME, CBOE, and other exchanges in the US, futures and options exchanges have been established in many other countries.

Table 1.1. Total equity options contract volume and growth rates.

Year	Total contracts	Annual growth (%)	Year	Total contracts	Annual growth (%)
1991	104850686	23.15	1981	109405782	13.1
1990	111785744	28.05	1980	96728546	50.5
1989	141839748	33.94	1979	64264863	12.3
1988	114927638	28.14	1978	57231018	44.4
1987	164431851	34.30	1977	39637328	22.4
1986	141930945	39.69	1976	32373927	78.8
1985	118556094	43.09	1975	18102569	218.5
1984	118925239	51.52	1974	5682907	407.9
1983	135658976	62.88	1973	1119177	
1982	137264816	25.20			

Data source: market statistics 1991, The Chicago Board of Options Exchanges

Table 1.2 gives the names and their corresponding countries of 42 major futures and options exchanges around the world. We can readily observe from Table 1.2 that the US has 13 out of the 42 exchanges. Besides the US, Britain, Japan, the Netherlands, most industrialized countries have active futures and options exchanges.

1.2. PATH-DEPENDENT OPTIONS

The payoff of a vanilla option depends only on the relative magnitude of its underlying asset price at maturity and its strike price, no matter whether the price of the underlying asset at maturity is reached from above, below, or in a zigzag way. The way the settlement price is reached should be of significant relevance to option values; vanilla options have limitations in not capturing how settlement prices are reached. Path-dependent options are, however, designed to capture how the settlement prices of the underlying assets are reached. There are four popular kinds of path-dependent options: Asian options, barrier options, lookback options, and forward-start options.

As Asian options are options with payoffs determined by some averages of the underlying asset prices during a prespecified period of time before the option expiration, they are also called average-price or average-rate options. They also include average-strike Asian options in which strike prices are some averages of the underlying asset prices rather than fixed as in vanilla options. Asian options can be used by corporations with reasonably predictable cash flows to hedge conveniently as a cheaper alternative to a string of vanilla

Table 1.2. The major futures and options exchanges in the world.

Abbreviation	Exchange Name	Country
AMEX	American Stock Exchange	USA
ATA	Agricultural Futures Exchange, Amsterdam	Netherlands
BELFOX	Belgium Futures & Options Exchange	Belgium
BM&F	Bolsa De Mercardorias & Futuros, Brazil	Brazil
CBOT	Chicago Board of Trade	USA
CME	Chicago Mercantile Exchange	USA
COMEX	Commodity Exchange, Inc.	USA
CSCE	Coffee Sugar & Cocoa Exchange	USA
DTB	Deutsche Terminborse	Germany
EOE	European Options Exchange	Netherlands
FFMA	Financial Futures Market Amsterdam	Netherlands
FINEX	Financial Instrument Exchange	USA
FOX	London Futures and Options Exchange	UK
FUTOP	Gurantee Fund Danish Options and Futures	Denmark
HKFE	Hong Kong Futures Exchange Ltd.	HK
IFOX	Irish Futures and Options Exchange	Ireland
IPE	International Petroleum Exchange	UK
KCBT	Kansas City Board of Trade	USA
KRE	Kobe Rubber Exchange	Japan
LIFFE	London Int'l Financial Futures Exchange	UK
LME	London Metal Exchange	UK
MACE	Midamerica Commodity Exchange	USA
MATIF	Marche a Terme International de France	France
MEFFRF	Meff Renta Fija, Spain	Spain
MERFOX	Mercado de Futuros y Opciones S.A., Argentina	Argentina
MGE	Minneapolis Grain Exchange	USA
MONEP	Marche des Options Negociables de Paris	France
MONTREAL	Montreal Exchange	Canada
NYCE	New York Cotton Exchange	USA
NYFE	New York Futures Exchange	USA
NYMEX	New York Mercantile Exchange	USA
NZFE	New Zealand Futures & Options Exchange	New Zealand
OSAKA	Osaka Securities Exchange	Japan
PHLX	Philadelphia Stock Exchange	USA
SFE	Sydney Futures Exchange	Australia
SIMEX	Singapore Int'l Monetary Exchange	Singapore
SOFFEX	Swiss Options and Financial Futures Exchange	Switzerland
SOM	Stockholm Options Market	Sweden
TFE	Toronto Futures Exchange	Canada
TGE	Tokyo Grain Exchange	Japan
TIFFE	Tokyo Int'l Financial Futures Exchange	Japan
TOCOM	Tokyo Commodity Exchange	Japan
TSE	Tokyo Stock Exchange	Japan
WCE	Winnipeg Commodity Exchange	Canada

options. They are thus popular in the commodity and currency markets. Although Asian options can be either arithmetic or geometric depending upon the averages, traders almost exclusively use arithmetic averages to construct Asian options. However, geometric Asian options also have their attractiveness as their prices can be expressed in closed-form similar to the Black-Scholes formula, and can also be used to approximate arithmetic Asian option prices.

Most Asian options are based on equally weighted arithmetic averages of the underlying asset prices. Zhang (1993) introduced the concept of flexible Asian options which allocate different weights to various observations under consideration. Zhang (1994) provided a closed-form solution and examples for flexible Asian options based on geometric averages. The concepts of flexible Asian options were later extended to arithmetic Asian options [Zhang (1995a)] . Flexible Asian options are of interest to traders who wish to assign greater (lesser) emphasis to the role played by more (less) recently observed prices in the average. They are attractive because of the flexibility in weight distribution and are now being used by many institutions.

Barrier options are probably the oldest of all exotic options. Snyder (1969) discussed "down-and-out" options although he used the phrase limited-risk special options. Donaldson, Lufkin and Jenrette started to use "down-and-out" options as early as 1970 [*Fortune*, Nov., 1971, p. 213]. These options were geared to the needs of sophisticated investors such as managers of hedge funds. They provided investors with two things they could not get otherwise. One was that most "down-and-out" options were written on more volatile stocks where premiums on standard calls are normally high. The other was increased convenience during a time when trading volume of stock options was rather low. Barrier options are actually conditional options, conditioned on whether some barriers or triggers are reached or not during the lives of options. Barrier options are also called trigger options.

There are two kinds of barrier options: knock-in and knock-out options, or simply knock-ins and knock-outs. A knock-in is an option whose holder is entitled to receive a European option if the barrier is hit, or a rebate at expiration if otherwise. A knock-out is an option whose holder is entitled to receive a rebate as soon as the barrier is hit, or a European option if otherwise. As it makes a difference whether the settlement price is breached from above or below, there are down knock-ins and down knock-outs as well as up knock-ins and up knock-outs, depending upon whether the barrier is below or above the current underlying asset price. Therefore, there are in total eight kinds of barrier options: down-in calls, up-in calls, down-out calls, up-out calls, down-in puts, up-in puts, down-out puts, and up-out puts. The advantage of barrier options is that they are cheaper than vanilla options as the sum of the premiums of a knock-in and its corresponding knock-out is the same as that of the corresponding vanilla option.

A lookback option is an option whose payoff is determined not only by the settlement price but also by the maximum or minimum prices of the underlying asset within the option's lifetime. There are two kinds of lookback options: floating-strike and fixed-strike lookback options. Floating-strike lookback options are true "no regret" options because their payoffs are the maximum. Specifically, the payoff of a floating-strike lookback call option is the difference between the settlement price and the minimum price of the underlying asset during the option's lifetime, and the payoff of a floating-strike lookback put option is the difference between the maximum price during the option's lifetime and the settlement price of the underlying asset. Thus the payoffs of these call and put options are the greatest that could be possibly achieved.

The payoff of a fixed-strike lookback call (put) option is the difference between the maximum price of the underlying asset and the fixed strike price (resp. the difference between the fixed strike and the minimum price) during the life of the option. Lookback options can somehow capture investors' fantasy of buying low and selling high, to minimize regret, as Goldman, Sosin, and Gatto (1979) argued. However, the "no-free-lunch" principle guarantees that these options are expensive to buy. The high premiums of lookback options prevent them from being widely used.

Forward-start options are options with up-front premium payments, yet they start in prespecified future time with strike prices equal to the starting underlying asset prices. Thus, forward-start options can be considered as simple spread options in which the spreads are the differences between the prices of the same underlying asset at two different time points compared to standard simple spread options over the differences of two underlying assets. Forward-start options normally exist in the interest-rate markets where investors can use them to bet on interest-rate fluctuations.

1.3. CORRELATION OPTIONS

Correlation options are options whose payoffs are affected by more than one underlying asset, unlike vanilla options. These underlying assets can be of either the same or different asset classes, for example, equity, bond, currency, commodity, and so on. If the two underlying assets are of different asset classes, the correlation options are often called cross-asset options. With the development of international finance, cross-market products have become more and more important. Market integration has stimulated growth in cross-market products, and financial market globalization has accelerated investments across national boundaries. The New York Stock Exchange has been considering trading foreign stocks in different currencies in an effort to maintain its prestige in the global marketplace (*Wall Street Journal*, Wednesday, March 23, 1994, C1&C20). Such measures are designed to prevent the market from becoming a regional exchange in a global marketplace. The National Association of Securities Dealers (NASDAQ), the second largest stock market in the world measured by dollar trading volume, is also making such efforts.

Correlation options can be divided into first-order and second-order correlation options according to the ways correlation affects option payoffs. Correlation has first-order or primary effect if it directly influences option payoffs, as in spread options and out-performance options. They are thus first-order correlation options. Differential swaps and quanto options have second-order correlation effect because correlation merely modifies the payoffs of options. An option can reflect both first- and second-order correlation effects, for example, an out-performance option on the DAX-30 [1] and CAC-40[2] denominated in British Sterling. The first-order effect is on the covariance of these indices. The second-order effect comes from the degree of relationship between movements in both of these indices (and their covariance) and changes in the French Franc/Sterling and German Mark/Sterling exchange rates.

Spread options are simple correlation options. A spread option is written on the difference of two indices, prices, or rates, for example, the spread between refined and crude oil prices which fluctuates due to international economic and financial information. Options written on this spread can be used by oil refiners to hedge the risks of their gross profit. Another popular spread option involve spreads between a long-term treasury interest rate and a short-term rate. In the early stage when spread options were used, the spread was regarded as some imaginary single asset price, and the well-known Black-Scholes formula was used to approximate the spread option price. This method is the so-called one-factor model. Garman (1992) pointed out the limitations and problems of this model and discussed how to

[1]The German Share Index (*Deutscher Aktienindex* or DAX) of 30 of the most heavily traded stocks listed on the Frankfurt Stock Exchange (*Frankfurt Wertpapierborse* or FWB), representing over 75% of the total turnover in German equities.

[2]The French Stock Market Index (*Cotation Assistee en Continue* or CAC) of 240 (CAC-240) stocks trading at the Paris Bourse (*Bourse de Paris*), reflecting the price activity of the 700 plus listed stocks on the Paris Bourse.

price spread options with a two-factor model. Ravindran (1993) attempted to price spread options with a two-factor model using statistical procedures and numerical analysis. We will provide closed-form formulas in a two-factor model for spread options in a Black-Scholes environment in this book.

Spread options written on more than two underlying assets or indices are less known even to many who have some knowledge of the OTC exotics. For example, options can be written on the spread between $X_1 + X_2$ and $X_3 + X_4$, where X_i is either an asset price or index. We may call this kind of spread options multiple spread options, as compared to standard spread options written on two underlying assets. Multiple spread means that the spread is between at least three underlying assets. Multiple spread options are used by quite a few institutions in the OTC marketplace. With further development in OTC derivatives, increasing sophistication in risk management, and accelerating globalization of international capital market, multiple spread options will certainly rise in popularity.

An out-performance option is a special call option which allows investors to take advantage of the expected differences in the relative performance of two underlying instruments or indices. The payoff of an out-performance option at maturity is the performance of one instrument minus the performance of a second instrument, multiplied by a fixed notional or face amount. The performance is normally measured by the rate of return as a percentage. The underlying instruments may be any combinations of stocks, bonds, currencies, commodities, or indices. One popular out-performance instrument might be on the spread between a bond index and a stock index or vice versa. Out-performance options are also often used to capitalize the relative performance of two stock markets such as the US market (measured by Standard & Poor's 500) relative to the Japanese market (measured by the Nikkei 225). From the above description, an out-performance option can be viewed as a spread option between the returns of two instruments rather than the actual instrument values.

An exchange option is an option which exchanges one underlying asset for another. Exchange options were first studied by Margrabe (1978). An exchange option can be interpreted either as a call option on asset one with a strike price equal to the future price of asset two at the option maturity, or a put option on asset two with a strike price equal to the future price of asset one at the option maturity. Exchange options are basic correlation options. They can be used to construct many other exotic options such as rainbow options or two-color rainbow options, since payoffs of options written on the better or worse performing of two assets can be valued in terms of their corresponding exchange options.

Complex options are options written on the better or worse performing (the maximum or minimum) of two or more underlying assets. These options are often called two-color rainbow options, or simply rainbow options in practice, because the maximum and the minimum prices of two assets look very much like the shape of a rainbow in a two-dimensional diagram, with the two asset prices as the two axes. Rainbow options can be either valued directly or in terms of the corresponding exchange options. They are useful in many financial applications such as pricing foreign currency debts, compensation plans, and risk-sharing contracts.

Currency-translated options have been created to meet investors' increasing demand in the international equity market as they can link foreign equity and currency exposures. They can be either foreign equity options with strike prices in domestic currency, domestic equity options with strike prices in foreign currency, foreign equity options translated into domestic currency, or domestic equity options translated into foreign currency. In all four cases, both equity and currency risks are involved. The most popular kind of currency-translated options is quanto options, or simply quantos. Quantos are foreign equity options with fixed exchange rates. With them, an investor can capture the upward potential on his foreign equity investment by hedging away all currency risks through fixing the exchange rate. The payoff will then be paid in domestic currency. Quantos are traded in the OTC market as well as in the American Stock Exchange.

A basket option is written on a basket of assets rather than one single asset. Basket options are also called portfolio options. The popular basket options are those written on baskets of currencies. As correlations among various components in a basket largely determine the characteristic of the basket, basket options are correlation options. They can be used by portfolio managers to hedge their positions on the basis of their whole portfolio performance, instead of individual assets within the portfolio. Or they can be used to speculate based on the same information about their portfolios.

1.4. OTHER EXOTIC OPTIONS

It is not easy to classify existing exotic options into small groups according to their characteristics. Besides the two groups we have described above, there are quite a few other kinds of exotic options popular in the OTC marketplace. We will introduce these options in this section.

Because of their simple payoff patterns and unique characteristics, digital options are especially attractive to many participants in the OTC marketplace. They are also called binary options or bet options because their payoffs are either something or nothing. Generally speaking, the payoff of a digital option can be either a fixed amount of cash, an asset, or the difference between an asset price and a prespecified level which is often different from the strike price. These digital options are often called cash-or-nothing, asset-or-nothing, and gap options, respectively. Cash-or-nothing and asset-or-nothing options are similar to betting in daily usage. When the gap is zero, a gap option becomes exactly the same as a vanilla option. In other words, vanilla options are gap options with zero gaps. In general, digital options can be used to capitalize investors' views of market movements. These digital options are popular mainly because they are easy to use. However, they cannot be hedged easily in practice because of limited liquidity, although there are some theoretical methods to hedge them.

Compound options are options written on other standard options. As there are two kinds of vanilla options, calls and puts, there are four kinds of compound options: a call option written on a call option, a call option written on a put option, a put option written on a call option, and a put option written on a put option. Compound options are often used to hedge difficult investments which are contingent on other conditions. The buyer of a compound option normally pays an initial up-front premium for an option which he/she may need later on. The buyer will have to pay an additional premium only if this option is needed. If the buyer finds that the option is not necessary, he/she can simply give up the right.

Chooser options, or as-you-like options, are options which permit the holders to choose between a vanilla call and a vanilla put at some prespecified time during the life of the option. The buyer of the option pays some up-front premium to the writer but does not specify whether the option is a call or a put until the prespecified time. At this time the buyer can decide between a vanilla call and a vanilla put. Therefore, chooser options are also called pay-now-choose-later options. Obviously, chooser options can reduce option buyers' regret resulting from mistakes in buying vanilla calls or puts with specific views of the underlying market.

Nonlinear payoff options, as their name implies, are options with nonlinear payoffs compared to linear payoffs of vanilla options. The popular nonlinear options are power options which exhibit payoffs as power functions of the underlying asset prices. These power functions can be either concave or convex. For example, let S_T stand for the underlying asset price at maturity and K the strike price of the option. The payoff of a power option can then be expressed as $(S_T)^p - K$ for those in-the-money options, where p can be any real number. Obviously, when $p = 1$, the payoff of the power option becomes precisely the same as that of a vanilla call option.

When $p > (<) \ 1$, the payoff of the power option is a convex (concave) function of the underlying asset price at maturity, and is always greater (smaller) than that of the corresponding vanilla call option. As a convex (concave) power option always has higher (lower) expected payoff than that of the corresponding vanilla option, power options are normally more expensive (cheaper) than the corresponding vanilla options. If an investor believes that the underlying asset is going to be bullish, she can buy a power option with $p > 1$, say $p = 2$, and obtain a payoff of $(S_T)^2 - K$ instead of $S_T - K$ as with a vanilla call option. From the above discussion, we can readily infer that power options can be used to take better advantage of investors' views.

Contingent premium options are also called pay-later options. The holders of pay-later options, as the name implies, do not pay the writers any up-front premiums. Actually, the holders of pay-later options do not pay any money at all if the options expire out-of-the money. Nevertheless, they need to pay the writers a prespecified premium only when the option turns out to be in-the-money. Clearly, pay-later options capture investors' desires to avoid unnecessary payment for out-of-the-money options, as they have to pay only when the options are in-the-money. However, they are not riskless because the holder of a pay-later option has to pay a prespecified premium which is very often more expensive than the otherwise equivalent vanilla option even when the option is slightly in-the-money. In other words, the gain from the next-in-the-money option may not be enough to cover the prespecified premium.

Mid-Atlantic options are known as Bermuda options or limited exercise options. As the phrase Mid-Atlantic indicates something between America and Europe, Mid-Atlantic option is a hybrid of American and European options. Instead of being exercised any time before maturity as standard American options, they can be exercised only at discrete time points before maturity. Thus, Bermuda options are quasi-American options. They are sometimes called modified American options because of this quasi-American property. At the inception of a Mid-Atlantic option, besides the regular specifications on a vanilla option, the discrete dates of exercise must also be specified. As Mid-Atlantic options possess the properties of both American and European options, their premiums are thus between those of the corresponding American and European options.

Installment options allow investors to pay the premiums in installments, therefore offering the flexibility of canceling the options if necessary. An installment option can be considered as a series of compound options or a string of extendable calls on a put option. After paying a minimum up-front premium, the investor has a choice of making the installment payments or letting the option expire. A typical installment option calls for a buyer to make four equal payments on a quarterly basis. If a payment is not made, then the option expires worthless automatically.

1.5. INSTITUTIONS INVOLVED IN EXOTIC OPTIONS

Banks have responded to the loss of some of their best customers in old businesses by embracing new products and taking efforts to create new ones, trying to entice them back with a new concept — financial risk management using derivatives. Most leading international financial institutions have paid a lot of attention to product development. Table 1.3 provides seventeen active product developer banks with established derivatives operations in London. Although product development is not solely for exotic options, exotic options represent a significant portion in product development. It should be noticed that Bankers Trust, the major product developer in derivatives business, is not included in Table 1.3 because its product development for derivatives was predominantly undertaken in the US.

Table 1.3. Active product developer banks with established derivatives operations in London.

	Location of head office
Barclays Bank	England
Chase Manhattan Bank	US
Chemical Bank	US
Citicorp Investment Bank	US
Credit Suisse First Boston	US
First National Bank of Chicago	US
Goldman Sachs International	US
Hambros Bank	England
Morgan JP Securities	US
Midland Montagu	England
Morgan Stanley	US
National Westminister Bank	England
Noimura Bank International	England
Salomon Brothers International	US
Societe Generale	France
Swiss Bank Corporation	Switzerland
Union Bank of Switzerland	Switzerland

Source: Telerate Bank Register (1991) and field study areas. The above 17 banks were identified by industry experts as innovative in terms of product development undertaken from a London base. Bankers Trust was excluded because its product development for derivatives was reported to us as predominantly undertaken in the US.

Table 1.4. Leading players in the second generation derivatives by September 1993.

Bankers Trust	(19+37+21+16+24+26+30+19+24)	216
JP Morgan	(11+10+22)	43
Credit Swiss Financial Products	(12+11+10+10)	42
General Re FP	(19+7)	26
Societe Generale	(10+8)	18
Union Bank of Switzerland	(15)	15
Mitsubishi Finance	(12)	12
Merrill Lynch	(11)	11
Swiss Bank Corporation	(11)	11
Solomon Brother	(10)	10
Barclay's Bank	(8)	8
Morgan Stanley	(8)	8
Goldman Sachs	(7)	7

Source: RISK Magazine.

Table 1.5. Leading players in the second generation derivatives by September 1994.

Bankers Trust	(26+17+11+16+11+23+17+16+10+11)	158
Swiss Bank Corp	(21+14+14+11+15+11)	86
Goldman Sachs	(12+11+14+11)	58
CSFP	(18+14+14+11)	57
Union Bank of Switzerland	(27)	27
Morgan Stanley	(11+14)	25
Merrill Lynch	(11+10)	21
Solomon Brother	(13)	13
JP Morgan	(12)	12
Societe Generale	(10)	10

Source: RISK Magazine.

Tables 1.4 and 1.5 list the institutions which are the active players in the second-generation derivatives market. The tables are based on the information from risk rankings from the *Risk* magazine in 1993 and 1994. The rankings were done for the ten most popular second-generation derivatives, six of them being second-generation options. These second-generation options include Asian (average) options, spread options, lookback options, barrier options, quanto options, and compound options. The original risk rankings gave a percentage for the top three institutions which are the most active in one particular product. In order to compare various institutions easily, we provide the sums of all percentages for all the ten products for each institution in Table 1.4. These tables clearly show that Bankers Trust, the New York based investment bank, is by far the leader in the second-generation derivatives. The tables also show that the dominant position of Bankers Trust declined significantly from 1993 to 1994, and other houses such as Swiss Bank Corporation, Goldman Sachs and Credit Swiss Financial Products are quickly catching up.

1.6. SUMMARY

We presented a bird's eye view on the world of exotic options. After a brief description of vanilla options, we described the most popular group of exotic options — path-dependent options which include four popular exotic options: Asian options, barrier options, lookback options, and forward start options. We then introduced another popular group of exotic options — correlation options which include six popular exotic options: spread options, out-performance options, exchange options, rainbow options, currency-translated options, and basket options. And in Section 1.4, we explained seven other popular exotic options: digital or binary options, compound options, chooser's options, nonlinear payoff options, pay-later options, Mid-Atlantic options or Bermuda options, and installment options. Although there are a few other exotic options besides those options we have described in this chapter, they are not as popular.

There are other types of exotic options which we have not discussed in this chapter. Although we try to provide a complete source on the existing exotic options, it is impossible to cover all of them because of the opaque nature of the OTC marketplace. What is more, product development is an on-going process. We will introduce and analyze the major kinds of exotic options and show how new exotic options can be made with a mixture of these existing products in the following chapters.

The majority of this chapter has been based on Zhang (1995b) which appeared in *European Financial Management.*

QUESTIONS

1.1. Describe briefly how exotic options are different from standard options or vanilla options.
1.2. What are path-dependent options? Give three examples of path-dependent options.
1.3. What are correlation options? Why are they becoming more important these days and will be more important in the future?

1.4. Do correlation coefficients play important roles only in correlation options? If not, give a few examples of exotic options in which correlation coefficients play important roles in determining their values.

1.5. Find an exotic option that may be considered as either a path-dependent or correlation option.

1.6. What are Asian options? What are average-strike options?

1.7. Are Asian options always cheaper than their corresponding vanilla options? Why?

1.8. What are barrier options? Why are they becoming more popular these days?

1.9. Are barrier options always cheaper than their corresponding vanilla options?

1.10. Are exotic options more expensive, or cheaper than vanilla options? Give two exotic options according to your intuition which are more expensive and two cheaper than vanilla options.

1.11. What are the three basic kinds of digital options? Describe them graphically.

1.12. Why is it true that exotic options are as old as exchange-traded options and why do they become very popular only in recent years?

1.13. Describe how you can use two types of existing exotic options to structure a new kind of option. What purpose may this new option serve?

1.14. If the strike price is specified in a contract and the option is to become valid in three months, is this option a forward-start option?

1.15. Why are floating lookback options true "no-regret" options?

1.16. Chooser options can somewhat minimize option buyer's regret, why?

1.17. Are chooser options "no-regret" options because they can minimize buyers' regret?

1.18. Is it true that pay-later options are riskless for buyers because they do not have to pay money up-front and do not have to pay at all if the options turn out to be out-of-the money?

1.19. Is it true that Bermuda options are more expansive than their corresponding European options and cheaper than their corresponding American options? Why?

1.20. Are exotic options exclusively traded in the OTC marketplace?

1.21. What exotic options are traded in both the OTC marketplace and exchanges?

1.22 What are the three most popular path-dependent options?

1.23 What are the most popular correlation options?

1.24 What are the six most popular exotic options?

Chapter 2

OPTION PRICING METHODOLOGY

2.1. EQUILIBRIUM AND ARBITRAGE

2.1.1. Equilibrium

Equilibrium is very likely the most often used term in modern economics. It appears in college and even high-school economics text books. In general, an equilibrium stands for the state of a market when supply is matched by demand. In other words, we say that a market is in equilibrium when the supply is precisely the same as the demand for a product. Relating to equilibrium of a particular market, general equilibrium means that all markets in an economy are in equilibrium simultaneously. The concept of equilibrium is so important that almost all existing economic theories are based on it.

Not only most economic theories but also most nonderivative financial theories are based on equilibrium. The first financial theory in modern finance, the theory of capital structure by Modigliani and Miller (1958, 1963), was based on capital-market equilibrium in perfect market. The celebrated capital asset pricing model or CAPM by Sharpe (1964), Lintner (1965), and Mossin (1966) is an equilibrium model which determines the expected returns of risky assets in equilibrium. Every MBA (Master of Business Administration) student or college student with a business major knows that the expected return of a risky asset in equilibrium is largely determined by its beta, or the ratio of its market risk measured by the covariance of its return and the market return, and the variance of market return. Thousands of papers have been written on these two subjects to extend the basic results in the past two to three decades and many more are still coming. Most of these studies are also based on the concept of equilibrium.

2.1.2. Arbitrage

Arbitrages can simply be considered as "free lunches". The everyday economic principle "there is no such thing as a free lunch" simply indicates that there exists no arbitrage. Arbitragers are traders who search for price differentials in different markets, buying at low prices in some markets and selling at high prices in others, thus making net profits without investing in any assets. Such profit opportunities are called arbitrage opportunities. Everyone would like to be an arbitrager to make money quickly out of nothing, yet it is not easy to find such opportunities without good understanding of how different markets are working individually and interactively. It is difficult to find such opportunities because there are many professionals using newly developed computer systems to process large amount of on-line data to search for such opportunities. More frequently, arbitrages are very important concepts in derivative theory. In this chapter, we will turn to how the "arbitrage-free" opportunity argument could be used to price derivative securities.

Arbitrage opportunities are "free lunch" opportunities. Such opportunities are too good to exist for long, because many people are looking for such opportunities and thus, price differentials diminish as the lower prices are bid up with more buying pressures and the higher prices are reduced with more selling pressures. Therefore, we can say that arbitraging activities help improve market efficiency through eliminating irrational price differentials.

2.1.3. Relationship Between Equilibrium and Arbitrage

Equilibrium analysis has been used to price many financial assets. In general, investors' preferences such as risk aversion, the distribution of endowments and preference across investors, and demand conditions, normally enter pricing expressions as they determine the equilibrium parameter values. The arbitrage-free argument, or no-arbitrage argument, has been used to price all derivative securities. In all no-arbitrage pricing models, investors' preferences such as risk aversion and demand conditions are irrelevant. In particular, expected returns of the underlying assets do not appear in pricing formulas for derivative securities but normally appear in equilibrium-based theories. Therefore, no-arbitrage argument is different from equilibrium analysis.

However, these two kinds of analyses may yield the same results under certain conditions. Rubinstein (1976) showed that the arbitrage-free pricing formula of Black-Scholes still holds under four conditions: (1) single-price law of markets, (2) non-satiation, (3) the marginal utility of consumption and the stock price are jointly lognormal, and (4) investors agree on the volatility of the underlying asset. We need to review two economic concepts before we can further explain the relationship between equilibrium and arbitrage analyses. When an investor's utility function is a power function $W^{1+\alpha}/(1+\alpha)$ or his marginal utility is W^{α}, where α is a constant parameter and W stands for his total wealth, we say that this investor exhibits constant proportional risk aversion (CPRA). When an investor's utility function is a negative exponential with constant coefficient $-\exp(-\alpha W)$, or its marginal utility is $\exp(-\alpha W)$, we say that this investor exhibits constant absolute risk aversion (CARA). Risk aversion is an economic term meaning disliking risk. Brennan (1979), and Stapleton and Subrahmanyam (1984) showed that the representative investor exhibits constant proportional (resp. absolute) risk aversion for arbitrary bivariate lognormal (resp. normal) distributions of the price of the underlying asset, and the aggregate wealth is necessary and sufficient for the equilibrium and arbitrage methods to yield the same results.

As it will be shown in the following chapters in the book, the no-arbitrage argument has replaced the equilibrium analysis in pricing all types of derivative securities, and become the single most important principle in pricing options. Any pricing model which includes arbitrage can no longer be easily accepted by researchers and professionals.

2.2. BASIC OPTION TERMINOLOGY

Let us start with an example. Suppose you believe that a stock price will rise from \$100 per share today to \$125 half a year from now, what would you like to do with your \$10,000? You would most likely buy 100 shares with all the \$10,000 and make 25% return on the stock in half a year if the stock turns out to be \$125 as you expect. You may also buy the stock by margin. Suppose that you can borrow \$10,000 from your broker and you pay 20% annual interest on the money you borrow. With the additional \$10,000, you can buy 200 shares, these 200 shares will make \$5000 for you. After paying \$1000 interest (the annual interest rate is 20%, the semiannual rate is 10%) for the money you borrow, you net \$4000 profit, thus making 40% in half a year on your initial \$10,000. Clearly, the second strategy works better than the first, yet you involve higher risks with the second strategy because you have the interest obligation. In general, the amount of money you can borrow is limited, so you cannot take the best advantage of your expectation even if you are right. With the existence of options market, you have another much better way to make higher profit with your money. We will return to this after clarifying some jargon of options.

There are different ways to classify options according to their characteristics. Most often, they are grouped as call options and put options. A call option is a financial contract that gives its buyer or holder the right to buy the underlying asset at a prespecified price within a prespecified time period. This prespecified price is called the exercise price or strike price of the option, and the prespecified time is called the time-to-maturity, time-to-expiration, or tenor of the call option. A put option, on the other hand, is a financial contract that gives its buyer or holder the right to sell the underlying asset at a prespecified price within a prespecified time. The underlying market is often called the cash market. If an option can only be exercised at maturity or expiration, the option is called a European option. If it can be exercised any time before or on the expiration day, it is called an American option.

Options are somewhat similar to futures — they are both financial contracts which can be exercised at a prespecified time in the future. However, there is a very important difference between them. A futures contract represents obligations on both sides of the contract, whereas an option contract represents rights for the holder to buy and obligations for the seller to sell. Thus, if the price of the underlying asset is higher than the strike price at or before the expiration time, the call option holder can simply exercise his/her rights to make a profit by buying the underlying stock at the strike price and selling it at a higher market price. If the market price falls below the strike price, he/she can simply let the call option expire without doing anything. Figure 2.1 represents the payoff pattern of a European call option. Obviously, there is one kinked point at the strike price in the option's payoff diagram. If the stock price at expiration is at or below the strike price, then the call option will expire worthless, and if it is greater than the strike price, then the payoff of the option is simply the difference between the stock price at expiration and the strike price.

Analytically, the payoff of a European call option (EC) can be formally expressed as follows:

$$EC = \text{Max}[S(\tau) - K, 0]\,, \tag{2.1}$$

where $S(\tau)$ and K stand for the underlying asset price at maturity and the strike price of the option respectively; $\tau = t^* - t$ is the time to maturity, where t and t^* stand for the current time and time to maturity of the option, respectively; and Max [.,.] is the mathematical function which gives the larger of two numbers involved.

Similarly, the payoff of a European put (EP) option is given by

$$EP = \text{Max}[K - S(\tau), 0]\,, \tag{2.2}$$

where $S(\tau)$, K, and Max [. , .] are the same as in (2.1).

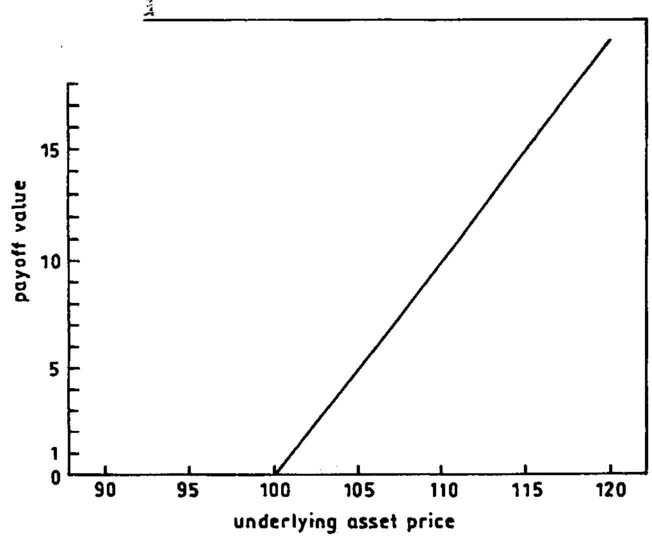

Fig. 2.1. Payoff of a European call, given spot price = strike price = $100.

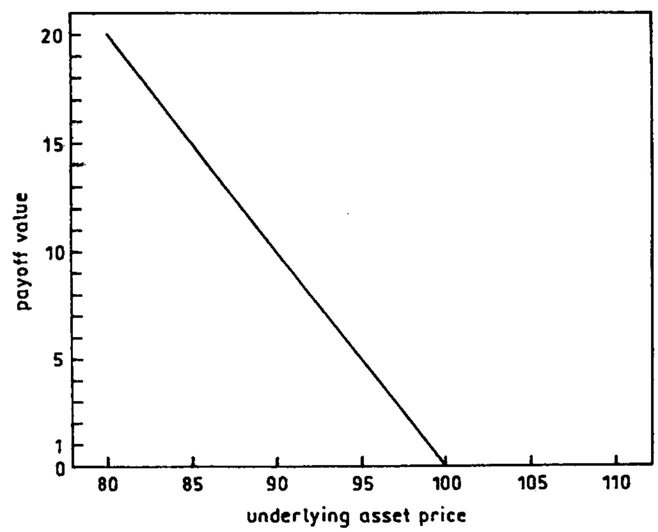

Fig. 2.2. Payoff of a European put, given spot price = strike price = $100.

Figure 2.2 depicts the payoff pattern of a European put option. Compared to that of a European call option given in Figure 2.1, there is also one kinked point at the strike price in the put option's payoff diagram. However, the curve is upward to the left rather than to the right as in the case of a European call option.

From the definition of a call option, we know that its buyer has a positive expected payoff, as the worst case is the situation when the option expires worthless and the best case is when he/she can obtain unlimited payoff if the stock price rises unlimitedly. On the other hand, the seller or writer of the call option has a negative expected payoff, as he/she has to meet the obligation to sell the underlying asset at the strike price even if it is lower than the market price. There is no such thing as a free lunch for either the buyer or the seller. The seller of an option is often called the writer of the option as he/she simply writes a selling contract. Unlike the forwards or futures buyer, the option buyer has to pay some money to the writer to compensate the expected loss. The amount of money the option buyer pays the writer is called the premium of the option. Thus, the worst case for the option buyer is that he loses the premium when the option expires worthless. That is why it is often said that call options have limited liabilities (the premiums) and unlimited payoffs or returns. In order to obtain the right to buy or sell the underlying assets, option buyers have to pay some premiums to the writer. This is another important difference between option contracts and futures contracts.

There are a few other terms often used in option literatures and professional discussions. If the current underlying asset price is greater (resp. smaller) than the strike price of a call option, the call option is said to be in-the-money or ITM (resp. out-of-the-money or OTM); if the underlying asset price is smaller (resp. greater) than the strike price of a put option, then the put option is called ITM (resp. OTM); and if the underlying asset spot price is the same as the strike price of an option, then the option is said to be at-the-money or ATM. In general, when the option is ITM, it is more likely that it will be worth exercising. The word moneyness stands for the relative magnitudes of the underlying spot and strike prices or whether an option is ITM, ATM, or OTM.

We can now return to the example at the beginning of this section. As you believe that the stock will go up from \$100 per share today to \$125 half a year from now, you can simply buy a European call option to expire half a year from now. You may need to pay a few dollars premium to buy a call option that gives you the right to buy the underlying stock at a prespecified price. To make our example simple, we assume that the price for a call option with exercise price \$105 written on the stock is \$5 (we will study how option prices are determined in the following section). Thus, you can buy 10,000/5 = 2000 call options written on the stock. As one option contract normally controls 100 shares of the underlying stock, buying 2000 call options is the same as buying 20 call option contracts. If the stock price rises to \$125 per share at maturity as you expected, you exercise your right to buy 2000 shares of the underlying stock at \$105 per share, selling these shares at the price of \$125 per share, thus making a net profit of $2000 \times (\$125 - \$105) = \$40,000$. The \$40,000 is equivalent to 300% return on your initial \$10,000 capital.

The above example shows how to make profits with call options if you believe that the underlying asset will become more expensive. When you forecast that the price of the underlying asset will go up, buying call options is an alternative way to profit from the appreciation of the underlying asset and very often a cheaper one. Put options, on the other hand, can be used to protect you from potential losses in the underlying market. Because of this protective nature, buying put options are very similar to buying insurance for the underlying assets.

2.3. THE BLACK-SCHOLES OPTION PRICING MODEL

In the previous section, we came across the problem of how to determine the premium of an option. Finding appropriate prices for options is the crucial part in options trading, because trading can be carried out by comparing the option market prices with these appropriate prices or true values. If the market price of an option is smaller than the true value of the option, we say that the option is undervalued, and it is profitable to buy the option; and if its market price is greater than the true value of the option, we say that the option is overvalued, and it is profitable to write or sell the option. Therefore, to determine the fair prices for all kinds of options and all other derivative securities is the central topic in the option business.

The concepts of options we have discussed so far are pretty straightforward. What makes them appear difficult are the techniques used to represent the arbitrage conditions from which we can find their appropriate values. Thus, we suggest that those first-time readers of a derivative book skip some sections of this chapter, and concentrate on the general ideas and arguments first. After finishing reading the whole book, they may have a much better understanding when returning to these sections.

Before any houses or buildings can be built, constructors normally examine the underlying land very carefully. They have to know the underlying land very well before they decide whether they can build a house or a building on it. The same is true for constructing derivative securities. In order to price any derivative securities, we have to know the underlying assets very well. Unfortunately, our knowledge of the stock market, currency market, debt market, and other underlying markets is still so limited that we can hardly describe the fluctuations of these underlying assets satisfactorily. Therefore, we have to make some assumptions about the fluctuations of these underlying asset prices. Any imaginary world in which a certain number of assumptions are satisfied is often called a model as in natural science and engineering. Two models are different if the set of assumptions is different from each other.

Although there are many option pricing models, the Black-Scholes lognormal model, in which the underlying asset price is assumed to be lognormally distributed, is still by far the most popular one. To some degree, the Black-Scholes option pricing model has facilitated option trading and helped the growth of the whole financial derivative market. Most option pricing models either follow it directly or indirectly. The Black-Scholes pricing model is rather difficult to be described thoroughly without using some advanced mathematics. In order to concentrate on the basic arguments in this section, we will explain the assumptions that are made in the Black-Scholes model and the properties that the pricing formula possesses, and leave the derivation of the formula in the following sections.

Black and Scholes (1973) pioneered the modern option pricing theory. Interestingly enough, this model came into existence in the same year that the Chicago Board of Options Exchange (CBOE) was established, in 1973. This model, as its name implies, was the joint work of the late Fisher Black, a former Professor at the University of Chicago and a former partner at Goldman Sachs, and Myron Scholes, a research Professor at Stanford University and now with Long Term Capital Management, a hedge fund in Connecticut. Hundreds of academic papers have been published to extend this model in various directions. Professionals started to use this model to evaluate options soon after it was published. However, it took some time for this model to be well understood. Although different firms may use their own models which reflect their beliefs of the fluctuations of the underlying assets, most of these models can, to a large degree, be considered as extensions of the Black-Scholes model. Although there are all kinds of extended versions of the Black-Scholes model, the original model is still by far the most widely used by professionals as a benchmark.

In a typical Black-Scholes environment the underlying asset return is assumed to follow a lognormal random walk. Suppose that the underlying asset price S follows the geometric Brownian motion (named after the English botanist R. Brown):

$$dS = \mu S dt + \sigma S dz(\tau), \tag{2.3}$$

where $z(\tau)$ is a standard Gauss-Wiener process,[1] and μ and σ are the instantaneous mean and standard deviation of the underlying asset price, respectively. The instantaneous standard deviation σ is more often called the volatility of the underlying asset.

Solving equation (2.3) using the standard method (see Appendix for the procedure) yields

$$S(\tau) = S \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma z(\tau)\right] . \tag{2.4}$$

Taking natural logarithm to both sides of (2.4), we can readily obtain

$$\ln\left\{\frac{S(\tau)}{S}\right\} = \left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma z(\tau) ,$$

which is normally distributed because $z(\tau)$ is normally distributed with zero mean and variance τ. That is why the Black-Scholes model is also called a lognormal model. Figure 2.3 depicts the curve of the density function of a standard normal distribution which looks very much like a bell. Figure 2.4 depicts the density function of the lognormal distribution. Figure 2.3 is

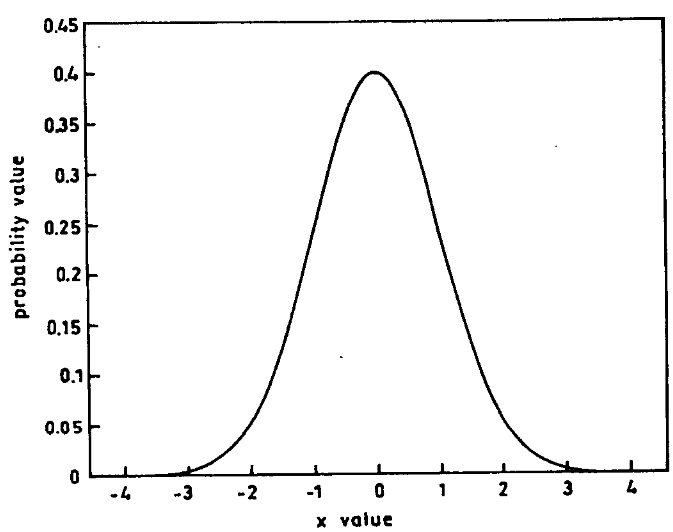

Fig. 2.3. Density function of the standard normal distribution.

[1] A standard Gauss-Wiener process can be simply understood as a normal variable with zero mean and variance which equals the difference between the future time and current time. Thus, the standard Gauss-Wiener process $z(\tau)$ is normally distributed with zero mean and variance τ.

symmetric around zero and Figure 2.4 is skewed to the left. From these two figures, we can observe that the variables can be either negative or positive in the normal distribution, yet they cannot be negative in the lognormal distribution. Thus lognormal distributions are used in pricing options instead of normal distributions because lognormal distributions can overcome the negativeness problem associated with normal distributions.

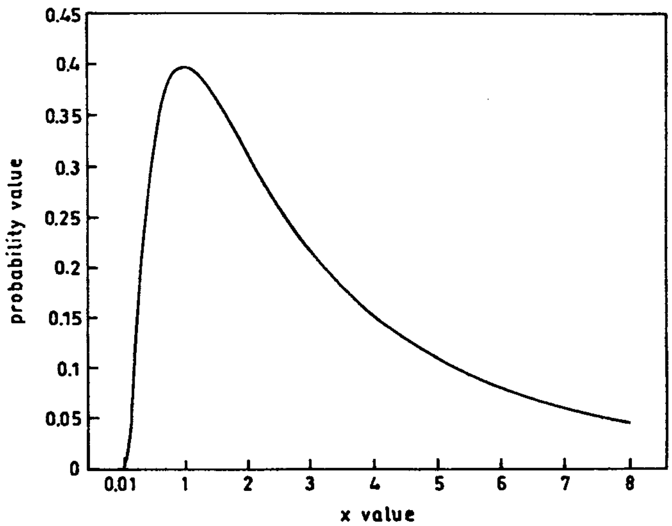

Fig. 2.4. Density function of the lognormal distribution.

As in most other theoretical models, the Black-Scholes model requires many assumptions. Besides the lognormal assumption, Black and Scholes made a few others:

(i) the short-term interest rate is known and is constant through time;
(ii) the underlying asset pays no dividend;
(iii) the option is European;
(iv) there are no transaction costs in buying or selling the underlying asset or option;
(v) it is possible to borrow any fraction of the price of a security to buy it or hold it, at the short-term interest rate;
(vi) trading can be carried on continuously; and

(vii) there are no penalties to short selling.[2]

Of all the above assumptions, continuous trading is a major one. It makes it possible to use continuous time calculus conveniently. With these assumptions, Black and Scholes obtained the amazing formula for the European call option price using the no-arbitrage argument. This formula is given as follows (see the following sections for the derivation):

$$C = SN(d_1) - Ke^{-r\tau}N(d_2)\,, \tag{2.5}$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_2 + \sigma\sqrt{\tau}\,,$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\,,$$

K is the exercise price of the option, r is the risk-free rate of return, σ is the volatility of the return of the underlying asset, e^x is the natural exponential function with power x, $e^{-x} = 1/e^x$, $\ln(x)$ is the natural logarithm function, and $N(x)$ is the value of the cumulative function of the standard normal distribution at x.

The cumulative function of the standard normal distribution $N(x)$ gives the probability that all normal random variables are not greater than x, or $N(x)$ = Probability ($X \leq x$). Figure 2.5 depicts the cumulative function value for various values of x. As $N(x)$ stands for probability, it is always between zero and one. We can observe that $N(x)$ approaches zero when x becomes smaller than -4, and $N(x)$ approaches one when x becomes greater than 4. Figure 2.5 also indicates that $N(x)$ is an increasing function of the argument x. Appendix I at the end of this book provides values of $N(x)$ for $0 \leq x \leq 3.99$. To find the value of $N(1.38)$, for example, we need to locate 1.3 in the first column and then locate 0.08 from the first row. The number at the intersection of column 1.3 and row 0.08 gives the probability $N(1.38) = 0.9162$.

It is easy to find the value of $N(x)$ when x is negative. As the standard normal distribution is symmetric about zero, we use the identity $N(x) = 1 - N(-x)$ to find $N(x)$ when x is negative. For example, $N(-1.38) = 1 - N[-(-1.38)] = 1 - N(1.38) = 1 - 0.9126 = 0.0874$.

[2]Simply speaking, short selling means selling some securities the seller does not own or selling other people's securities. Specifically, a seller who does not own a security accepts the price of the security from a buyer and agrees to settle the buyer on some future date by paying him an amount equal to the price of the security.

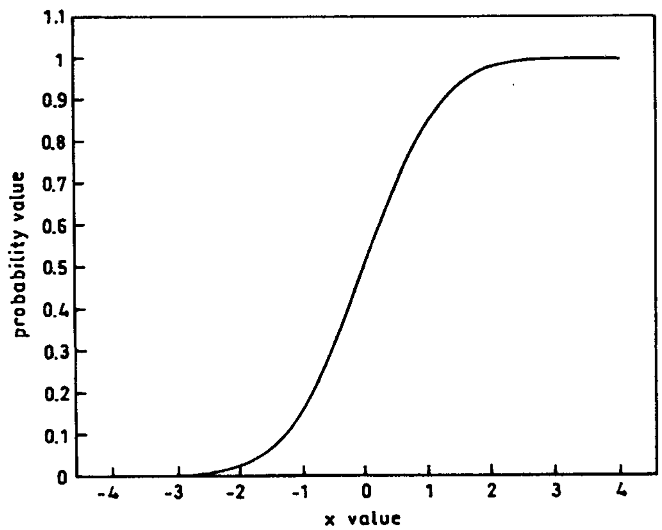

Fig. 2.5. Cumulative function of the standard normal distribution.

The Black-Scholes formula given in (2.5) is clearly a function of five factors: (1) underlying asset spot price S; (2) option strike price K; (3) time to maturity τ; (4) risk-free rate of return r, and (5) volatility of the return of the underlying asset σ. The first and the fourth factors can be observed from the market, and the second and the third are specified in the option contract. The last factor, the volatility of the return of the underlying asset, or the annualized standard deviation of the return of the underlying asset, is neither specified in the option contract nor directly observable from the market. We have to estimate this volatility value using historical data of the underlying asset in order to use the Black-Scholes formula.

Although there are various ways to estimate the volatility parameter depending upon the particular markets and particular problems, most often daily prices or index data from certain number of days back are used to estimate the volatility. The daily prices or index data, often called level prices, are converted into daily gross returns — ratios of each daily price over the previous daily price. The standard deviation of these daily returns, or more specifically the logarithm of these daily gross returns is calculated. The standard deviation thus calculated is the daily standard deviation. The last step is to annualize the standard deviation: multiply the daily standard deviation by the square root of 253 because there are approximately 253 business days in a year.

Example 2.1. Suppose that the underlying spot price is \$100, the strike price \$105, interest rate 20%, the volatility of the underlying stock 30%, what is the price of the call option to expire in half a year?

Substituting $S = \$100$, $K = \$105$, $\sigma = 30\%$, $r = 20\%$, and $\tau = 0.50$ into (2.5) yields the two arguments d_1 and d_2:

$$d_1 = d_2 + \sigma\sqrt{\tau} = 0.14 + 0.30 \times \sqrt{0.5} = 0.35\,,$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$= \frac{\ln(100/105) + (0.20 - 0.30^2/2) \times 0.5}{0.30\sqrt{0.5}} = 0.14\,.$$

Using the table of the cumulative function for the standard normal distribution at the end of this book, we get $N(d_2) = N(0.14) = 0.5557$ and $N(d_1) = N(0.35) = 0.6368$. Substituting these values into (2.5), we obtain the call option price:

$$C = SN(d_1) - Ke^{-r\tau}N(d_2) = 100\times 0.6368 - 105\times e^{-0.2\times 0.5}\times 0.5557 = \$10.89\,.$$

Following the same procedure, we can obtain the call option prices with various current underlying asset prices. Figure 2.6 depicts the call option prices for $\$80 \leq S \leq \$120$, $K = \$100$, $\sigma = 15\%$, interest rate $r = 10\%$, and time to maturity $\tau = 1$ year and three months. The top curve represents various prices of call options to expire in one year, the dotted curve represents various prices of call options to expire in three months, and the kinked line represents the value of call options at maturity. It is obvious that the call option price curves are concave and well above the kinked payoff line below. This is because the time to maturity is one year and three months, i.e., greater than zero. The difference between the concave curve and the kinked payoff line is called the time value of the call options, since the concave curve approaches the kinked payoff line as the time to maturity approaches zero.

If we increase or decrease both the spot and strike prices by the same percentage, say $\lambda > 0$, the call option price would increase or decrease by the same percentage. In other words, the pricing formula given in (2.5) is homogeneous of degree one for spot and strike prices.[3] If we use $C(S, K, \sigma, r, \tau)$ to stand for the call option price given in (2.5), the call option price with

[3] A function $F(x, y)$ is said to be homogeneous of degree k for the variables x and y if $F(\lambda x, \lambda y) = \lambda^k F(x, y)$.

spot λS and strike λK can be expressed as

$$C(\lambda S, \lambda K, \sigma, r, \tau) = \lambda C(S, K, \sigma, r, \tau)\,, \tag{2.6}$$

which is the so-called scaling property of the Black-Scholes formula. This is true for both call and put options. The scaling property is always valid simply because the scaling parameter λ cancels out in $\ln[(\lambda S)/(\lambda K)]$ in the d_1 and d_2 functions. The scaling property is a very useful characteristic of the pricing formula which can be used to simplify many problems significantly. We will use it in the following chapters.

2.3.1. Limiting Cases of the Black-Scholes Model

From the pricing formula given in (2.5), we can readily find the following always holds:

$$C \le SN(d_1) \le S \tag{2.7}$$

which indicates that the price of a European call option can never surpass that of its corresponding underlying asset.

The implication of (2.7) is also intuitive because the value of any call option is derived from that of its underlying asset, and therefore should not be greater than its underlying asset price.

The inequality given in (2.7) gives the upper limit of the price of a European call option in relation to its underlying asset price. There are a few limiting cases of the Black-Scholes model in which a European call option price could exactly reach this upper limit values.

2.3.1.1. When time to maturity is infinity

Figure 2.6 illustrates the time value of a call option, which increases with longer time to maturity. What should be the price of a European call option if its time to maturity goes to infinity, given other parameters?

Substituting $\tau \to +\infty$ into (2.5). We can readily find (we leave this as an Exercise by the end of this chapter)

$$C = S\,. \tag{2.8}$$

2.3.1.2. When interest rate is infinity

Substituting $r \to +\infty$ into (2.5) yields (we leave this as an Exercise by the end of this chapter)

$$C = S\,. \tag{2.9}$$

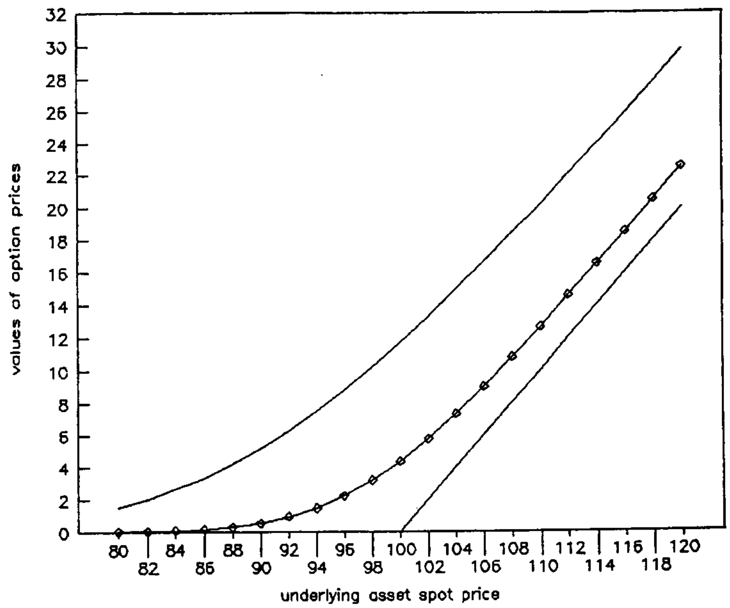

Fig. 2.6. Call option values for various spot prices and maturity.

2.3.1.3. When volatility is infinity

Substituting $\alpha \to +\infty$ into (2.5) yields (we leave this as an Exercise by the end of this chapter)

$$C = S\,. \tag{2.10}$$

2.4. PRICING OPTIONS USING THE ARBITRAGE-FREE ARGUMENT

We described the concept of arbitrage at the beginning of this chapter. In this section, we will illustrate how the arbitrage-free or "no-free-lunch" argument can be used to obtain the celebrated Black-Scholes formula. Black and Scholes (1973) constructed a portfolio including one unit of the underlying asset long and w/w_1 units of call options written on the underlying asset short, where w stands for the value of a call option which is assumed to be a function of the underlying spot price x and the time to maturity of the option τ; and w_1 stands for the first-order partial derivative of the call option value with respect to its underlying spot price. Following Black-Scholes, the value of the hedged portfolio is given by

$$x - \frac{w}{w_1}\,, \tag{2.11}$$

and the small change of the value of the hedged portfolio given in (2.11) in a very short period of time Δt can be obtained as

$$\Delta x - \frac{\Delta w}{w_1}, \tag{2.12}$$

where Δ stands for an infinitesimal change in value.

Using stochastic calculus, or more specifically Itõ's lemma,[4] we can express the change in the call option value in terms of its partial derivatives:

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} \sigma^2 x^2 \Delta t + w_2 \Delta t, \tag{2.13}$$

where σ stands for the constant standard deviation or volatility of the underlying asset, and w_2 and w_{11} represent the first-order partial derivative of the option value function with respect to the time to maturity and the second-order partial derivative of the call option value with respect to the underlying spot price.

Substituting (2.13) into (2.12) yields the change in the hedged portfolio value as follows:

$$\Delta\left(x - \frac{w}{w_1}\right) = -\left(\frac{1}{2} w_{11} \sigma^2 x^2 + w_2\right) \frac{\Delta t}{w_1}. \tag{2.14}$$

As trading of both the underlying asset and the call option is assumed to be continuous and thus the hedging ratio can be adjusted continuously without any cost in the Black-Scholes model, the risk of the hedged portfolio is very small and can be diversified away totally. Therefore, the change in the hedged portfolio value must equal the interest made with the hedged portfolio over the infinitesimal period of time Δt:

$$\left(x - \frac{w}{w_1}\right) r \Delta t = -\left(\frac{1}{2} w_{11} \sigma^2 x^2 + w_2\right) \frac{\Delta t}{w_1}, \tag{2.15}$$

where r is the net risk-free rate of return which is assumed to be a constant.

[4]Itõ's lemma is the basic stochastic calculus rule for computing stochastic differentials of composite stochastic functions. Specifically, for any function $y(t) = u[t, Z(t)]$, where $Z(t)$ is a stochastic process $d_Z(t) = f(t)dt + \sigma(t)d_u(t)$, Itõ's lemma states that the process $y(t)$ has a differential on $[0, T]$ given by

$$dy(t) = \{u_t[t, Z(t)] + u_z[t, Z(t)]f(t) + \frac{1}{2} u_{zz}[t, Z(t)]\sigma^2(t)\}dt + u_x[t, x(t)]\sigma(t)d_u(t).$$

In our example w is a function of time and the underlying asset price which is also a stochastic process.

Deleting Δt from both sides of (2.15) and multiplying w_1 to both sides of it yields the following:

$$(xw_1 - w)r + w_{11}\sigma^2 x^2/2 + w_2 = 0\,, \tag{2.16}$$

which is a second-order partial differential equation with the boundary condition as the final payoff of the European call option:

$$\begin{aligned} w(x, t^*) &= x - K \quad \text{for} \quad x \geq K \quad \text{and} \\ &= 0 \quad \text{otherwise}\,, \end{aligned} \tag{2.17}$$

where t^* and K stand for the time at maturity and strike price of the option, respectively.

Solving the second-order partial differential equation (2.16) for the option value w under the boundary condition given in (2.17) gives us precisely the European call option pricing formula (2.5) in terms of the time to maturity $\tau = t^* - t > 0$, strike price K, volatility σ, spot price x, and interest rate r. In order to keep the transparency of how the arbitrage-free argument is used to price options, we leave the question of how to solve the partial differential equation to the following section.

2.5. SOLVING PARTIAL DIFFERENTIAL EQUATIONS

Following Black and Scholes (1973), we established the second-order partial differential equation (PDE) (2.16) using the arbitrage-free argument in the previous section. The solution with the corresponding boundary condition (2.17) should be the pricing formula for the European-style call option in terms of the current stock price S, exercise price K, interest rate r, volatility of the underlying asset σ, and time to maturity $\tau = t^* - t$. In order to concentrate on the way the arbitrage-free argument is used in our analysis and to separate financial argument with mathematical solution, we did not solve the PDE. We concentrate on how to solve the PDE in (2.16) in this section, with both analytical and numerical methods.

2.5.1. Analytical Method

Many analyses in physics, chemistry, engineering, and other fields of science involve the PDEs with some specific initial and boundary conditions. A lot of effort has been taken to find efficient methods to solve PDEs. In general, it is not easy to solve second-order or higher order PDEs. The general method is to transform or simplify relevant PDEs into some standard PDEs whose solutions can be more easily found. In order to be consistent with our notation, we use S to stand for the current spot price of the underlying asset. Let's make the following substitution first:

$$w(S,t) = e^{-r\tau} y(z_1, z_2)\,, \tag{2.18}$$

where

$$z_1 = \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\left[\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau\right],$$

$$z_2 = \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 \tau, \quad \text{and} \quad \tau = t^* - t\,.$$

With the substitution given in (2.18), the PDE given in (2.16) can be shown to be transformed to the following standard form:

$$\frac{\partial^2 y}{\partial u^2} = \frac{\partial y}{\partial t}\,, \tag{2.19}$$

and the boundary condition given in (2.17) becomes the following initial condition

$$\begin{aligned} y(u,0) &= 0 \quad \text{for} \quad u < 0 \quad \text{and} \\ &= K\left[e^{u(\frac{1}{2}\sigma^2)/(r-\frac{1}{2}\sigma^2)} - 1\right] \quad \text{for} \quad u \geq 0\,. \end{aligned} \tag{2.20}$$

Equation (2.19) with the initial condition (2.20) is the heat-transfer equation in physics. Black and Scholes used the existing solution of Churchill (1963):

$$y(u,s) = \frac{K}{\sqrt{2\pi}} \int_{-u/\sqrt{2s}}^{\infty} \left[e^{(u+\sigma q\sqrt{2s})(\frac{1}{2}\sigma^2)/(r-\frac{1}{2}\sigma^2)} - 1\right] e^{-q^2/2} dq\,. \tag{2.21}$$

Substituting (2.21) into (2.18) yields

$$\begin{aligned} w(S,t) &= e^{-r\tau} \frac{K}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left[e^{(\ln(S/K)+\sigma q\sqrt{\tau}+(r-\frac{1}{2}\sigma^2)\tau} - 1\right] e^{-q^2/2} dq \\ &= K\left[\frac{S}{K}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(\sigma q\sqrt{\tau}-\frac{1}{2}\sigma^2\tau-\frac{1}{2}q^2)} dq - e^{-r\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-q^2/2} dq\right] \\ &= S\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(q-\sigma\sqrt{\tau})^2/2} dq - Ke^{-r\tau}[1 - N(-d_2)]\,, \end{aligned} \tag{2.22}$$

where

$$d_2 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau\right],$$

and $N(x)$ is the cumulative function of the standard normal distribution.

Using the substitution $v = q - \sigma\sqrt{\tau}$ and the identity $N(x) + N(-x) = 1$, we can simplify (2.22) to

$$\begin{aligned} w(S,t) &= S[1 - N(-d_2 - \sigma\sqrt{\tau})] - Ke^{-r\tau}[1 - N(-d_2)] \\ &= SN(d_2 + \sigma\sqrt{\tau})] - Ke^{-r\tau}N(d_2)\,, \end{aligned} \tag{2.23}$$

which is precisely the Black-Scholes formula given in (2.5).

2.5.2. Numerical Method

The above analytical derivation involves one critical step — the transformation given in (2.18). For a general PDE with arbitrary boundary conditions, it can be rather difficult to find an appropriate transformation so that a standard form can be obtained. What makes the matter worse is that analytical transformations do not exist for many PDEs with given boundary conditions. Thus the corresponding analytical solutions do not exist. Although analytical solutions are beautiful and convenient, this method is very limited in practical use.

From the rapid progress in computer technology, a lot of studies have been done to solve PDEs numerically. Brennan and Schwartz (1978) first introduced finite-difference — a numerical method to solve PDEs to obtain option prices. This method has also been widely used in pricing various kinds of complex derivatives. Wilmott, Dewynne, and Howison (1993) is a good source for explaining various aspects of finite-difference and its applications in pricing derivatives.

2.5.3. Finite-Difference

The basic idea of finite-difference approximation is very simple: to replace partial derivatives used in any PDEs with their corresponding finite-difference approximations. More specifically, instead of treating the change of the independent variable infinitesimal as in the definition of partial derivatives, the change is small and its magnitude depends on the currency level required in a specific problem. Put mathematically, the partial derivative of a function $f(x,y)$ with respect to y is given

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}\,, \tag{2.24}$$

which is approximated with

$$\frac{\partial f(x,y)}{\partial y} \cong \frac{f(x, y + \delta y) - f(x,y)}{\delta y} \tag{2.25}$$

in finite-difference approximation, where δy represents a small change of the variable y.

The difference given in (2.25) is called a forward difference since the difference is in the forward direction. Similarly, the following finite-difference is called a backward difference

$$\frac{\partial f(x,y)}{\partial y} \cong \frac{f(x,y) - f(x,y-\delta y)}{\delta y}, \tag{2.26}$$

since the difference is in the backward direction. The central difference is similarly defined as follows

$$\frac{\partial f(x,y)}{\partial y} \cong \frac{f(x,y+\delta y) - f(x,y-\delta y)}{2\delta y}. \tag{2.27}$$

The central finite-difference approximation given in (2.27) is not used in practice as it often leads to unstable numerical schemes. Instead, the following central finite-difference approximation is often used

$$\frac{\partial f(x,y)}{\partial y} \cong \frac{f(x,y+\delta y/2) - f(x,y-\delta y/2)}{\delta y}. \tag{2.28}$$

Similarly, second-order partial derivatives can also be approximated with the finite-difference method. For example, the second-order partial derivative of the function $f(x,y)$ with respect to y may be approximated using (2.28)

$$\begin{aligned}\frac{\partial^2 f(x,y)}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f(x,y+\delta y/2)}{\delta y} - \frac{\partial f(x,y-\delta y/2)}{\delta y}\right] \\ &\cong \frac{f(x,y+\delta y) - 2f(x,y) + f(x,y-\delta y)}{(\delta y)^2}. \end{aligned} \tag{2.29}$$

Substituting the above approximations into specific second-order PDEs, we can find the corresponding finite-difference equations for the approximated values of the desired variables.

2.5.4. Finite-Element

Besides the finite-difference method in solving PDEs, there is a more sophisticated method which involves the finite-element approximation. The finite-element method is currently used by mathematicians and engineers to solve complicated PDEs with several dimensions. The advantages of the finite-element method over the finite-difference method are that the former is more accurate and can handle more complicated boundary conditions than the latter. The finite-element method derives necessary discrete equations automatically following a set of rules which are programed into the computer. For most existing options, both vanilla and exotic, only two dimensions (normally time and asset price) are involved; the boundary conditions are most often straightforward, therefore the finite-difference method can solve most problems and the finite-element method may not show its advantages. However, the finite-element method may find its use with further development of the derivatives industry to solve more complicated financial problems.

Although both the finite-difference and finite-element methods are powerful enough to solve most of the problems in pricing derivatives products, they share one obvious disadvantage: lack of intuition. Unlike the lattice or the tree method (to be discussed in Section 2.8), intuition is somewhat buried in the mathematics or computer programs with these two methods. Thus, these methods are like a "black box" which is supposed to give correct answers to some specific problems. Yet whenever problems arise, they are not as easily fixed as with the tree or lattice method.

2.6. RISK-NEUTRAL VALUATION RELATIONSHIP

2.6.1. Risk-Neutral Valuation Relationship

Cox and Ross (1976) analyzed the structure of option valuation models and developed an intuitive technique to solve many option pricing problems. Essentially, they showed that as long as hedge positions can be constructed, the values of European call options can be obtained by discounting the expected payoffs of the options at maturity by the risk-free rate of return. They argued that whenever a portfolio can be constructed, which includes a contingent claim and its underlying asset in such proportions that the instantaneous return on the portfolio is non-stochastic, the resulting valuation relationship is risk-neutral. A risk-neutral valuation relationship (RNVR) is a formula relating the value of the contingent claim to the value of the underlying asset and other directly or indirectly observable exogenous variables. Under risk neutrality, values of any contingent claims do not involve any parameters of investor's preferences such as risk aversion as if investors were risk-neutral.

Harris and Kreps (1979) showed that the RNVR holds in general. Their theory is based on a somewhat abstruse statistic concept called martingales. Loosely speaking, a martingale represents a sequence of events; the expected value of every next trial is, on average, to be neither larger nor smaller than the value of the current trial. If each trial represents the return of a trial in a game, the gambler is expected to be neither wealthier nor poorer in the next trial than he was before this trial. Thus, a martingale measures a fair game as the gambler's fortune on the next play is on average his current fortune and is not otherwise affected by the previous history. Harris and Kreps showed that as long as the underlying security price model does not permit free lunches or arbitrages, arbitrage pricing methods hold; or there exists one single value for a derivative product if and only if there exists a unique equivalent martingale measure.

2.6.2. Compounding and Discounting Factors

Before we derive the Black-Scholes formula using the RNVR, it is useful for us to review the concepts of compounding and discounting. Compounding relates to the way in which interest is calculated. If the annual interest rate r is constant and the number of interest calculation is n in a year, then the interest rate per calculation period is r/n, and the principle plus interest from one dollar invested today by the end of the first calculation period is simply $(1+r/n)^1$, the principle plus interest by the end of the second period is $(1+r/n)^2$, and that by the end of one year from now is $(1+r/n)^n$. The principle plus interest by the end of t year(s) from now is

$$CPDF(n,t) = \left(1 + \frac{r}{n}\right)^{tn}, \tag{2.30}$$

where $CPDF(n,t)$ stands for the compounding factor with n calculations in a year in t years, the number $(1+r/n)^{tn}$ is called the compounding factor for n calculations per year and t years from today, because it represents the amount of money that will be available t years from now; for any amount of deposit today is simply the amount deposited today multiplied by this factor.

If the compounding frequency is annual, semiannual, quarterly, monthly, weekly, daily, and minute by minute, we need only set $n = 1$, 2, 4, 12, 52, 365, 8700, and 525600, respectively. Table 2.1 gives the compounding factors with daily, weekly, monthly, semiannual, and annual compounding and for various interest rates $r = 5$, 10, 15, 20, 25, 30, 40, 50, 60, 70, and 80%. We can readily see that the more frequently compounding is carried, the larger the compounding factor because interest is put to make more interest faster with higher compounding frequency.

For example, if you deposit \$1000 in your bank for one year with the annual interest rate 15% and monthly compounding, the money you will have in the bank by the end of one year is the product of 1000 and the compounding factor 1.1608, or \$1160.80.

Table 2.1. Compounding factors for various interest rates and compounding frequencies.

n	1 annual	2 semi-an	4 quarterly	12 monthly	52 weekly	365 daily	8760 hourly	525600 minute	Infinity continuous
interest									
5%	1.05	1.050625	1.050945	1.051162	1.051246	1.051267	1.051271	1.051271	1.0512711
10%	1.1	1.1025	1.103813	1.104713	1.105065	1.105156	1.10517	1.105171	1.1051709
15%	1.15	1.155625	1.15865	1.160755	1.161583	1.161798	1.161833	1.161834	1.1618342
20%	1.2	1.21	1.215506	1.219391	1.220934	1.221336	1.2214	1.221403	1.2214028
25%	1.25	1.265625	1.274429	1.280732	1.283256	1.283916	1.284021	1.284025	1.2840254
30%	1.3	1.3225	1.335469	1.344889	1.348696	1.349692	1.349852	1.349859	1.3498588
40%	1.4	1.44	1.4641	1.482126	1.489543	1.491498	1.491811	1.491824	1.4918247
50%	1.5	1.5625	1.601807	1.632094	1.644788	1.648157	1.648698	1.648721	1.6487213
60%	1.6	1.69	1.749006	1.795856	1.81587	1.821221	1.822081	1.822118	1.8221188
70%	1.7	1.8225	1.906125	1.974557	2.004371	2.012403	2.013696	2.013752	2.0137527
80%	1.8	1.96	2.0736	2.169425	2.212025	2.223593	2.22546	2.22554	2.2255409

If you deposit \$1000 in your bank for one year with the annual interest rate 15% and continuous compounding, the money you will have in the bank by the end of one year is the product of 1000 and the continuous compounding factor 1.1618, or \$1161.80. Calculations show that daily compounding is rather close to continuous compounding and hourly compounding is very close to continuous compounding. For instance, the compounding factor for $r = 15\%$ and $t = 1$ is 1.161833 with hourly compounding, as can be seen in Table 2.1, and it is 1.161798 with daily compounding, the corresponding continuous compounding factor being 1.161834.

If the compounding is continuous, the number of compounding n will approach infinity. Therefore, the following limit result represents the compounding factor for continuous compounding:

$$CPDF(\infty, t) = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{tn} = \left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = e^{rt}. \qquad (2.31)$$

Table 2.1 provides the continuous compounding factors for the chosen interest rates. We can readily observe that compounding factors with daily compounding frequency are almost the same as those of continuous compounding frequency.

Discounting is the opposite process of compounding. As compounding gives the future value of some current investment, discounting gives the present value of some future value. If we deposit A dollars in the bank today, there will be Ae^{rt} dollars in the account t years in the future if interest is compounded continuously and the interest rate r is constant. Discounting solves the opposite problem: how much money should we deposit today at the constant annual interest rate r compounding continuously in order to have B dollars in the account t years from today? The problem can be readily solved by assuming that x dollars need to be deposited today, then xe^{rt} will be the amount of money available in the account t years from today using (2.31), thus $xe^{rt} = B$, therefore $x = Be^{-rt}$, which implies that in order to have one dollar t years from today, we need to deposit e^{-rt} in the account today. The amount of money x is called the present value of the future value B, and B is called the future value of the present value x. The factor e^{-rt} is called the discounting factor, which is the reciprocal of the compounding factor e^{rt}.

Therefore, we have the continuous discounting factor (CDCF) at a constant rate r in t years:

$$CDCF(r,t) = 1/CPDF(\infty, t) = e^{-rt}, \tag{2.32}$$

which will be used repeatedly throughout this book.

Whenever we need to discount some expected payoffs at a constant interest rate r continuously, we simply multiply the expected payoffs by the continuous discounting factor given in (2.32).

Similarly, the discrete discounting factor (DDFT) corresponding to (2.31) is given

$$DDFT(n,t) = 1/CPDF(n,t) = \left(1 + \frac{r}{n}\right)^{-tn}, \tag{2.33}$$

which is often used in discrete models such as the tree-related models to be introduced later in Section 2.8.

Table 2.2. lists all the discounting factors corresponding to those compounding factors in Table 2.1. Because compounding factors are always greater than one with positive interest rates, discounting factors are always between zero and one.

2.6.3. Black-Scholes Formula Using RNVR

In the remaining of this section, we will show how the Black-Scholes formula can be derived using the RNVR discussed in 2.6.1 by discounting the expected payoff of the European call option at the risk-free rate of return rate r.

Table 2.2. Discounting factors for various interest rates and discounting frequencies.

n	1 annual	2 semi-an	4 quarterly	12 monthly	52 weekly	365 daily	8760 hourly	525600 minute	Infinity continuous
interest									
5%	0.9252381	0.951814	0.951524	0.951328	0.951252	0.951233	0.95123	0.951229	0.9512294
10%	0.909091	0.907029	0.905951	0.905212	0.904924	0.90485	0.904838	0.904837	0.9048374
15%	0.869565	0.865333	0.863073	0.861509	0.860894	0.860734	0.860709	0.860708	0.860708
20%	0.833333	0.826446	0.822702	0.820081	0.819045	0.818776	0.818733	0.818731	0.8187308
25%	0.800000	0.790123	0.784665	0.780804	0.779267	0.778867	0.778804	0.778801	0.7788008
30%	0.769231	0.756144	0.748801	0.743556	0.741457	0.74091	0.740822	0.740818	0.7408182
40%	0.714286	0.694444	0.683013	0.674706	0.671347	0.670467	0.670326	0.67032	0.670320
50%	0.666667	0.640000	0.624295	0.61271	0.607981	0.606738	0.606539	0.606531	0.6065307
60%	0.625000	0.591716	0.571753	0.556837	0.5507	0.549082	0.548823	0.548812	0.5488116
70%	0.588235	0.548697	0.524624	0.506443	0.49891	0.496918	0.496599	0.496586	0.4965853
80%	0.555556	0.510204	0.482253	0.460952	0.452074	0.449722	0.449345	0.449329	0.449329

Taking natural logrithm to both sides of (2.4) yields

$$\ln\left[\frac{S(\tau)}{S}\right] = \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma z(\tau)\,. \tag{2.34}$$

Equation (2.34) shows that $\ln[S(\tau)/S]$ is normally distributed with mean $(\mu - \sigma^2/2)\tau$ and variance $\sigma^2\tau$. Denoting $x = \ln[S(\tau)/S]$, then x is a normal distribution with mean $\mu_x = (\mu - \sigma^2/2)\tau$ and variance $\sigma_x^2 = \sigma^2\tau$ because $z(\tau)$ is a standard Gauss-Wiener process and $z(\tau)$ is normally distributed with zero mean and variance τ.

It is straightforward to obtain the expected payoff of the call option as follows

$$E(PFC) = \int_0^\infty \max[S(\tau) - K, 0]dG[S(\tau)] = \int_{\ln(K/S)}^\infty (Se^x - K)f(x)dx\,, \tag{2.35}$$

where $G[S(\tau)]$ is the cumulative function of the underlying asset price at maturity, and $f(x)$ is the standard normal density function as discussed above.

Making the standard substitution $u = (x - \mu_x)/\sigma_x$, the lower bound of the integration on the right-hand side of (2.35) becomes

$$\left[\ln\left(\frac{K}{S}\right) - \mu_x\right]/\sigma_x = -\left[\ln\left(\frac{S}{K}\right) + \mu_x\right]/\sigma_x\,.$$

Letting $d_2 = [\ln(\frac{S}{K}) + \mu_x]/\sigma_x$ and carrying out the integration in (2.35) yields

$$\begin{aligned} E(PFC) &= S\int_{-d_2}^{\infty} e^{\mu_x + u\sigma_x} f(u)du - K\int_{-d_2}^{\infty} f(u)du \\ &= Se^{\mu_x}\int_{-d_2}^{\infty} e^{u\sigma_x} f(u)du - K[1 - N(-d_2)] \\ &= Se^{\mu_x}\int_{-d_2}^{\infty} e^{u\sigma_x}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du - K[1 - N(-d_2)] \\ &= Se^{(\mu_x + \sigma_x^2/2)}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{(\sigma_x u - \frac{1}{2}\sigma_x^2 - \frac{u^2}{2})}du - K[1 - N(-d_2)] \\ &= Se^{\mu\tau}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(u-\sigma_x)^2/2}du - K[1 - N(-d_2)]\,. \end{aligned} \tag{2.36}$$

Making the substitution $v = u - \sigma_x$ to the last step of (2.32) yields

$$\begin{aligned} E(PFC) &= Se^{\mu\tau}\int_{-d_2-\sigma_x}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}dv - K[1 - N(-d_2)] \\ &= Se^{\mu\tau}[1 - N(-d_2 - \sigma_x)] - K[1 - N(-d_2)] \\ &= Se^{\mu\tau}N(d_1) - KN(d_2)\,, \end{aligned} \tag{2.37}$$

where $d_2 = [\ln(\frac{S}{K}) + \mu_x]/\sigma_x = [\ln(\frac{S}{K}) + (\mu - \sigma^2/2)\tau]/(\sigma\sqrt{\tau})$, $d_1 = d_2 + \sigma_x = d_2 + \sigma\sqrt{\tau}$.

The last step in (2.33) was carried out using the identity $N(z) + N(-z) = 1$ or

$$N(z) = 1 - N(-z)\,,$$

for any real number z.

Using the RNVR, the expected return of the underlying asset μ must be the same as the risk-free rate of return r. Substituting $\mu = r$ into (2.33) and discounting the expected payoff of the European call option given in (2.33) at the continuously compounding risk-free rate of return r [using the discounting factor given in (2.28)] yields the following

$$C = SN(d_1) - Ke^{-r\tau}N(d_2)\,, \tag{2.38}$$

where $d_2 = [\ln(\frac{S}{K}) + (r - \sigma^2/2)\tau]/(\sigma\sqrt{\tau})$, $d_1 = d_2 + \sigma\sqrt{\tau}$, which is exactly the same as the Black-Scholes formula given in (2.5).

2.7. MONTE CARLO SIMULATIONS

Boyle (1977) first introduced the Monte Carlo simulation method into finance. Essentially, Monte Carlo methods involve generating large numbers of numerically simulated realizations of some random walks followed by the underlying asset prices, and these simulated realizations are used to price derivative products. Monte Carlo simulation is simple and flexible in the sense that it can be easily modified to accommodate different processes governing the underlying instrument movement. The use of Monte Carlo simulations to price path-dependent derivatives has increased because products have become more complex in nature and it is difficult to obtain closed-form solutions for many of these complicated products, or closed-form solutions simply do not exist. Another obvious advantage of the Monte Carlo simulation method over other procedures is that it can value derivative products with several underlying assets more efficiently.

However, the potential drawback of the Monte Carlo method is that the standard deviation error of estimate is inversely proportional to the square root of the number of simulation trials. Although any desired level of accuracy can be obtained by increasing the number of simulated trials, there are more efficient ways to reduce the standard deviation error. There are two techniques often used in simulations which can reduce variances quickly, namely the control-variate method and the antithetical variate method. These techniques are normally called variance-reduction techniques. The former is often used when a pair of similar problems which possess similar characteristics can be easily found, and the solution of the easier one, usually in closed-form, is used to solve the other relatively more difficult one. The solution to the easier problem is often called a control variate. The efficiency in reducing standard deviation error depends on the degree to which the control variate mimics the behavior of the other problem. Thus, the efficiency of the control-variable method depends on how well the control variate mimics the target problem. For example, the price of any geometric Asian option can be expressed in closed-form and the solution is usually used as a control variate for its corresponding arithmetic Asian option for which the closed-form solution has yet to be found. The difficulties in selecting a control variate are that it must mimic the target function well and must give rise to an integral that is easy to evaluate.

The antithetical variate method always calculates two values of a derivative security, one being calculated in the normal way and the other calculated through changing the sign of all the samples from standard normal distributions. That is, if the first is calculated using the normal sample y, then the other value is calculated by using the sample $-y$. The average of the two values is considered as the value of the derivative security from the sample y. The final estimation of the value of the derivative security is the average of all the averages of the pairs of values. The total standard error is significantly reduced using the antithetical variate method.

The antithetical variate method is more general than the control-variate method for it can be used to solve many problems without additional conditions. This is because the control-variate method is efficient only when we know a similar problem which mimics the target problem well. Therefore, the antithetical variate method is more reliable for a very new product which we know little about, whereas the control variate method can be more efficient if we can find a good control variate. There are many other procedures which can either reduce standard errors significantly or lead to rapid convergence. These topics are beyond the scope of this section.

2.8. LATTICE- AND TREE-BASED METHOD

The binomial model was originally developed to price standard options. It was then extended to trinomial tree model in which the underlying asset price is assumed to follow three different paths in each following period. Since the late 1970s, the lattice- and tree-based method has been widely used in pricing essentially all kinds of derivative products, especially path-dependent and other complicated products such as interest-rate derivatives involving the term structure of interest rate. It has become a powerful and efficient method because of its intuitive nature.

There are essentially two types of models relating to the lattice- and tree-based method: the recombining tree model and the bushy tree model. In models of the first type, any upward move followed by a downward move is indifferent to the downward move followed by the corresponding upward move. In other words, the total number of upward moves and that of the downward moves determine a path completely. The recombining tree is often used to price many derivatives. The popular term structure of interest rate model of Black, Derman, and Toy (1991) utilizes the recombining tree model. The recombining tree model is convenient, yet it does not accurately represent reality as an upward move followed by a downward move is normally different from a downward move followed by the corresponding upward move.

The second type of lattice- and tree-based models, the bushy tree model, is used to overcome the order-indifference limitation of the recombining model. It obviously possesses advantages over the recombining tree model, yet this advantage is achieved with many more paths and thus in general, involves much more computing time. For a n-period model, there are 2^n paths in a bushy tree model and only $(n + 1)$ paths in a recombining tree model, the former increasing exponentially and the latter only linearly with the number of periods. Because our purpose in this chapter is to overview various kinds of methods used to price derivatives but not to analyze these methods in depth, we simply close this section and return to other binomial tree models in Chapter 3 and the binomial tree model in Chapter 4 to price American options.

2.9. METHOD USED IN THIS BOOK

As the universal principle in pricing all kinds of derivative products is the so-called arbitrage-free principle or "no-free lunch" argument, it should also be the principle to price all kinds of exotic options. Of all the pricing methods described above, the risk-neutral valuation method is the most intuitive and convenient one. This is because the likelihood of the underlying instrument being within a certain range can be calculated conveniently, and the characteristics of the underlying instrument distribution can be seen more easily than with other methods such as finite-difference or partial-differential equations. As the focus of this book is to introduce and price various forms of exotic options but not to illustrate various methods in pricing options, we mainly use the risk-neutral valuation relationship to find option prices by discounting their expected payoffs at the risk-free rate of return.

Solutions in terms of univariate integrations should be considered as closed-form solutions because of two reasons. One is that univariate integrations can be carried out quickly with very high accuracy with any computer system or personal computer. The other is that the Black-Scholes formula requires univariate integrations to calculate the two cumulative function values of the standard normal distribution, since normal distribution tables do not provide values for arbitrary arguments and are thus inconvenient to use. We provide, in this book, closed-form solutions for almost all European-style exotic options in terms of the cumulative functions of the standard normal distribution as in the Black-Scholes formula. For a few exotic options such as spread options, alternative options, and dual-strike options, we provide closed-form solutions in terms of univariate integrations first and then approximate these univariate integrations in terms of the standard normal cumulative functions. Integration solutions and their corresponding analytical approximations is a major characteristic of this book.

Closed-form solutions in terms of univariate integrations have clear advantages over their corresponding simulation methods because sensitivities can also be expressed in univariate integrations, and thus sensitivities of various degrees can be obtained quicker than in the corresponding simulation methods. We mainly work with European-style options in this book in order to illustrate the basic concepts of all kinds of exotic options simply because European options are easier to work with and closed-form solutions are more likely to be obtained. American-style exotic options can be priced using the binomial method, to be introduced in Chapter 4.

QUESTIONS AND EXERCISES

Questions

2.1. What is an arbitrage opportunity?

2.2. What is an equilibrium? What is a general equilibrium?

2.3. What are the major characteristics of the solutions of arbitrage and equilibrium models?

2.4. What is the relationship between an equilibrium model and its corresponding arbitrage model?

2.5. What are ITM, ATM, and OTM options?

2.6. Is it always true that when calls are ITM, the corresponding puts are OTM? Is it true that whenever calls are ATM, puts are also ATM?

2.7. What is the time value of an option? What is the intrinsic value of an option?

2.8. What is the difference between an European option and its corresponding American option? Why are American options generally more expensive than their corresponding European options?

2.9. Is the time value of an option larger or smaller with longer time to maturity? Why?

2.10. Why is it more difficult to find the values of American options?

2.11. What is a compounding factor? What is a discounting factor?

2.12. What is the relationship between a discounting factor and its corresponding compounding factor?

2.13. What are Monte Carlo simulations?

2.14. What is the problem often associated with Monte Carlo simulations?

2.15. What are the two popular methods to reduce variance when using Monte Carlo simulations?

2.16. What is finite-difference method? What is finite-element method?

2.17. What is the common disadvantage of finite-difference and finite-element methods?
2.18. What is the most important advantage of the binomial tree method?
2.19. What is a risk-neutral valuation relationship (RNVR)?
2.20. Why are RNVRs useful in pricing options?

Exercises

2.1. Find the call option price using the Black-Scholes formula, given the annual interest rate 10%, time to maturity one year, strike price \$110, current stock price \$105, and volatility of the return of the underlying asset $\sigma = 20\%$.
2.2. Find the call option price using the Black-Scholes formula if the spot price is \$100 and other parameters are the same as in Exercise 2.1.
2.3. Find the call option price using the Black-Scholes formula if the volatility is 10% and other parameters are the same as in Exercise 2.1.
2.4. Find the call option price using the Black-Scholes formula if the interest rate is 6% and other parameters are the same as in Exercise 2.1.
2.5. Find the prices of the call options to expire in one and three months if other parameters are the same as in Exercise 2.1.
2.6. Find the prices of the ATM call options in Exercise 2.5.
2.7. Show that the scaling property in (2.6) always holds for the European call option pricing formula in the Black-Scholes model.
2.8. Find the call option price if the strike price is \$121, current stock price \$115.5, and other parameters are the same as in Exercise 2.1 (Hint: use the scaling property of the Black-Scholes formula given in (2.6)).
2.9. Find the call and put option prices if the strike price is \$99, current stock price \$94.5, and other parameters are the same as in Exercise 2.1.
2.10. What is the compounding factor in 8 months if annual interest rate is 10% and compounding is monthly?
2.11. What is the compounding factor in 8 months if annual interest rate is 10% and compounding is weekly?
2.12. What is the annual compounding factor when annual interest rate is 8% and compounding is monthly?
2.13. What are the annual compounding factors when annual interest rate is 8% and compounding is weekly and daily?
2.14. Find the discounting factors for the two cases in Exercises 2.10 and 2.11.
2.15. Find the corresponding discounting factors in Exercises 2.12 and 2.13.

2.16. Show (2.8) is true when $\tau \to +\infty$.
2.17. Show (2.9) is true when $r \to +\infty$.
2.18. Show (2.10) is true when $\sigma \to +\infty$.

APPENDIX

Solving Equation (2.3)

Let $y(t) = \ln[S(t)]$. Equation (2.10) indicates that $S(t)$ depends on the Wiener process $z(t)$, thus $y(t)$ also depends on the Wiener process $z(t)$. Using Itõ's lemma, we can obtain the change in $y(t)$ as follows:

$$dy(t) = y_s dS(t) + \frac{1}{2} y_{ss}[dS(t)]^2 = \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{[dS(t)]^2}{S^2(t)}. \tag{A2.1}$$

Substituting (2.10) into (A2.1) yields the following

$$dy(t) = \mu dt + \sigma dz(t) - \frac{1}{2}[\mu dt + \sigma dz(t)]^2, \text{ or}$$

$$dy(t) = \mu dt + \sigma dz(t) - \frac{1}{2}\{\mu^2(dt)^2 + 2\mu dt\sigma dz(t) + \sigma^2[dz(t)]^2\}. \tag{A2.2}$$

In standard stochastic calculus, $[dz(t)]^2$ is treated as dt and terms higher than dt is assumed to be zero. Substituting $[dz(t)]^2 = dt$, $(dt)^2 = 0$, and $dtdz(t) = 0$ into (A2.2) yields [see Protter (1992) for further information on stochastic calculus]

$$dy(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz(t). \tag{A2.3}$$

Equation (A2.3) is a standard stochastic equation which can be solved by stochastic integration as follows:

$$\begin{aligned} y(\tau) - y(t) &= \int_t^{t^*} \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \int_t^{t^*} dz(t) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma dz(\tau), \text{ or} \\ \ln S(\tau) - \ln S(t) &= \left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma dz(\tau). \end{aligned} \tag{A2.4}$$

Rearranging (A2.4) yields the solution given in (2.4).

Approximating $N(x)$

Abramowitz and Stegun (1972) provided a polynomial approximation for the cumulative function of a standard normal distribution $N(x)$. With this approximation, we can find the values of the cumulative function for the standard normal distribution much more conveniently and thus increase the calculating efficiency in using the Black-Scholes formula. This approximation is given as follows:

$$N(x) = 1 - (a_1 y + a_2 y^2 + a_3 y^3) f(x) \quad \text{for} \quad x \geq 0, \quad \text{and}$$
$$N(x) = 1 - (N - x) \quad \text{for} \quad x < 0\,, \tag{A2.5}$$

where

$$y = \frac{1}{1 + a_0 x}\,,$$
$$a_0 = 0.33267\,,$$
$$a_1 = 0.4361836\,,$$
$$a_2 = -0.1201676\,,$$
$$a_3 = 0.9372980, \quad \text{and}$$

$f(x) = e^{-\frac{x}{2}}/\sqrt{2\pi}$ is the density function of a standard normal distribution.

The approximation given in (A.25) are normally accurate to four decimal places and are always accurate to 0.0002. For a more accurate approximation, we can use the following

$$N(x) = 1 - (a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5) f(x) \quad \text{for} \quad x \geq 0, \quad \text{and}$$
$$N(x) = 1 - N(-x) \quad \text{for} \quad x < 0\,, \tag{A2.6}$$

where

$$y = \frac{1}{1 + a_0 x}\,,$$
$$a_0 = 0.2316419\,,$$
$$a_1 = 0.319381530\,,$$
$$a_2 = -0.356563782\,,$$
$$a_3 = 1.781477937\,,$$
$$a_4 = -1.821255978\,,$$
$$a_5 = 1.330274429\,,$$

The approximation given in (A2.6) can be accurate to six decimal places.

PART II: STANDARD OPTIONS

Because each kind of exotic options differs in one or two aspects from vanilla options, it is very efficient to learn exotic options by comparing them to their corresponding vanilla options. Thus, a systematic review of vanilla options is highly necessary as a reference. Chapter 3 first extends the Black-Scholes model to incorporate the payout rate of the underlying asset, then extends the Black-Scholes model to price futures options, or options written on futures. We will review other popular extensions of the Black-Scholes model in Chapter 3. We will review standard Greeks representing sensitivities of option values to various parameters and also higher sensitivities such as speed, charm, and color. The brief description of the term structure of volatility and volatility smile is also helpful in pricing exotic options.

Chapter 3 is almost exclusively on European options. We will concentrate on American options in Chapter 4. We will show that American options can be priced using the well-known binomial model. As a by-product, options on underlying assets with less liquidity can be obtained using the binomial method. As an example to show how American option prices can be approximated analytically, we will describe a popular quadratic method or quasi-quadratic method in Chapter 4.

Chapter 3

VANILLA OPTIONS

In describing option pricing methodology using the arbitrage-free or "no-free-lunch" argument in the previous chapter, we described the European options and how to price them in the Black-Scholes model. There are many other kinds of standard options and many extensions of the Black-Scholes model in various directions. These standard options and extensions of the Black-Scholes model are very useful for our description and analysis of exotic options in later parts of this book. The purpose of this chapter is to introduce other kinds of standard options and to price them in the Black-Scholes environment, to introduce the major extensions of the Black-Scholes model, and to review other aspects of vanilla options.

3.1. EQUITY OPTIONS WITH DIVIDEND AND FOREIGN CURRENCY OPTIONS

Obviously, the yield of the underlying asset during the life of an option is not considered in the Black-Scholes model. Yet most assets have significant yields: stocks have dividend yields, foreign currencies have yields equal to foreign interest rates, and so on. In this section, we try to show how the Black-Scholes model can be immediately extended to incorporate the payout of the underlying asset.

3.1.1. Equity Options with Dividend

Let g stand for the annual continuous dividend yield on the underlying asset. The stochastic process which governs the underlying asset price movement given in (2.3) becomes

$$dS = (\mu - g)Sdt + \sigma Sdz(t)\,, \tag{3.1}$$

where all other parameters are the same as in (2.3).

With the same method as that used to derive the standard Black-Scholes model in the previous chapter, the Black-Scholes formula given in (2.5) can be modified as follows:

$$C = Se^{-g\tau}N(d_1) - Ke^{-r\tau}N(d_2), \tag{3.2}$$

where

$$d_1 = \frac{\ln(S/K) + (r - g + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_2 + \sigma\sqrt{\tau},$$

$$d_2 = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

and other parameters are the same as in (2.5).

Comparing (3.2) with (2.5), we can easily find that there are two differences between them. Firstly, the constant interest r is replaced by the difference $r - g$ in the expressions of d_1 and d_2, and secondly, there is a discounting factor in the first term at the payout rate g. Obviously, (3.2) degenerates to (2.5) when $g = 0$.

Example 3.1. What is the European call option price in Example 2.1 if the underlying stock has a continuous dividend of 5%?

Substituting interest rate $r = 20\%$, $g = 5\%$, time to maturity $\tau = 0.50$, strike price $K = \$105$, current stock price $S = \$100$, and volatility of the return of the underlying asset $\sigma = 30\%$ into (3.2) yields:

$$\begin{aligned}
d_2 &= \frac{\ln(S/K) + (r - g - \sigma^2)\tau}{\sigma\sqrt{\tau}} \\
&= \frac{\ln(100/105) + (0.20 - 0.05 - 0.30^2/2) \times 0.50}{0.30\sqrt{0.50}} = 0.0175, \\
d_1 &= d_2 + \sigma\sqrt{\tau} = 0.0175 + 0.30 \times \sqrt{0.50} = 0.2296, \\
C &= Se^{-g\tau}N(d_1) - Ke^{-r\tau}N(d_2) \\
&= 100e^{-0.05\times 0.5} \times 0.5908 - 105e^{0.2\times 0.5} \times 0.2296 = \$9.455.
\end{aligned}$$

3.1.2. Foreign Currency Options

The Black-Scholes formula for pricing foreign currency options is precisely the same as the one given in (3.2) if we substitute the annual continuous payout rate on the underlying asset g with the foreign interest rate r_f. Because the foreign interest rate is exactly the yield on the underlying asset — foreign currency, the pricing formula for options written on an underlying asset with continuous yield (3.2) can be used directly without any modification for currency options.

Example 3.2. The US dollar/Japanese yen exchange rate is ¥85 per dollar. The Japanese interest rate is 3%, the US interest rate $r = 8\%$, the volatility of the dollar/yen exchange rate is 15%. What is the European call option on the yen/dollar exchange rate with strike price ¥90 per dollar to expire in half a year?

Substituting interest rate $r = 0.08$, $r_f = 0.03$, spot price $S = 1/85 =$ \$0.0175, strike price $K = 1/90 =$ \$0.01111, time to maturity $\tau = 1/2 = 0.50$, and volatility $\sigma = 15\%$ into (3.2) yields

$$
\begin{aligned}
d_2 &= \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
&= \frac{\ln(0.0175/0.01111) + (0.08 - 0.03 - 0.15^2/2) \times 0.50}{0.15\sqrt{0.50}} = 4.466\,, \\
d_1 &= d_2 + \sigma\sqrt{\tau} = 4.466 + 0.15 \times \sqrt{0.50} = 4.572\,, \\
C &= Se^{-r_f\tau}N(d_1) - Ke^{-r\tau}N(d_2) \\
&= 0.0175 \times e^{-0.03\times 0.5} \times N(4.572) - 0.0111 \times e^{-0.08\times 0.5} \times N(4.466) \\
&= \$0.00657\,.
\end{aligned}
$$

3.2. FUTURES AND FUTURES OPTIONS

3.2.1. Futures

Before we introduce futures options, it is necessary for us to review the concepts of forwards and futures. A forward contract is a financial contract which involves two parties, one agreeing to deliver a certain amount of the underlying commodity of a certain quality at a prespecified price to a prespecified place at some specific time in future, and the other agreeing to buy the same amount of the commodity as specified. This specified price is called the forward price, and the specified time is called the expiration time, or maturity time of the contract. The forward contract carries obligations for both parties involved. The underlying commodity does not have to be a physical commodity. It can also be currency, bonds, or indexes. The settlement does not have to be actual delivery of the underlying commodity. The contract can be settled by cash according to the then market price of the underlying commodity and the prespecified forward price. The largest forward market in the world is the currency forward market.

Futures contracts are standardized forward contracts in which the quality of commodity, the amount of commodity, the place, and the time to deliver the commodity are all standardized by an exchange. Or simply, futures contracts are exchange-traded forward contracts. Buyers and sellers of the futures contracts do not meet to make their transactions. They buy or sell through exchanges. Through a clearing house, the exchange guarantees the delivery of the underlying products. According to a very recent report by the Group of Thirty, the notional amount of futures traded annually is now estimated to be several times larger than the total Gross National Product in the world.

Futures contracts trade in exchanges with daily market-to-market settlement. Like all other assets trading in exchanges with prices quoting on real time, futures contracts also have on-line prices reflecting markets' assimilation of expected information and/or heterogeneous expectation of perspective market movement. Prices of most active futures trading in all the major exchanges around the world are available on most on-line financial service systems such as Telerate, Bloomberg, Reuters, Knight Ridder, and so on. All these systems charge certain service fees, therefore they are not available for the general public. Daily close prices of active futures can be obtained in financial newspapers such as the *Wall Street Journal*.

We have so far described forward and futures contracts and their basic properties. Many may wonder how futures prices are determined in general. In the remaining of this section, we will discuss how to price futures using the arbitrage-free or "no-free-lunch" argument. Futures prices are normally different from forward prices on the same underlying assets even with the same time to expiration because of taxes, transaction cost, and other factors. Fortunately, prices of forward and futures contracts with the same time to expiration are generally very close to each other. Thus we can simply regard them as the same for convenient understanding and analysis. Forwards are easier to analyze than futures because no daily settlement is involved. We will treat the two concepts as the same in the rest of this book.

To show how a forward price is determined in general, we take an example of a foreign-currency forward, or more specifically, a US dollar–Japanese yen forward contract which governs ¥1 million. Suppose the US interest rate r is 10% constant, the Japanese interest rate r_f 5% constant, and the current US dollar–Japanese yen exchange rate $S = \$10$ per ¥1000 or $S = \$0.01$ per ¥. Let the forward dollar-yen exchange rate in one year be represented by F dollar per yen. Let's consider the following steps:

(1) Short a forward contract which controls ¥1 million to expire in one year. We can obtain US\$$F$ million in one year by selling ¥1 million at the prespecified exchange rate F;
(2) Converting the US\$$F$ million into the present value. US\$$F$ million in one year is actually US\$$Fe^{-r}$ million today because we need to discount the future value US\$$F$ million into the present value using the discounting factor given in (2.28);
(3) Buy Japanese yen with US\$$Fe^{-r}$ at the current dollar-yen exchange rate $S = \$0.01$. We can buy ¥$Fe^{-r}/S$ because S is the current price of each yen in US dollars;
(4) Deposit the ¥Fe^{-r}/S million into a bank. We can make interest on the Japanese yen bought in step (3) and the value of the yen will be $(Fe^{-r}/S)e^{r_f}$ million using the continuous compounding factor in (2.27).

If the forward price F is greater than $Se^{r-r_f} = 0.01 \times e^{0.10-0.05} = 0.010512$, say $F = 0.0106$, the value of the yen obtained in step (4) will be

$$(0.0106 \times e^{-0.10}/0.01)e^{0.05} = ¥1.0083 \text{ million},$$

which is ¥8300 more than the initial ¥1 million to be sold in one year. This ¥8300 is arbitrage or "free lunch" because we would be able to make ¥8.3 billion in one year if we follow the above four steps selling one million such contracts with no initial cost. Certainly this is too good to be true. To eliminate such free lunches, the forward price F cannot be greater than Se^{r-r_f}. The arbitrage comes from the fact that the forward price is too high compared to the current exchange with given home and foreign interest rates. That is why we chose to shorten the forward contract.

On the other hand, if the forward price F is less than $Se^{r-r_f} = 1.0512$, say $F = 1.05$, we find that we could make ¥1209 by buying one forward contract with notional value of one million. Again, this is "free lunch". To eliminate free lunches, the forward price cannot be smaller than Se^{r-r_f}. As the forward price can be neither greater nor smaller than Se^{r-r_f} in order to be arbitrage free, it has to be equal to Se^{r-r_f}.

Generalizing the above example with arbitrary time to expiration t in number or fraction of year(s) and interest rates r and r_f using the discounting and compounding factors developed in Appendix I, we can obtain the following expression:

$$F = Se^{(r-r_f)\tau}, \tag{3.3}$$

where F and S stand for the current futures and spot prices of the exchange rate per unit of foreign currency, respectively; r and r_f are domestic and foreign interest rates, respectively, and τ is the time to expiration of the forward contract in number or fraction of year(s).

Equation (3.3) connects the forward price and the spot price given the time to expiration of the contract and both the domestic and foreign interest rates. It is called the forward or futures pricing formula. It is often called the interest-rate parity condition in economics. It actually indicates that the future price of the commodity should be the same as the present value compounding continuously at the interest rate spread $r - r_f$.

The pricing formula in (3.3) for currency futures can be readily generalized for futures on underlying assets with continuous dividend yield g

$$F = Se^{(r-g)\tau}, \tag{3.4}$$

where all parameters are the same as in (3.3).

3.2.2. Futures Options

Options written on forwards are called frations and options written on futures are called futures contracts. Because forwards are not traded in exchanges, frations are thus not as popular as futures options. For many underlying markets, both futures and futures options exist, and futures options are very often more popular than options written on the underlying assets directly. This is especially true for foreign-currency options as volumes of options on foreign currencies futures trading at the International Monetary Market (IMM) in Chicago Mercantile Exchange far out pace those of options on foreign currencies trading at Philadelphia Stock Exchange. The most important reason is that futures markets often exhibit more volatility than their corresponding underlying markets and thus creating more room for option trading activities.

Using the futures pricing formula in (3.4), we can readily express the spot price S in terms of the futures price F, $S = Fe^{(g-r)\tau}$. Substituting $S = Fe^{(g-r)\tau}$ into (3.2), we can easily obtain the pricing formula for a call futures option

$$C = e^{-r\tau}[FN(d_1) - KN(d_2)], \tag{3.5}$$

where

$$d_1 = \frac{\ln(F/K) + \tau\sigma^2/2}{\sigma\sqrt{\tau}} = d_2 + \sigma\sqrt{\tau},$$

$$d_2 = \frac{\ln(F/K) - \tau\sigma^2/2}{\sigma\sqrt{\tau}},$$

and other parameters are the same as in (3.2).

Example 3.3. The September 1995 S&P-500 futures price is \$510, the volatility 18%, the US interest rate 7%. What is the S&P-500 futures call option price with strike price \$515 to expire in five months?

Substituting $F = \$510$, $\sigma = 0.18$, $r = 0.07$, $K = \$515$, and $\tau = 5/12$ year into (3.5) yields

$$d_2 = \frac{\ln(F/K) - \tau\sigma^2/2}{\sigma\sqrt{\tau}} = \frac{\ln(510/515) - 5 \times 0.18^2/(2 \times 12)}{0.18\sqrt{5/12}} = -0.14\,.$$

$$d_1 = d_2 + 0.18 \times \sqrt{5/12} = -0.03\,,$$

Using the table of the cumulative function for the standard normal distribution given in Appendix I at the end of the book, we get $N(d_2) = 1 - N(0.03) = 1 - 0.512 = 0.488$ and $N(d_1) = 1 - N(0.14) = 1 - 0.5557 = 0.4443$. Substituting these values into (3.6), we obtain the futures call option price:

$$\begin{aligned} C &= e^{-r\tau}[FN(d_1) - KN(d_2)] \\ &= e^{-0.07\times 5/12}[510 \times 0.488 - 515 \times 0.4443] = \$11.27\,. \end{aligned}$$

3.3. OTHER POPULAR MODELS

Following the celebrated work of Black and Scholes (1973), many researchers have extended it in several directions. We briefly introduce these extensions in this section so that it will help us understand exotic options and extend them along these directions.[1]

3.3.1. Uncertain Strike Prices

The strike price is assumed to be constant in the Black-Scholes model. Fisher (1978) extended the model to include uncertainties in strike prices. Fisher assumed that the strike price follows a geometric Brownian process similar to the underlying asset price given in (2.10) with the current strike price K:

$$dK = \alpha_x K dt + \sigma_x K dz_x(\tau)\,, \tag{3.6}$$

where $z_x(\tau)$ is a standard Gauss-Wiener process, and α_x and σ_x are the instantaneous mean and standard deviation of the stated strike price, respectively.

[1]There is another direction relating to the Black-Scholes model which is worth mentioning here. Rubinstein (1976), Brenan (1979), Stapleton and Subrahmanyam (1984), and others showed that with certain restrictions of the preferences of the representative investor, the Black-Scholes pricing formula can still hold even if trading is not continuous.

The standard Gauss-Wiener process $z_x(\tau)$ in (3.6) for the strike process is assumed to be correlated to the standard Gauss-Wiener process $z(\tau)$ in (2.3) for the underlying asset price with an instantaneous constant correlation coefficient ρ_x, and

$$dz_x(\tau)dz(\tau) = \rho_x dt\,. \tag{3.7}$$

With the assumption of the strike price in (3.6), the current strike price K, the correlation coefficient in (3.7), Fisher obtained the following pricing formula for a European call option:

$$C = SN(\hat{d}_1) - Ke^{-\hat{r}\tau}N(\hat{d}_2)\,, \tag{3.8}$$

where

$$\hat{d}_1 = \hat{d}_2 + \hat{\sigma}\sqrt{\tau}\,, \quad \hat{d}_2 = \frac{\ln(S/K) + (r - \hat{\sigma}^2/2)\tau}{\sigma\sqrt{\tau}}\,,$$
$$\hat{\sigma} = \sqrt{\sigma^2 - 2\rho_x\sigma\sigma_x + \sigma_x^2}\,,$$
$$\hat{r} = \rho_{mx}\sigma_x(r_m - r)/\sigma_m\,,$$

r_m and σ_m stand for the expected return of the market and the standard deviation of the market return, respectively and ρ_{mx} is the correlation coefficient between the market return and the stated strike price.

The pricing formula in (3.8) is of Black-Scholes type with modified volatility parameter. When $\sigma_x = \alpha_x = 0$, $r = r$, the formula (3.8) degenerates to the Black-Scholes formula. The only drawback of this model is that it depends on the expected market return, market volatility, and the correlation between the market return and the strike price.

3.3.2. Constant Elasticity Model

In the general constant elastic volatility (CEV) model, the underlying asset price is assumed to be governed by the diffusion process

$$dS = \mu S dt + \delta S^{\beta/2} dz\,, \tag{3.9}$$

where β is the elasticity parameter which reflects the sensitivity of the underlying asset price with respect to the spot price S.

Obviously, the process governed by (3.9) becomes identical to the lognormal process given in (2.3) in the well-known Black-Scholes model when $\beta = 2$. We need only to consider the two cases of $\beta < 2$ and $\beta > 2$.

The CEV Model for $\beta < 2$

Cox and Ross (1976) obtained an expression for the value of a European call option, and Schroder (1989) simplified the formula in terms of the non-central χ^2 cumulative functions

$$C = SQ[2y; 2 + 2/(2-\beta), 2\xi] - Ke^{-r\tau}\{1 - Q[2\xi; 2/(2-\beta), 2y]\}, \quad (3.10)$$

where

$$y = \kappa K^{2-\beta};$$
$$\kappa = \frac{2\mu}{\delta^2(2-\beta)(e^{(2-\beta)\mu\tau} - 1)},$$
$$\xi = \kappa S_t^{2-\beta} e^{(2-\beta)\mu\tau},$$

$Q[s; \nu, \lambda]$ is the complementary distribution function of the non-central χ^2 distribution $F(s; \nu, \lambda)$ with ν degrees of freedom, non-central parameter λ, the lower integrand limit s, and $Q[s; \nu, \lambda] = 1 - F(s; \nu, \lambda)$.

The cumulative function for the non-central χ^2, $F(s; \nu, \lambda)$, can be obtained as follows[2]

$$F(s; \nu, \lambda) = \sum_{j=0}^{\infty} \left[\frac{(\lambda/2)^j}{j!}\right] \cdot P_r[\chi^2_{\nu+2j} \leq s], \quad (3.11)$$

where $P_r[\chi^2_{\nu+2j} \leq s]$ is the cumulative distribution function of the central χ^2 distribution;[3] $\nu = 2 + 2/(2-\beta)$ and $\lambda = 2\xi$. $F(s; \nu, \lambda)$ can be considered as a mixture of the central χ^2 distribution and the Poisson distribution.

The CEV Model for $\beta > 2$

Emanuel and MacBeth (1982) showed that, for a CEV process with $\beta > 2$, the density function of $S(\tau)$ conditional on the current stock price S in a risk-neutral world is

$$f[S(\tau)] = (\beta - 2)\kappa^{1/(2-\beta)}(\xi w^{1-2\beta})^{1/(4-2\beta)} e^{-\xi - w} I_{1/(\beta-2)}(2\sqrt{\xi w}), \quad (3.12)$$

where κ and ξ are the same in (3.10), $\omega = \kappa S_t^{2-\beta}$, and $I_{1/(\beta-2)}$ is the modified Bessel function of the first kind of order $1/(\beta - 2)$. See Appendix of this chapter for the specific functional form of $I_{1/(\beta-2)}$.

[2]See Johnson and Kotz (1970), p. 132 for the cumulative distribution function of the non-central χ^2 distribution.

[3]See Appendix of this chapter for the formulae to calculate the cumulative functions of the central χ^2 distribution.

It can be shown that using the same parameters κ, ξ, and w as in the case of $\beta < 2$, the value of a European call option can be expressed as follows:

$$C = S\{1 - Q[2\xi; 2/(\beta - 2), 2y] - Ke^{-r\tau}Q[2y; 2 + 2/(\beta - 2), 2\xi], \quad (3.13)$$

where $y = \kappa K^{2-\beta}$ and $Q[s; \nu, \lambda]$ is the same complementary distribution function as given in (3.10).

The expression is the same as (3.10) as it is indicated in Schroder (1989). The reason why we use this formula is to avoid the negativeness of the parameter of degrees of freedom $2 + 2/(2 - \beta)$ for some $\beta > 2$. The formula can be readily obtained by using the identity

$$Q(2y, 2 - 2/(\beta - 2), 2\xi)] + Q(2\xi, 2/(\beta - 2), 2y] = 1 .$$

3.3.3. Brownian Motion with Jumps

Merton (1976) extended the Black-Scholes model to include situations when the underlying asset returns are discontinuous. As in many economic models, the discontinuity is modeled with a Poisson process. Assuming that the underlying asset returns are discontinuous resulting from arrivals of important information. The Poisson-distributed "event" is the arrival of an important piece of information about the underlying instrument. The arrivals of information are assumed to be independently and identically distributed. The probability of an event during a time interval of length h (h can be as small as possible) can be written as

Prob. [the event does not occur in the time interval $(t, t + h)] = 1 - \lambda h + O(h)$,

Prob. [the event occurs in the time interval $(t, t + h)] = \lambda h + O(h)$,

Prob. [the event occurs more than once in the time interval $(t, t + h)] = O(h)$,

where $O(h)$ represents a function of h which goes to zero faster than h.

With the above description of the Poisson distribution, Merton (1976) assumed the following stochastic process for the underlying asset:

$$dS/S = (\mu - \lambda k)dt + \sigma dz + dq , \quad (3.14)$$

where μ and σ are the instantaneous mean and standard deviation of the underlying asset return without jumps; dz is a standard Gauss-Wiener process; q is the independent Poisson process described above; dq and dz are assumed to be independent; λ is the mean number of arrivals per unit of time; $k = E(Y-1)$, where $(Y-1)$ is the random variable percentage change in the underlying asset price if the Poisson event occurs, and E is the expectation operator over the random variable Y; and $Y \geq 0$, $\{Y\}$ from successive jumps are independently and identically distributed.

With the Poisson distribution assumption of information arrivals and the underlying asset return distribution process described in (3.14), Merton obtained a pricing expression for a European call option with strike price K as follows

$$ECOPWJPS = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \left\{E_n[C(SX_n e^{-\lambda k\tau}, \tau, K, r, \sigma]\right\}, \quad (3.15)$$

where X_n has the same distribution as the product of n independently and identically distributed variables $Y, X_0 = 1$; E_n represents the expectation operator over the distribution of X_n; $n!$ is the factorial function, meaning the product of all integers from 1 through n; and $C(W, K, \tau, r, \sigma)$ is the standard Black-Scholes formula for a European call option given in (3.5) with spot price W and strike price K.

The expression given in (3.15) is not in closed-form and it is not easy to use because the distribution of X_n can be rather complicated. It can be readily shown that when there are no jumps, or when $\lambda = 0$, (3.15) degenerates to the standard European call option pricing formula $C(S, K, \tau, r, \sigma)$ as $X_0 = 0! = \lim_{x \to 0} x^x = 1$.

3.3.4. A Pricing Model with Transaction Cost

As the Black-Scholes model is a frictionless model, transaction cost invalidates the Black-Scholes assumption for option pricing (for continuous revision implies infinite trading which in turn implies infinite transaction cost). There have been many studies incorporating transaction cost into the standard option pricing theory. In the first, and very likely, the most popular study, Leland (1985) developed a technique for replicating option returns in the presence of transaction cost. The strategy depends upon the level of transaction cost and the time period between portfolio revisions. The additional parameters enter in a very simple way, through adjustment of the volatility in the Black-Scholes formula.

Within the same framework as in the Black-Scholes model, with the only exception that transaction cost is included, Leland considered that the hedging strategy depends on a percent transaction cost and the revision frequency. The central point in his model is the following modified variance function as a function of the transaction cost:

$$\hat{\sigma}^2(\sigma^2, \kappa, \Delta t) = \sigma^2 \left[1 + \kappa E\left|\frac{\Delta S}{S}\right| / (\sigma^2 \Delta t)\right]$$
$$= \sigma^2 \left[1 + \kappa\sqrt{(2/\pi)}/(\sigma\sqrt{\Delta t})\right], \qquad (3.16)$$

where κ is the round-trip transaction cost measured as a fraction of the volume of transactions, σ^2 is the same variance of the underlying asset, and Δt is the revision interval.

With the above modified variance function and a modified replicating strategy, Leland obtained the following option pricing formula with transaction cost

$$\hat{C}(S; K, \sigma^2, r, \tau, \kappa, \Delta t) = SN(\hat{d}_1) - Ke^{-r\tau}N(\hat{d}_1 - \hat{\sigma}\sqrt{\tau}), \qquad (3.17)$$

where

$$\hat{d}_1 = \left[\ln\left(\frac{S}{K}\right) + r\tau\right] / (\hat{\sigma}\sqrt{\tau}) + \frac{1}{2}\hat{\sigma}\sqrt{\tau},$$

$S, K, r,$ and τ are the same as in the Black-Scholes model described previously.

It is obvious that the pricing formula given in (3.17) becomes exactly the same as the standard Black-Scholes formula if $\kappa = 0$, because the modified variance becomes the same as the standard variance. Another interesting fact about this model is that although transaction cost is path-dependent, a path-independent net result can be achieved with probability one in the limiting case of zero revision time interval.

There are a few limitations with this model. First of all, there is no convenient way to calculate the transaction parameter κ which is the most important parameter in this model. Another limitation of this model is that transaction cost may become arbitrarily large with very short revision periods as $\Delta t \to 0$. In practice, revision is discrete. Discrete revision generates hedging errors which are correlated with the market, and do not approach zero with more frequent revision when transaction cost is included.

3.3.5. Stochastic Volatility Model

The Black-Scholes model and almost all other extended models assume constant volatility. As a matter of fact, volatilities change dramatically from time to time in all markets. Several researchers have attempted to incorporate the fluctuation of volatility into option pricing models. Coincidentally, three well-known papers were published in 1987 on this very topic: Hull and

White (1987), Scott (1987), and Wiggins (1987). Among these studies, Hull and White (1987) may be the most popular and easiest to follow. In this section, we illustrate how option prices are affected by variable volatility based on the Hull and White model.

Hull and White assumed that the underlying asset price follows the same process described in (2.3) with the only exception that the volatility parameter σ is not constant. They assumed that the instantaneous variance, $V = \sigma^2$, follows the stochastic process:

$$dV = \eta V dt + \xi V dw\,, \tag{3.18}$$

where η and ξ stand for the instantaneous drift and standard deviation of the variance V, w is a standard Gauss-Wiener process which is assumed to be correlated with the standard Gauss-Wiener process z given in (2.3) with a correlation coefficient ρ.

The variables η and ξ may depend on σ and t, but it is assumed that they do not depend on S. The actual process that a stochastic variance follows is probably fairly complex. As variance is lognormally distributed, it cannot take on negative values. The average variance can be defined as follows:

$$\bar{V} = \frac{1}{\tau}\int_0^{\tau} \sigma^2(t)dt\,. \tag{3.19}$$

Since $\ln(S_T/S_0)$ conditional on the mean variance $\bar{V}$ is normally distributed with variance $\bar{V}_\tau$ when S and V are instantaneously uncorrected, the option value given the mean volatility can be shown to be the following expression:

$$C(\bar{V}) = SN(\bar{d}_1) - Xe^{-r\tau}N(\bar{d}_2)\,, \tag{3.20}$$

where

$$\bar{d}_1 = \frac{\log(S/K) + (r + \bar{V}/2)\tau}{\sqrt{\bar{V}\tau}}, \text{ and}$$

$$\bar{d}_2 = \bar{d}_1 - \sqrt{\bar{V}\tau}\,.$$

With the conditional option pricing formula given in (3.20), the option value can be given by

$$C(S, \sigma_t^2) = \int C(\bar{V})h(\bar{V}|\sigma_t^2)d\bar{V}\,, \tag{3.21}$$

where $h(\bar{V}|\sigma_t^2)$ is the density function of the mean volatility given the current variance.

Although formula (3.21) is concise, it is not convenient to use because the functional form of $h(\bar{V}|\sigma_t^2)$ is not known. However, the moments of $\bar{V}$ can be obtained and $C(\bar{V})$ can be expanded in a Taylor series about its mean $E(\bar{V})$:

$$\begin{aligned} C(S_t, \sigma_t^2) &= C(\bar{V}) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{V}} \int [\bar{V} - E(\bar{V})]^2 h(\bar{V}) d\bar{V} + \cdots \\ &= C(\bar{V}) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{V}} \mathrm{Var}(\bar{V}) + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}\bigg|_{\bar{V}} \mathrm{Skew}(\bar{V}) + \cdots, \end{aligned}$$

where $\mathrm{Var}(\bar{V})$ and Skew $(\bar{V})$ are the second and third central moments of $\bar{V}$, respectively. For sufficiently small values of $\xi^2\tau$, this series converges very quickly. Using the moments for the distribution of $\bar{V}$, the above series becomes, when $\eta = 0$:

$$\begin{aligned} C(S, \sigma^2) = C(\sigma^2) &+ \frac{S\sqrt{T-t}N'(d_1)(d_1 d_2 - 1)}{4\sigma^3}\left[\frac{2\sigma^4(e^k - k - 1)}{k^2} \sigma^4\right] \\ &+ \frac{S\sqrt{T-t}N'(d_1)[(d_1 d_2 - 3)(d_1 d_2 - 1) - (d_1^2 + d_2^2)]}{8\sigma^5} \\ &\times \sigma^6 \left[\frac{e^{3k} - (9 + 18k)e^k + (8 + 24k + 18k^2 + 6k^3)}{3k^3}\right] + \cdots, \end{aligned}$$

where $k = \xi^2(t^* - t) = \xi^2\tau$.

The above analysis is very good, yet the results are not concise and therefore not convenient to use. It is beyond the scope of this book to discuss this topic in more detail.

3.3.6. The Term Structure of Interest Rate

In the original Black-Scholes model and its extensions described above, interest rate is assumed to be constant. Although it is probably not so problematic to assume constant interest rate for short-term equity options with time to maturity less than one year, it is not reasonable to assume constant interest rate for long-term options such as LEAPS (long-term equity anticipation securities) with time to maturity of at least two years currently trading in major exchanges. Furthermore, it is not reasonable even for short-term interest-rate options which comprise the bulk of derivatives industry because their values are much more sensitive to interest-rate fluctuations.

Merton (1973) first established a pricing model with stochastic interest rates. In Merton's model, stochastic interest rates are indirectly modeled on the prices of a discount bond which follows a geometric Brownian motion.

Option prices are given in closed-forms in this model. It was used as the basis for many studies on interest-rate derivatives. Jamshidian (1989) provided closed-form solutions for European options on discount bonds in a mean-reverting model[4] of interest rate. The early stage of the studies on the term structure of interest rate was mainly based on equilibrium, such as Vasick's (1977) "An Equilibrium Characterization of the Term Structure" and Cox, Ingersoll, and Ross's (1985) or simply CIR's "A Theory of the Term Structure of Interest Rate".

Ho and Lee (1986) marked a milestone in the study of the term structure of interest rates by including the arbitrage-free argument. As in the original Black-Scholes model where the current spot is given and option prices are derived from the no-arbitrage argument, Ho and Lee took the current yield curve as given and applied the no-arbitrage argument to price all kinds of interest-rate derivatives. Black, Derman, and Toy (1990) extended this argument by assuming that short rates are lognormally distributed in a single discrete time binomial process with equal probability. Hull and White's (1990) model is essentially the extension of Vasick (1977) and CIR (1985), incorporating the mean-reverting process of interest rates and taking the current yield curve as given. The Ho-Lee, Black-Derman-Toy, and Hull-White models are all one-factor models. Heath, Jarrow, and Merton (HJM) (1987) provided a multiple model in which multiple random factors are introduced so that default-free bonds of different maturities can have positive but not perfectly correlated returns, and continuous trading is introduced so that estimating parameters becomes easier. The HJM model utilizes forward rates instead of spot rates as in most other models. Theoretically, it is currently the most general model which includes known models of the term structure of interest rate as special cases and allows flexible term structure of volatility.[5]

The most important characteristic of the modern study on the term structure of interest rate is that the current yield curve is taken as given and the no-arbitrage principle is used to price all kinds of interest-rate derivatives.

[4]A mean-reverting process is a stochastic process in which the stochastic variable tends to move to a mean target value.

[5]Although it is general theoretically, the HJM model has been found to have certain limitations in practical implementation. Flesaker and Hughston (1996) described a new theory of interests which prevents the possibility of negative rates in the general HJM model. As Flesaker and Hughston's approach is consistent with the economic arguments of the HJM model, particularly with the methodology for contingent claims valuation, the Flesaker and Hughston's approach can be regarded as a precise identification and characterization of the subclass of the HJM model for positive interest rates. Based on their academic paper in 1995, Li, Ritchken and Sankarasubramanian described another subclass of the HJM model which requires a finite number of state variables with only slight restrictions in the class of volatility functions for forward rates.

This characteristic is clearly in contrast to the current spot as given, as in the original Black-Scholes and all its following models. The current yield curve is implied from the prices of government bonds with various maturities or from forward or futures prices of these bonds. In other words, information from government bond markets with various maturities or from government bond forward or futures markets is used to price various kinds of derivatives in the absence of arbitrage. As we have argued above, the Black-Scholes and most of its extended models are not problematic for pricing options written on individual stocks for which no futures or forward markets exist.

However, they can be problematic for currency options, stock-index options such as options written on the S&P-100 Index,[6] and commodity options such as gold options because futures markets exist for both S&P-100 Index and gold, and both futures and forward markets exist for the major currencies. The information contained in the futures or forward markets is not used in the original Black-Scholes model and most of its extensions. Taking the spot price as given without considering information from the futures or forward markets in the Black-Scholes and many of its extended models, arbitrage is absent among the underlying cash market, option market, and the bond market. Existing models price futures and forwards based on the arbitrage-free relationship between the underlying cash market and the futures/forward market [see Eq. (2.15)], and value options based on the arbitrage-free relationship between the underlying cash market and the option market. Thus, no arbitrage condition is imposed between the option market and the futures market. In other words, the information contained in the futures or forward market is not used to price options on the same underlying cash market. The arbitrage-free method in studying the term structure of interest rate discussed in this section can be used to utilize this information to price stock-index options, currency options, commodity options, and some other options.

3.4. PUT-CALL PARITY

Curious readers may wonder why we've only covered pricing call options in Chapter 2 and the previous sections of this chapter. Actually, the same method can be used to price put options. In this section, we are going to introduce an important relationship between a call and its corresponding put option prices with the same strike price. The put option price can then

[6]Standard and Poor's 100 Index is a US stock index for 100 large US stocks trading in the New York Stock Exchange.

be conveniently calculated from this relationship.

Put-call parity is actually an equality that connects the current prices of a call option and a put option with the same strike price and time to maturity. In other words, it gives a condition under which there is no arbitrage or "no-free-lunch" between a call option and its corresponding put option with the same strike price. This parity relationship can be demonstrated by considering the portfolio: *a long call* (C) *and a short put* ($-P$) *with the same strike price and on the same underlying asset.*

If the underlying asset price at the expiration of the option $S(\tau)$ is greater than the strike price K, the payoff of the call option is $S(\tau) - K$ and the payoff of the put option is zero. Thus the net payoff of the portfolio becomes

$$S(\tau) - K - 0 = S(\tau) - K\,, \tag{3.22}$$

from the payoffs of a call and a put given in (2.1) and (2.2), respectively. Similarly, if the underlying asset price at the expiration of the option $S(\tau)$ is smaller than the strike price K, the payoff of the call option is zero and the payoff of the put option is $K - S(\tau)$, and thus the net payoff of the portfolio becomes

$$0 - [K - S(\tau)] = S(\tau) - K\,,$$

which is exactly the same as the payoff of the portfolio given in (3.22) if the underlying stock price is greater than the strike price.

The above analysis shows that the portfolio has the same payoff $S(\tau)-K$ regardless of whether the underlying stock price is greater or smaller than the strike price at maturity. Because the portfolio $C - P$ always has the same future payoff $S(\tau) - K$, the value of the portfolio must be the same as the present value of the payoff $S(\tau) - K$, otherwise there would be arbitrage. Equalizing the present value of the portfolio $C - P$ and the present value of the future payoff $S(\tau) - K$,[7] we can readily obtain

$$\begin{aligned} C - P &= S - Ke^{-r\tau}\,, \text{or} \\ P &= C - S + Ke^{-r\tau}\,, \end{aligned} \tag{3.23}$$

where C, P stand for the call option and put option prices, respectively. S is the spot stock price, K represents the strike price of the options, and r and τ represent the interest rate and time to maturity, respectively.

[7]Using the solution of the standard geometric Brownian motion given in (2.4), we can readily find that the expected value of the underlying asset price at maturity $S(\tau)$ is $Se^{r\tau}$. Discounting the expected payoff $Se^{r\tau} - K$ by the continuous factor $e^{-r\tau}$ yields $S - Ke^{-r\tau}$.

The equality given in (3.23) is called the put-call parity in options literature. Substituting the call option pricing formula (2.5) into (3.23) yields the following pricing formula for the put option:

$$P = -SN(-d_1) + Ke^{-r\tau}N(-d_2)\,, \tag{3.24}$$

where d_1 and d_2 and all other parameters are the same as in (2.5).

Example 3.4. What is the corresponding put option price in Example 2.1?

The put option price P can be calculated directly by substituting $C =$ \$10.89, $S =$ \$100, $K =$ \$105, $r = 20\%$, and $\tau = 0.50$ into the put-call parity (3.23):

$$\begin{aligned} P &= C - S + Ke^{-r\tau} \\ &= 10.89 - 100 + 105e^{-0.5\times0.20} = \$5.898\,. \end{aligned}$$

Alternatively, substituting $d_1 = 0.35$, $d_2 = 0.14$ (from Example 2.1), $S =$ \$100, $K =$ \$105, $r = 20\%$, and $\tau = 0.50$ into the put option pricing formula (2.38) yields

$$\begin{aligned} P &= -SN(-d_1) + Ke^{-r\tau}N(-d_2) \\ &= -100N(-0.35) + 105e^{-0.5\times0.20}N(-0.14) = \$5.898\,, \end{aligned}$$

which is precisely the same value obtained using the put-call parity.

The put-call parity in (3.23) and (3.24) can be readily extended to incorporate the yield on the underlying asset using the call option pricing formula given in (3.14):

$$P = C - Se^{-g\tau} + Ke^{-r\tau}\,, \tag{3.25}$$

where C, P stand for the call option and put option prices, respectively, and all other parameters are the same as in (3.2).

Alternatively, the equality in (3.25) can be given as

$$P = -Se^{-g\tau}N(-d_1) + Ke^{-r\tau}N(-d_2)\,, \tag{3.26}$$

where d_1 and d_2 and all other parameters are the same as in (2.5).

Example 3.5. What are the corresponding put option prices in Examples 3.1 and 3.2?

The price of the corresponding put option in Example 3.1 can be calculated directly by substituting $C =$ \$9.455, $S =$ \$100, $K =$ \$105, $r = 20\%$, $g = 5\%$, and $\tau = 0.50$ into the put-call parity (3.25)

$$\begin{aligned} P &= C - Se^{-g\tau} + Ke^{-r\tau} \\ &= 9.455 - 100e^{-0.5\times0.05} + 105e^{-0.5\times0.20} = \$6.932 \end{aligned}$$

and the price of the corresponding put option in Example 3.2 can be calculated directly by substituting $C = \$0.00657$, $S = \$0.0175$, $K = \$0.0111$, $r = 8\%$, $r_f = 3\%$, and $\tau = 0.50$ into the put-call parity (3.25)

$$\begin{aligned} P &= C - Se^{-r_f\tau} + Ke^{-r\tau} \\ &= 0.00675 - 0.0175e^{-0.5\times 0.03} + 0.0111e^{-0.5\times 0.08} = \$0.000005\,. \end{aligned}$$

Following the similar method as in deriving the put-call parity given in (3.23) and (3.24), we can obtain the put-call parity between the prices of a futures call option and its corresponding put option:

$$P = C - Fe^{-r\tau} + Ke^{-r\tau}\,, \tag{3.27}$$

where all parameters are the same as in the futures call option pricing formula (3.5).

Similarly, we can obtain the put option pricing formula using (3.2) and (3.27):

$$P = e^{-r\tau}[-FN(-d_1) + KN(-d_2)]\,, \tag{3.28}$$

where d_1, d_2, and other parameters are the same as in (3.5).

3.5. MODERN GREEKS

In the previous sections, we studied how to price call and put options. We know that both the European call and put option prices are affected by the five factors in the Black-Scholes model without considering the payout rates of the underlying assets. It is useful to know how sensitive call and put option prices change with these factors. As a matter of fact, there are a few popular terms characterizing the sensitivities of option prices with respect to these factors. These sensitivities are often named by Greek alphabets. They play an important role in both trading activities and risk management in all financial institutions with any derivative securities. In this section, we first introduce the traditional sensitivities which are more familiar to most people in the derivatives industry, and then some higher sensitivities.

3.5.1. Traditional Greeks

Delta(δ)

An option's delta measures how fast an option price changes with the price of the underlying asset. There are other explanations to an option's delta: mathematicians interpret an option's delta as the first-order partial derivative of the option price with respect to the price of its underlying asset, and economists interpret it as the sensitivity of the option price toward the price of its underlying asset. The call option's delta in the Black-Scholes model can be obtained by taking partial derivative of (2.5) with respect to S

$$\begin{aligned}\frac{\partial C}{\partial S} &= N(d_1) + Sf(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r\tau}f(d_2)\frac{\partial d_2}{\partial S} \\ &= N(d_1) + [Sf(d_1) - Ke^{-r\tau}f(d_2)]\frac{1}{S\sigma\sqrt{\tau}}\,. \end{aligned} \tag{3.29}$$

The delta given in (3.29) can be simplified using the following identity:

$$\begin{aligned}\frac{f(d_1)}{f(d_2)} &= \left(\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\right)\Big/\left(\frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}\right) = e^{(d_2^2-d_1^2)/2} \\ &= e^{-(d_1-d_2)(d_1+d_2)/2} = e^{-\sigma\sqrt{\tau}(2d_2+\sigma\sqrt{\tau})/2} \\ &= e^{-\sigma\sqrt{\tau}(d_2+\sigma\sqrt{\tau}/2)} = e^{-d_2\sigma\sqrt{\tau}-\sigma^2\tau} \\ &= e^{-[\ln(S/K)+(r-\sigma^2/2)\tau]+(\sigma^2\tau/2)} \\ &= e^{-r\tau}e^{-\ln(S/K)} = e^{-r\tau}e^{\ln(K/S)} = \frac{K}{S}e^{-r\tau}\,,\end{aligned}$$

thus,

$$Sf(d_1) = Ke^{-r\tau}f(d_2)\,. \tag{3.30}$$

Substituting (3.30) into (3.29) yields the delta of the call option in the Black-Scholes model as $N(d_1)$. Similarly, the put option's delta is $-N(-d_1)$, the negative sign implying that the put option price declines as the underlying asset spot price goes up. The identity given in (3.30) is used to simplify most other Greeks significantly.

In Examples 2.1 and 3.4, the delta of the call option is $N(d_1) = N(0.35) = 0.6368 = 63.68\%$ and the delta of the put option is $-N(-d_1) = -0.3632 = -36.32\%$. Normally, the negative sign is omitted because every person knows that the put option price changes at the opposite direction with its underlying asset price. Thus we simply say that the call option has a delta of 63.68% and the put has a delta of 36.32%. In general, deep-out-of-the-money options have deltas close to zero, implying that these option prices change little with the underlying asset prices; deep-in-the-money options have deltas close to one, indicating that these option prices change about the same amount with the underlying asset prices.

Vega (ν)

An option's vega measures how fast an option price changes with its volatility. Mathematically, an option's vega is the first-order partial derivative of the option price with respect to the volatility of its underlying asset. Vega is important because the volatility of the underlying asset is of vital importance to option trading. Volatility to options is what wind to kites. Kites cannot fly without wind, and they tend to crash if there is too much wind. Options would not exist without volatility, and they cannot trade smoothly if there is too much volatility. From previous sections, we learned that hedging is the most important reason for most derivative securities to exist. If there does not exist enough noise in the market, the prices of the underlying assets can remain relatively stable, then there is really not much need for options to be traded on these assets. Even if options exist on these assets, trading volumes are likely to be rather thin. If there are more uncertainties in the market, there will be more risks for the option writers as there is more possibility for them to lose. Thus, option writers normally charge more for options with higher volatility, other things being equal. Therefore, vegas of all options are always positive.

The formula for vega in the Black-Scholes model is simply $S\sqrt{\tau}f(d_1)$ or $K\sqrt{\tau}e^{-r\tau}f(d_2)$ regardless of whether it is for a call or put. In Examples 2.1 and 3.4, $S = \$100$, $\tau = 0.5$, $d_1 = 0.35$, then vega is

$$\text{Vega} = 100 \times \sqrt{0.50} \times \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} = 26.53\,,$$

which implies that the call or put option values increase (resp. decrease) 26.53% for each percent increase (resp. decrease) of the underlying volatility.

Theta (θ)

An option's theta measures the sensitivity of its price with respect to time to maturity. It is also called the time decay of an option. We know that the value of a European option at expiration depends on the relative price level of the underlying asset and the strike price of the option. Option values at expiration are called intrinsic values of options. The intrinsic value is only one part of an option's value because it is a measure with zero time to maturity. For an option with positive time to maturity, it also has value changing with time, and this part of value is called the time value of an option. Options have time values because there is always possibility for the prices of the underlying assets to change when there is time left before expiration. An option's theta is always positive because there is always more possibility for the prices of the underlying assets to change whenever there is more time before expiration. The positiveness of thetas is also shown in Figure 2.6, because the time value of options is larger with more time to maturity for all spot prices under consideration.

The formula for theta in the Black-Scholes model is

$$\text{Theta} = S\sigma \frac{1}{\sqrt{2\pi}} f(d_1) + rKe^{-r\tau}N(d_2) \text{ for a call, and}$$

$$\text{Theta} = S\sigma f(d_1) \frac{1}{\sqrt{2\pi}} - rKe^{-r\tau}N(-d_2) \text{ for a put}.$$

In Examples 2.1 and 3.4, $S = \$100$, $\tau = 0.5$, $d_1 = 0.35$, the theta for the call is 21.79 and that for the put is 2.79, implying that the call and put option values increase (resp. decrease) 21.79% and 2.79% for each percent increase (resp. decrease) of the time to maturity.

Rho (ρ)

An option's rho measures the sensitivity of the option's value with respect to the fluctuation of interest rate. Interest level reflects the opportunity cost of holding options. The higher the interest rate, the higher the opportunity cost for a call option, thus the higher the price of a call option. The formula of rho is $\tau Ke^{-r\tau}N(d_2)$ for a European call option, and $-\tau Ke^{-r\tau}N(-d_2)$ for a put. In Examples 2.1 and 3.4, the rho is 26.31 for the call option and −21.19 for the put option, implying that the call option will appreciate (resp. depreciate) approximately 26.31% for each percent increase (resp. decrease) of the interest rate and the put option will depreciate (resp. appreciate) approximately 21.19% for each percent increase (resp. decrease) of the interest rate.

Lamda (λ)

We have discussed the sensitivities of option values with respect to four important parameters: spot price, volatility, time to maturity, and interest rate. Besides these four measures, there is another important measure, often called the lamda of an option. The lamda of an option measures how much the option's price changes in percentage for each percent change in the price of the underlying asset. Clearly, an option's lamda is related to its delta. Simple mathematical manipulation shows that the lamda of an option equals the option's delta multiplied by the ratio of the spot prices of the underlying asset and the option. In Examples 2.1 and 3.4, the delta of the call option is 63.68%, and −36.32% for the put option. The lamda of the call option is thus $0.6368 \times 100/10.89 = 5.85$, implying that the call option price will increase (resp. decrease) 5.85% if the underlying stock price increases (resp. decreases) 1%. And the lamda of the corresponding put option is equal to $-0.3632 \times 100/5.898 = -6.16$, implying that the put option price will decrease (resp. increase) 6.16% if the underlying stock price increases (resp. decreases) 1%. These examples show that options are like amplifiers that can expand the significance of the changes of the underlying asset prices. Thus, we may also say that lamdas measure the leverage effects of options.

Gamma (γ)

Gamma is another important sensitivity measure for an option. It measures how fast the option's delta changes with the price of its underlying asset. Gamma is clearly a second-order sensitivity. It is often used in option trading strategies such as gamma hedging. An option's gamma is $f(d_1)/(S\sigma\sqrt{\tau})$, the same for both call and put options with the same strike price. In Examples 2.1 and 3.4, the gamma is 0.018 or 1.8%, implying that the delta of both the call and put options will increase (resp. decrease) approximately 1.8% for one dollar increase (resp. decrease) of the spot price.

With the above descriptions, we can express the total change of a vanilla option price (VOP) as follows:

$$\Delta PVOP = \text{Delta}(\Delta S) + \text{Vega}(\Delta\sigma) + \text{Theta}(\Delta\tau) + \text{Rho}(\Delta r) + O(\Delta y^2)\,, \tag{3.31}$$

where $O(\Delta y^2)$ stands for the higher terms of the changes in the current spot S, volatility, time to maturity, and interest rate, with the Greek alphabets given as follows:

$$\text{Delta} = \omega e^{-g\tau} N(\omega d_1)\,, \tag{3.32}$$

$$\text{Vega} = S\sqrt{\tau} e^{-g\tau} f(d_1) = K e^{-r\tau}\sqrt{\tau} f(d_2)\,, \tag{3.33}$$

$$\text{Theta} = K e^{-r\tau}\left[\frac{\sigma}{2\sqrt{\tau}} f(d_2) + \omega r N(\omega d_2)\right] - \omega S g e^{-g\tau} N(\omega d_1)\,, \tag{3.34}$$

$$\text{Rho} = \omega\tau K e^{-r\tau} N(\omega d_2)\,, \tag{3.35}$$

$$\text{Lamda} = S e^{-g\tau} N(\omega d_1)/[S e^{-g\tau} N(\omega d_1) - K e^{-r\tau} N(\omega d_2)]\,, \tag{3.36}$$

$$\text{Gamma} = e^{-g\tau}\frac{f(d_1)}{S\sigma\sqrt{\tau}}\,. \tag{3.37}$$

The traditional sensitivities named after these Greek alphabets are most often used in risk management of most derivatives portfolios.

3.5.2. Higher Sensitivities

The traditional sensitivities discussed above are very useful for most trading strategies. Their sensitivities to the underlying asset price and time to maturity are therefore of significant importance. Recently, higher sensitivities, or sensitivities of the traditional sensitivities with respect to time and underlying asset price are used by many traders. We now introduce these higher sensitivities in the Black-Scholes model.

Speed

Speed measures the rate at which the gamma of a derivative asset changes with one underlying asset price. Economically, it is the sensitivity of the gamma with respect to one underlying asset price. As there is only one underlying asset in the Black-Scholes model, we can easily obtain the speed formula of a vanilla option in the Black-Scholes model as

$$\begin{aligned}\text{Speed}_{\text{bs}} &= -\frac{(d_1+\sigma\sqrt{\tau})}{S^2\sigma\sqrt{\tau}}\,e^{-g\tau}f(d_1)\\ &= -\frac{d_1+\sigma\sqrt{\tau}}{S}\,\text{Gamma}\,. \qquad (3.38)\end{aligned}$$

In Examples 2.1 and 3.4, the speed of both the call and put options is -0.0001 or -0.01%, implying that the gamma will decrease (resp. increase) 0.01% for each dollar increase of the underlying spot price.

Charm

Charm measures the rate at which the delta of a derivative asset changes with its time to maturity. The charm of a vanilla option in the Black-Scholes model can be readily obtained as follows:

$$\text{Charm}_{\text{bs}} = \left(\frac{2r\tau - d_2\sigma\sqrt{\tau}}{2\sigma\tau\sqrt{\tau}}\right)f(d_1)\,. \qquad (3.39)$$

In Examples 2.1 and 3.4, the charm for the call and put options is 0.303, implying that the delta will decrease (increase) 30.3% as the options 1% closer to maturity.

Color

Color measures the rate at which the gamma of a derivative asset changes with respect to the time to maturity. The color of a vanilla option can be easily derived as follows:

$$\text{Color}_{\text{bs}} = -\left(\frac{\sigma + [\ln(K/S) + (r+\sigma^2/2)\tau]d_1}{2\sigma^2\tau^2 S}\right)f(d_1)\,. \qquad (3.40)$$

In Examples 2.1 and 3.4, the color of the call option is $-0.03 = -3\%$ for both the call and the put option, implying that their gamma will decrease 3% as the option is 1% closer to maturity.

Speed is obviously a third-order derivative with respect to the underlying asset price. Both charm and color are cross sensitivities as they measure how option values change, first with respect to the increase in the underlying spot price, and then to the time to maturity.

The traditional and higher sensitivities we described in this section are used to analyze risks of individual options or portfolios of options. There are other risk measures of options such as skewness, kurtosis, the systematic risk measured with beta, etc. For a systematic study of these measures and related literature, see Lee and Zhang (1995).

3.6. DELTA HEDGING AND GAMMA HEDGING

After describing the traditional and higher Greeks in the previous section, we can now introduce two important concepts in option trading: delta hedging and gamma hedging. Delta hedging is a trading strategy to make the delta of a portfolio neutral to the fluctuation of the underlying asset price. For example, consider the standard portfolio in option pricing theory which includes one unit of the underlying asset long, and $N(d_1)$ unit of a call option on the underlying asset short. The value of this portfolio can be expressed as $S(\tau) - N(d_1)C$. The delta of this portfolio can be easily obtained using the delta formula given in (3.32):

$$1 - N(d_1)\text{Delta of } C = 1 - N(d_1)/N(d_1) = 0\,,$$

because the delta of the underlying asset is always one.

The portfolio including one unit of the underlying asset long and $N(d_1)$ unit of a call option on the underlying asset short is a well-known example of delta hedging. If a portfolio has a positive (resp. negative) delta $DELTA$, to carry out delta hedging, we can simply write (resp. buy) $DELTA/N(d_1)$ units of call option so that the delta of the portfolio will be zero.

Whereas delta hedging is to make the delta of a portfolio neutral to the fluctuation of the underlying asset price, gamma hedging is to neutralize the gamma of a portfolio or to make the gamma of the portfolio zero. In the above example of delta hedging, the portfolio $S(\tau) - N(d_1)C$ is always delta-hedged, yet it is not gamma-hedged because its gamma is

$$0 - N(d_1)f(d_1)/(S\sigma\sqrt{\tau}) = -N(d_1)f(d_1)/(S\sigma\sqrt{\tau}) < 0\,,$$

because the gamma of the underlying asset is always zero (the second order derivatives of S with respect to S is always zero) and the gamma of the call option is given in (3.37) with $g = 0$. If a portfolio has a positive (resp. negative) gamma $GAMMA$, to carry out gamma hedging, we can simply write (resp. buy) $GAMMA/[f(d_1)/(S\sigma\sqrt{\tau})]$ units of call option so that the gamma of the new portfolio will be zero.

A portfolio may not be gamma-hedged when it is delta-hedged, as our above example showed, or it may not be delta-hedged when it is gamma-hedged. This is simply because when we change the composition of the portfolio to achieve the goal of either delta hedging or gamma hedging, the other is changed at the same time. However, this is not a serious problem because the need for one hedge often dominates the other, so it is alright to consider the more important issue and hedge it consequently.

3.7. IMPLIED VOLATILITY

In discussing the Black-Scholes model, we learned that all the parameters in the model can be either observed from the market directly, or specified in option contracts with one exception — volatility of the underlying asset. We learned that historical data can be used to estimate the volatility of the underlying asset. However, there is no general rule as to what kind of historical data and how far back in history the data should be used to estimate this parameter. Estimation can be very different using daily data of the immediate past three months, six months, one year, or two years. Thus, the prices of options can be different using different estimated volatility parameters. That is a problem with the Black-Scholes pricing model and all other models as well.

Academics have tried to overcome this problem. The market prices of options, like market prices of all other securities, are determined by the changing supply and demand conditions. The actual option prices can be observed from the markets. Using the actual market prices and the Black-Scholes formula inversely, we can solve for the value of the volatility parameter. The volatility value which equals the theoretical Black-Scholes formula value and the actual market price is called the implied volatility. Mathematically, the implied volatility is the solution of the inverse equation from the Black-Scholes formula.

Example 3.6. What is the implied volatility if the call option price is \$9.00 and other parameters are the same as in Example 2.1?

We can solve the implied volatility by trial and error. If we substitute $\sigma = 28\%$, the current underlying asset price $S = \$100$, the strike price $K = \$105$, interest rate $r = 20\% = 0.20$, time to maturity $\tau = 0.50$ into the Black-Scholes formula in (2.5), we would obtain $C = \$10.44 > \$9.00$, the market price of the call; if we substitute $\sigma = 20\%$, we would get $C = \$8.35 < \$9.00$. From these two trials, we know that the implied volatility must be between 20% and 28% because the Black-Scholes formula is a monotonically increasing function of volatility, or the call option's vegas are always positive in the Black-Scholes model. Continuing the trial and error procedure, we could obtain $\sigma = 0.2252$ or 22.52% with which the Black-Scholes formula yields the same price as the actual market price \$9.00. Therefore, the implied volatility is 22.52% if the actual option price is \$9.00.

As there is a one-to-one correspondence between the Black-Scholes option price and its volatility, it is equivalent to say that someone buys the call option at the premium of \$9.00 or at 22.52% implied volatility in the above example. As a matter of fact, most option traders prefer implied volatilities to premiums. The implied volatility can be used in several other ways. It can be interpreted as the average volatility that the underlying asset will have from now to the option's expiration time, it can be used to forecast the change of the underlying asset price in the short term. We will discuss this more in the next section.

Example 3.7. What is the implied volatility if the put option price is \$5.00 and other parameters are the same as in Example 3.4?

Following the same procedure as in Example 3.6, we can find the implied volatility of the put option price \$5.00 is 0.263 or 26.3%.

Examples 3.6 and 3.7 show that implied volatilities can be different for put and call options even with the same strike price and time to maturity. The difference may imply the imperfection of the actual market which violates the assumptions of the Black-Scholes model. The imperfect factors may include taxation, transaction cost, liquidity, and many others.

3.8. TERM STRUCTURE OF VOLATILITY AND VOLATILITY SMILE

In the previous section, we discussed the concept of implied volatilities and how to calculate them given the option prices and other parameters. Normally, there are options written on the same underlying instrument with various time to maturity. Using the same option pricing formula such as the Black-Scholes formula and observing the option prices with various maturities, we are able to find a set of implied volatilities with different time to maturity. In practice, these implied volatilities are different from one another. The reason for these differences is likely to be the market imperfection discussed at the end of the previous section. The volatility structure with different time to maturity is called the term structure of volatility. Normally, the implied volatilities are larger for options with shorter time to maturity. The term structure of volatility is very useful for exotic options with time to maturity different from that listed in exchanges. Reasonable implied volatilities can be found for exotic options with various time to maturity by interpolating along the implied volatility curve.

With the same time to maturity, there are often many options with various strike prices written on the same underlying asset. Theoretically, implied volatilities for options with various strike prices but the same maturity time should be the same. However, empirical studies show that implied volatilities of out-of-the-money options are on average higher than that of at-the-money options. From the author's experiences in currency futures options trading at the International Monetary Market (IMM) of the Chicago Mercantile Exchange (CME), the implied volatilities normally become higher for deeper out-of-the-money options. This phenomenon is called "volatility smile" by professionals. The volatility smile obviously violates the Black-Scholes assumption of constant volatility.

Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) independently constructed models to incorporate the smile effect into pricing models. The basic idea of these studies is to infer useful information about the distribution of the underlying asset prices from exchange-traded options and to price other derivatives using this information. These smile models are very useful because reasonable implied volatilities can be estimated using interpolation for exotic options with strike prices not listed in exchanges.

3.9. LIQUIDITY FACTOR

The Black-Scholes option pricing model has been extended in various directions. But there is one important factor that has yet to be captured in any pricing model — liquidity of the underlying market. Some may simply consider that liquidity is characterized by volatility because the lower the liquidity, the larger the bid-ask spread in general, and therefore the higher the volatility of the underlying asset. However, the above argument is not true because the volatility of the underlying asset can be low as a result of infrequent activities in the market even though its bid-ask spread is rather wide. Although liquidity may be, in general, correlated with volatility, these two concepts are very different and one of them could not replace the other. Volatility measures the degree of fluctuations of the underlying asset returns or prices and it has nothing to do with trading volumes of the underlying asset. However, liquidity measures the the trading frequency of the underlying asset. It is generally measured by the bid-ask spread per unit of trading volume of the underlying asset within a specified period of time. Thus, liquidity is normally determined by the average price spread and the average trading volume, and cannot be generally replaced by volatility.

As different assets possess different degrees of volatility, different markets normally exhibit different levels of liquidity. Some markets such as the US Treasuries may have near perfect liquidity and other markets such as stocks of some small firms or exotic currencies may have one transaction in one day or even one week, and many art markets may have transactions in every five to ten years. As liquidity can also be understood as the average time between two consecutive tradings, low liquidity generally implies longer average time between two consecutive transactions and in turn imply more difficulties in hedging the underlying asset and/or selling and buying it. Thus, there exist two kinds of risks resulting from low liquidity: one is that the underlying asset can neither be bought nor sold at the option maturity between two possible consecutive transactions, and the other is that desirable quantity of the underlying asset may not be bought or sold even if the underlying asset can be bought or sold at all. Therefore, liquidity is an important factor in determining derivatives values written on an underlying asset which is less than perfectly liquid. To date, there is no satisfactory model in the literature that has incorporated liquidity into an option pricing theory.

A theoretical closed-form solution to incorporate the liquidity factor may be hard to find, but the problem can be solved numerically using the popular tree-model to be described in the following chapter. We will show more specifically how liquidity affects option prices in Chapter 4 when we use the binomial tree extensively to price American options.

3.10. SUMMARY

We have reviewed various aspects of vanilla option theories and markets, from the well-known binomial model to the celebrated Black-Scholes model, and its various extensions. The review is useful because all exotic options, as we will illustrate later in this book, are extensions of vanilla options, which can be used as benchmarks for our understanding of exotic options. The term structure of volatility and volatility smile are of particular use to exotic options because they can be used to infer reasonable implied volatilities for exotic options with time to maturity and/or strike prices not listed in exchanges. The review is also useful because most of the jargons of vanilla options are also used for exotic options. This chapter can be used as a quick reference for our following ones. For those readers who have a good knowledge of vanilla options, this chapter may be used as a reference to check specific formulas and expressions.

Although the review has touched most topics about vanilla options, it is impossible to include all aspects of standard options in one chapter. Of the subjects on standard options we have not covered, option trading strategies are well known and used by many traders. The chart on option trading strategies made by the Chicago Mercantile Exchange (CME) is very popular in the professional world. The chart includes 29 popular strategies most often used in practice. The best book on this subject is Gary Geastiue's *The Stock Options Manual* (2nd ed., 1979).

QUESTIONS AND EXERCISES

Questions

3.1. What are the effects of the underlying payout rate on the call option?

3.2. What are frations?

3.3. What are futures options?

3.4. Why are futures options more popular than frations?

3.5. What is the obvious shortcoming of Fisher's extension of the Black-Scholes formula incorporating uncertain strike prices?

3.6. What is the advantage of the CEV model over the standard Black-Scholes model?

3.7. What is the major difference of the CEV models with positive and negative elasticity parameters?

3.8. What is the advantage of the pricing model with jump process over the standard Black-Scholes model?

3.9. What is the advantage of Leland's model with transaction cost over the standard Black-Scholes model?

3.10. Why does the volatility parameter needs to be randomized? What is the advantage of the stochastic volatility model over the standard Black-Scholes model?

3.11. What are the disadvantages of all the extended versions of the Black-Scholes model compared to the standard Black-Scholes model?

3.12. What is the term structure of interest rate? Why is it useful to price derivatives?

3.13. What is the contribution of Ho and Lee (1987) to the study of interest-rate derivatives?

3.14. What is the most important difference between interest-rate derivatives and equity derivatives?

3.15. What are implied volatilities? Why are they useful?

3.16. Are implied volatilities the same from the prices of a call and its corresponding put options with the same strike price? Why?
3.17. What is the term structure of volatility?
3.18. What is volatility smile? Why is it useful in practice?
3.19. Why is liquidity important in pricing options?
3.20. Has Leland's model with transaction cost captured liquidity factor in pricing options? Why?

Exercises

3.1. Find the call and put option prices, given the annual interest rate 10%, time to maturity four months, strike price \$110, current stock price \$105, the underlying stock has a constant dividend yield of 6%, and the volatility of the return of the underlying asset $\sigma = 20\%$.
3.2. Find the delta, vega, theta, gamma and lamda for the call and put options in Exercise 3.1.
3.3. Find the speed, charm, and color for both the call and put options in Exercise 3.1.
3.4. Show that the identity $Se^{-g\tau}N(d_1) = Ke^{-r\tau}N(d_2)$ is always true for the extended Black-Scholes model with constant payout of the underlying asset.
3.5. Find the call and put option prices if the payout rate of the underlying asset is 3% and other parameters are the same as in Exercise 3.1.
3.6. Find the European call and put option prices on the Japanese yen/US dollar exchange rate with strike price ¥88 per dollar to expire in half a year, given the spot US dollar/Japanese yen exchange rate ¥86 per dollar, the Japanese interest rate is 2.5%, the US interest rate 7%, and the volatility of the dollar/yen exchange rate 18%.
3.7. Find the European call and put option prices on the German mark/US dollar exchange rate with strike price 1.50 mark per dollar to expire in five months, given the spot US dollar/German mark exchange rate 1.45 mark per dollar, the German interest rate is 5%, the US interest rate 8%, and the volatility of the dollar/mark exchange rate 15%.
3.8. Find the prices of the European call and put options on the September 1995 S&P-500 futures with strike price \$520 to expire in four months, given the spot September 1995 S&P-500 futures price is \$510, the volatility of the September 1995 S&P-500 futures is 18%, the US interest rate is 7%.

3.9. Find the prices of the European call and put options on the December 1995 Nikkei-225 (Japanese stock market index) to expire in half a year, given the current futures price is ¥16500, the volatility of the futures price is 15%, interest rate is 3%, strike price ¥17000.

3.10. A straddle is a pair of call and put options with the same strike price. Find the price of the straddle including the call and put options in Exercises 3.1 and 3.5.

3.11. Find the price of the straddle including the call and put options in Exercises 3.7 and 3.8.

3.12. A strangle is a pair of call and put options with the call's strike price greater than that of the put. Thus, a straddle is a special case of a strangle when the strike prices of the two options are the same. Find the price of the strangle including a call with strike price $100 and a put with strike price $110 and other parameters the same as in Exercise 3.1.

3.13. Find the price of the strangle including a call with strike price $110 and a put with strike price $100 and other parameters the same as in Exercise 3.1.

3.14. Find the call and put option prices if the strike price is $121, the current stock price $115.5, and other parameters are the same as in Exercise 3.1. (hint: use the scaling property of the Black-Scholes formula).

3.15. Find the implied volatility of the call option if its price is $10 and other parameters are the same as in Exercise 3.1.

3.16. Find the implied volatility of the put option if its price is $8 and other parameters are the same as in Exercise 3.5.

3.17. Find the implied volatilities of the call options to expire in three months and half a year if their prices are $5 and $6, respectively, other parameters remaining the same as in Exercise 3.1.

3.18. If the implied volatilities for three call options written on one particular stock with time to maturity six, nine, and twelve months are 15%, 12%, and 10%, respectively, what are the implied volatilities for call options written on the same underlying stock with time to maturity seven, eight, ten, and eleven months? (hint: use the linear or quadratic extrapolation).

3.19. In the above exercise, what are the prices of call options if the annual interest rate 10%, time to maturity one year, strike price $110, and current stock price $105.

3.20. If the implied volatilities for three call options written on one particular stock with the same strike price; time to maturity six, nine, and twelve months are 15%, 12%, and 10%, respectively, what are the implied volatilities for call options written on the same underlying stock with time to maturity seven, eight, ten, and eleven months? (hint: use the linear extrapolation).

APPENDIX

Calculating the Cumulative Distribution Function of the Central χ^2 Distribution

The cumulative distribution function of the central χ^2 distribution $Q[\chi^2_{\nu+2j} \leq C]$ is given as follows:

$$Q[\chi^2_{\nu+2j} \leq C] = \frac{\gamma(j+\nu/2; C/2)}{\Gamma(j+\nu/2)}, \tag{A2.1}$$

where $\Gamma(j + \nu/2)$ is the standard gamma function and $\gamma(\frac{n}{2}, \frac{C}{2})$ is the incomplete gamma function which is defined as follows:

$$\gamma(\nu; C) = \int_0^C z^{\nu-1} e^{-z} dz, \ C \geq 0 \text{ and } \nu > 0. \tag{A2.2}$$

The incomplete gamma function can be approximated in a number of ways:

$$\gamma(\nu; x) = \frac{x^\nu}{\nu} - \frac{x^{\nu+1}}{1+\nu} + \frac{x^{\nu+2}}{2(2+\nu)} - \frac{x^{\nu+3}}{6(3+\nu)} + \cdots = x^\nu \sum_{j=0}^{\infty} \frac{(-x)^j}{j!(j+\nu)}, \text{ or} \tag{A2.3}$$

$$\gamma(\nu; x) = e^{-x}\left[\frac{x^\nu}{\nu} + \frac{x^{\nu+1}}{\nu(1+\nu)} + \frac{x^{\nu+2}}{\nu(\nu+1)(\nu+2)} + \cdots\right] = e^{-x} \sum_{j=0}^{\infty} \frac{x^{j+\nu}}{(\nu)_{j+1}}. \tag{A2.4}$$

Modified Bessel Function of the First Kind

I_q is the modified Bessel function of the first kind of order q, and can be expressed as follows:

$$I_q(x) = \frac{(x/2)^q}{\sqrt{\pi}\Gamma(q+1/2)} \int_{-1}^{1} (1-z^2)^{q-1/2} \exp(+xz) dz, \quad q > -1/2. \tag{A2.5}$$

The Bessel function given in (A2.5) can be approximated by the following:

$$I_q(x) = \frac{(x/2)^q}{\Gamma(q+1)} + \frac{(x/2)^{2+q}}{1!\Gamma(q+2)} + \frac{(x/2)^{4+q}}{2!\Gamma(q+3)} + \cdots = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+q}}{j!\Gamma(q+j+1)}. \tag{A2.6}$$

Chapter 4

AMERICAN OPTIONS

4.1. AMERICAN OPTIONS

A wide variety of options trading in exchanges such as commodity options, commodity futures options, call options on dividend paying stocks, put options on dividend or non dividend paying stocks, foreign-exchange options, index options, and so on, are American options and therefore may be exercised optimally before the expiration of the contracts. As a matter of fact, most options trading in exchanges in the US are American options. Due to the wide spread of American options in the marketplace, it is thus important to find appropriate ways to price them. However, the optimality of early exercise presents difficulties in evaluating them. All the pricing models covered in Chapters 2 and 3 are for European options — options which can only be exercised at maturity. The small conceptual difference between American- and European-style options causes a big difference when pricing them. Since American-style options can be exercised before maturity, the actual exercising time is uncertain when the option is bought. No efficient and accurate formulas have been found to price American options. Numerical methods have to be used to price them in practice.

Analytically, the payoff of an American option (PAO) can be formally expressed as follows:

$$PAO = \text{Max}[\omega S(T) - \omega K, 0]\,, \tag{4.1}$$

where $S(T)$ stands for the underlying asset price at any time between the present time t and the maturity time of the option $t^*, t < T < t^*$, ω is a binary operator (1 for a call option and -1 for a put option), and other parameters and functions are the same as in (2.1) and (2.2).

The payoff given in (4.1) is a function of the exercise time T which is chosen to maximize the payoff. Since the optimal time T is uncertain, it is more difficult to price American options than their European counterparts as there is an additional dimension of time involved.

A lot of efforts have been made in both academic and professional studies to price American options, and they are more than enough for a separate volume. Early efforts in pricing American options were taken in the case of discrete dividends for which analytical solutions were found [see Roll (1977), Geske (1979), and Whaley (1981) for such solutions]. These analytical solutions were obtained under special conditions; they cannot be obtained in general. When no analytical solutions are available, numerical methods have to be used. Schwartz (1977), and Brennan and Schwartz (1977, 1978) introduced the finite-difference method, and Cox, Ross, and Rubinstein (1979) introduced the binomial tree method to price American options. Amin and Khanna (1994) showed that the results from the binomial method converge. The binomial method was later extended to include multinomial methods in studies by Boyle (1988), Boyle, Evnine, and Gibs (1989), and others.

Besides the numerical methods above, various quasianalytical solutions have been developed. Geske and Johnson (1984) gave an exact analytical solution for pricing American options, but their formula is an infinite series that can only be evaluated approximately by numerical methods. MacMillan's (1986), and Barone-Adesi and Whaley's (1987) quadratic method is based on exact solutions to the approximated partial differential equations of the options. Various methods using Monte Carlo simulation have been found, the most popular one being Tilley (1992). Using a bundeling method and conditional expectation, Tilley found an efficient way to price American options with Monte Carlo simulation.

As our focus in this book is to illustrate various kinds of exotic options and how to price them, we will simply show how American options can be priced using the intuitive binomial tree method and how the option prices can be approximated with the quadratic methods.

4.2. THE BINOMIAL MODEL

Although there are many option pricing models, we may simply classify them into two major groups according to their assumption of the underlying asset price movement: the discrete model and the continuous model. The most popular continuous model is the Black-Scholes lognormal model we studied in Chapters 2 and 3, in which the underlying asset price is assumed to be lognormally distributed. The most popular discrete model is the binomial model in which the underlying asset price is assumed to either jump or fall. Both models have some advantages and disadvantages. Whereas the Black-Scholes lognormal model is concise in expression, the binomial model is intuitive and can be used to price many kinds of options. The binomial model approaches the lognormal model as the time between each two consecutive periods becomes very small.

The binomial tree model is the most widely used model because of its simplicity, intuitiveness, and convenience in handling many complicated problems. In the binomial option pricing model, the current asset price S is always assumed to follow a binomial process, either going up or falling down. More specifically, the spot price S is assumed to rise $100(u - 1)\%$ or fall $100(1 - d)\%$, both $u > 1$ and $1 > d > 0$ being known with certainty. Graphically, the next-period asset price $S(\tau)$ can be shown in Figure 4.1. In Figure 4.1, p stands for the probability that the spot price will rise and $1 - p$ the probability that the price will fall.

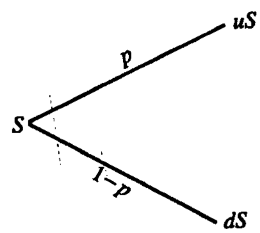

Fig. 4.1. Price movement in the single-period binomial model.

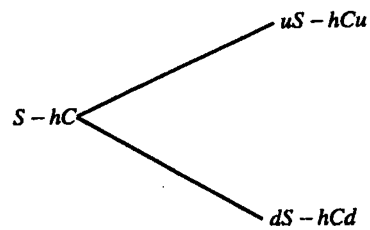

Fig. 4.2. Portfolio value movement in the single-period binomial model.

4.2.1. The Single-Period Binomial Model

With the stock price movement as described in Figure 4.1, we can consider a portfolio including one share of the underlying asset long, and h call options short. Let C stand for the current call option price, the current portfolio value is therefore $S - hC$, and the value of the portfolio in the next period can be shown in Figure 4.2, where

$$C_u = \max(S_u - K, 0) = \max(uS - K, 0)\,,$$
$$C_d = \max(S_d - K, 0) = \max(dS - K, 0)\,,$$

and K is the exercise price of the call option.

Figure 4.2 shows that there are two possible outcomes for the portfolio in the second period resulting from the two possible outcomes of the next-period asset price. It is possible to choose a h so that these two possible outcomes of the portfolio in the next period become the same. Equalizing these two outcomes $uS - hC_u$ and $dS - hC_d$ shown in Figure 4.2 and solving for h yields:

$$h^* = \frac{(u-d)}{C_u - C_d} S. \tag{4.2}$$

The value h^* given in Equation (4.2) is called the optimal hedge ratio. With this hedge ratio, the portfolio will be risk-free because the uncertainty in the two possible outcomes simply disappears as there will be only one certain outcome for the portfolio. Any risk-free portfolio should yield the same return as the risk-free asset, otherwise there would be arbitrage opportunities. Using the arbitrage argument, the risk-free portfolio $S - h^*C$ should yield the same return as the interest rate, thus

$$(S - h^*C)R = uS - h^*C_u, \tag{4.3}$$

where R equals the interest rate in the period under consideration plus one.

Substituting h^* in (4.2) into (4.3) and solving the equation for the current call option price C yields

$$C = \frac{1}{R}\{\pi_d C_d + \pi_u C_u\}, \tag{4.4}$$

where

$$\pi_d = \frac{u-R}{u-d}, \quad \pi_u = \frac{R-d}{u-d}, \quad \text{and} \quad \pi_d + \pi_u = 1.$$

Equation (4.4) is the one-period binomial option pricing formula which gives the value of the European call option with known parameters u, d, S, K, and R. The two intermediate parameters π_d and π_u can be considered as risk-neutral or quasiprobability parameters, for they are positive and add up to one. It is worth noticing that these two risk-neutral probability parameters are independent of the probability p that the spot will rise. With π_d and π_u interpreted as probability parameters in the risk-neutral world, Equation (4.4) can be readily interpreted as the expected payoff of a European call option in the risk-neutral world discounted at the risk-free rate of return.

Example 4.1. As in Example 2.1, the current stock price $S = \$100$, exercise price $K = \$105$, the upward stock price increase $u = 125/100 = 1.25$. If the annual risk-free rate of return is 20%, and the downward decrease of the stock $d = 0.80$, then what is the price of the call option to expire in one year?

To calculate the call option price using Equation (4.4), we need to find the appropriate value of R. As the annual risk-free rate of return is 20%, the risk-free rate of return $R = 1 + 0.20/2 = 1.10$ (one period is half a year). Thus, $C_u = \max(uS - K, 0) = \max(125 - 105, 0) = \$20$, and $C_d = \max(dS - K, 0) = \max(70 - 105, 0) = 0$. Substituting C_u, C_d, $u = 1.25$, $d = 0.70$, and $R = 1.10$ into (4.4) yields $\pi_d = 0.2727$, $\pi_u = 0.7273$, and $C = \$13.22$.

4.2.2. The Multiperiod Binomial Model

The single-period model we considered above is simple and intuitive, yet it is very restrictive. We will now extend it to the multiperiod case. In the standard multiperiod binomial model, the underlying asset price is assumed to follow the same binomial jump-fall process illustrated in Figure 4.3, with n periods from each period to the next. Figure 4.3 describes the possible movements of the underlying asset price in a 4-period binomial tree. There is only one path for the underlying asset price to rise four times consecutively, four paths for it to rise three times and fall once, six paths to rise twice and fall twice, four paths to rise once and fall three times, and one path to fall four times consecutively.

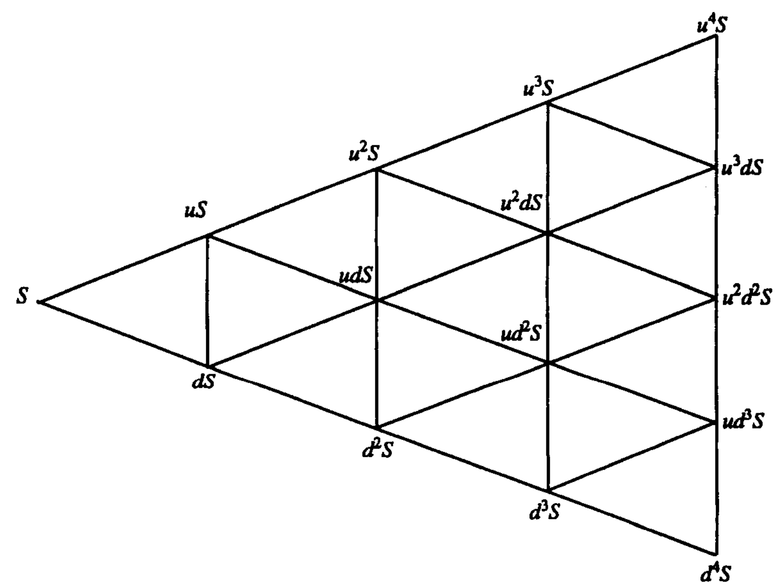

Fig. 4.3. Price movement in a 4-period binomial model.

In constructing Figure 4.3, we used an implicit assumption that an upward move followed by a downward move is the same as a downward move followed by an upward move. This is actually a path-independent assumption. Figure 4.3 is also called a lattice. As vanilla options are path-independent, it is reasonable to use such a method. In general, there are $\binom{n}{i}$ paths for the underlying asset price to rise $0 \le i \le n$ times and fall $n - i$ times out of n, where $\binom{n}{i}$ is the combinatorial number. Thus, we can obtain the payoff of a European call option as follows:

$$PFC(n) = \sum_{i=0}^{n} \pi_u^i \pi_d^{n-i} \max[Su^i d^{n-i} - K, 0], \tag{4.5}$$

where $0 \le i \le n$ is the total number of jumps leading to the final state, and

$$\pi_u = \frac{e^{\tau r/n} - d}{u - d} \quad \text{and} \quad \pi_d = \frac{u - e^{\tau r/n}}{u - d},$$

using the risk-neutral probability given in (4.4), r and τ representing the net interest rate $r = R - 1$ and time to maturity of the option, respectively.

It is obvious that the larger the number of jumps i is, the more the call option is in-the-money. There exists a smallest number of jumps j such that the call option is in-the-money and out-of-the-money with $j - 1$ jumps. Solving the following inequality

$$Sd^{n-j}u^j - K \ge 0$$

yields

$$j = \max\left\{0, \text{ the smallest integer greater than } \frac{\ln(K/S) - n \ln d}{\ln(u/d)}\right\}. \tag{4.6}$$

With the j given in (4.6), we can rewrite (4.5) as follows:

$$PFC(n) = S\left[\sum_{i=j}^{n} \pi_d^{n-i} \pi_u^i d^{n-i} u^i\right] - K\left[\sum_{i=j}^{n} \pi_d^{n-i} \pi_u^i\right]. \tag{4.7}$$

Equation (4.7) gives the expected payoff of a European call option. To find the call option price or the present value of the option, we need to discount the expected payoff at the interest rate r. Discounting the expected payoff in (4.7) with the discounting factor $(1 + r/n)^{-n}$ [see (2.29)] yields the call option pricing formula in a n-period binomial model:

$$C(n) = \frac{1}{(1 + r/n)^n}\left[S \sum_{i=j}^{n} \pi_d^{n-i} \pi_u^i d^{n-i} u^i - K \sum_{i=j}^{n} \pi_d^{n-i} \pi_u^i\right], \tag{4.8}$$

where

$$\pi_u = \frac{(r/n) - d}{u - d}, \quad \pi_d = \frac{u - (r/n)}{u - d},$$

and $r = R - 1$ is the net return of the risk-free asset, j is given in (4.6).

It is straightforward to check that when $n = 1$, Equation (4.8) becomes precisely the same as (4.4), the single-period model pricing formula.

Example 4.2. What is the call option price in Example 4.1 if we choose to use each month as the calculating period?

In Example 4.1, the current stock price $S = \$100$, the exercise price $K = \$105$, the upward stock price increase parameter $u = 1.25$, the downward parameter $d = 0.8$, interest rate $r = 10\%$, and time to maturity $\tau = 0.5$. The time per period is $\tau/n = 0.5/4 = 0.125$. We can find the smallest integer j using (4.6)

$$\begin{aligned} j &= \max\{0, \text{the smallest integer greater than } [\ln(K/S) - n \ln d]/[\ln(u/d)]\} \\ &= \max\{0, \text{the smallest integer greater than } 4 \times 0.35667/0.5798\} \\ &= \max\{0, \text{the smallest integer greater than } 2.46\} = 3\,. \end{aligned}$$

The risk-neutral probability can be found using (4.5):

$$\begin{aligned} \pi_u &= (e^{\tau r/n} - d)/(u - d) \\ &= (e^{0.5\times 0.2/6} - 0.70)/(1.25 - 0.70) = 0.576 \end{aligned}$$

and

$$\pi_d = 1 - \pi_u = 0.424\,.$$

Substituting π_u, π_d, u, d, $r = 0.1$, $n = 4$, and $S = K = \$100$ into (4.8) yields

$$\begin{aligned} C(4) &= \frac{1}{(1 + 0.1/4)^4}\left(100 \sum_{i=3}^{4} \pi_d^{n-i} \pi_u^i 1.25^{n-i} 0.70^i - 105 \sum_{i=3}^{4} \pi_d^{n-i} \pi_u^i\right) \\ &= 0.9056 \times (100 \times 0.22044 - 105 \times 0.083199) = \$12.05\,. \end{aligned}$$

4.3. PRICING AMERICAN OPTIONS IN THE BINOMIAL MODEL

As we discussed in the previous section, the binomial model is a very intuitive method and can be used to price essentially all kinds of derivatives. We illustrated how to price European options using the binomial tree approach in the previous section. We will now show how to use it to price

American options. The time to maturity is often divided into n equal periods $\Delta t = \tau/n$. The current underlying asset price S is assumed to follow a binomial process, or the current asset price can either go up to uS or fall down to dS from one period to the next. The two parameters u and d are specified as follows

$$u(\sigma, \tau, n) = e^{\sigma\sqrt{\tau/n}}, \tag{4.9}$$

and

$$d(\sigma, \tau, n) = e^{-\sigma\sqrt{\tau/n}}, \tag{4.10}$$

where σ and τ are the volatility of the underlying asset and the time to maturity of the option, and n is the number of periods in the binomial model.

We can readily observe that the specification of the upward and downward movement parameters u and d in (4.9) and (4.10) is path-independent, that is, an upward movement followed by a downward movement is the same as a downward movement followed by an upward movement, because the product of u and d is always unitary. As volatility is in the power of the two parameters u and d, it determines the degree of fluctuation in the binomial process.

Using the result in (4.4) and the specification of the two parameters in (4.9) and (4.10), we can obtain the probability that the underlying asset price will go up π_u and the probability that it will fall down π_d in the risk-neutral case:

$$\pi_u = \frac{a(r, n) - d(\sigma, \tau, n)}{u(\sigma, \tau, n) - d(\sigma, \tau, n)}, \; \pi_d = 1 - \pi_u, \tag{4.11}$$

where

$$a(r, \tau, n) = e^{r\tau/n}, \tag{4.12}$$

and d and u are given in (4.9) and (4.10), respectively.

The expression given in (4.12) is actually the continuous compounding factor from one period to the next consecutive period. We can observe directly how the prices are determined in a binomial tree with the movement parameters specified in (4.9) and (4.10) with an example.

Example 4.3. What are the prices of the asset in each period in a 4-period binomial model in four months if the spot asset price is \$100, the volatility of the asset 20%, and the interest rate 10%?

Using (4.9) and (4.10), we can find the upward and downward movement parameters as follows

$$u = e^{\sigma\sqrt{\tau/n}} = e^{0.2\times\sqrt{1/(4\times 3)}} = 1.0594\,,$$

and

$$d = \frac{1}{u} = 0.9439\,.$$

Following the price specification given in Figure 4.1 in a binomial model using the parameters u and d obtained above, we can find the prices of the asset in each period in a 4-period binomial tree. These prices are shown in Figure 4.4.

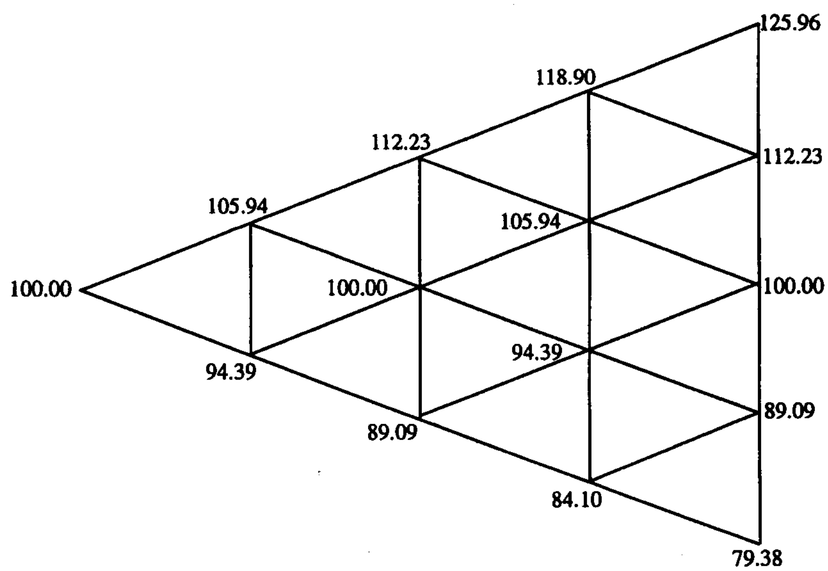

Fig. 4.4. Price movement in a 4-period binomial tree.

4.3.1. The Backward Method

We expressed European option values in terms of $n - j$ terms in Section 4.2 in the multiple-binomial model. When the number of periods n becomes extremely large, the number of terms in the summations in (4.8) also becomes extremely large. In practice, an alternative method called backward calculation is used instead of the expression given in (4.8). The idea of backward calculation is straightforward because we know that the value of any option at its maturity is simply its prespecified payoff. Discounting the expected payoff of the option one period backward using the discounting factor $1/a(r, \tau, n)$ given in (4.5), we can obtain its value one period away from its maturity. Following the same method to discount the value one period back, we can obtain the present value of the option. Let us see how the backward method is used to price a European option in an example.

Example 4.4. The current underlying price $S = \$100$, the volatility of the underlying asset is 20%, the interest rate $r = 10\%$, what is the price of the at-the-money European put option to expire in four months in a 4-period binomial model?

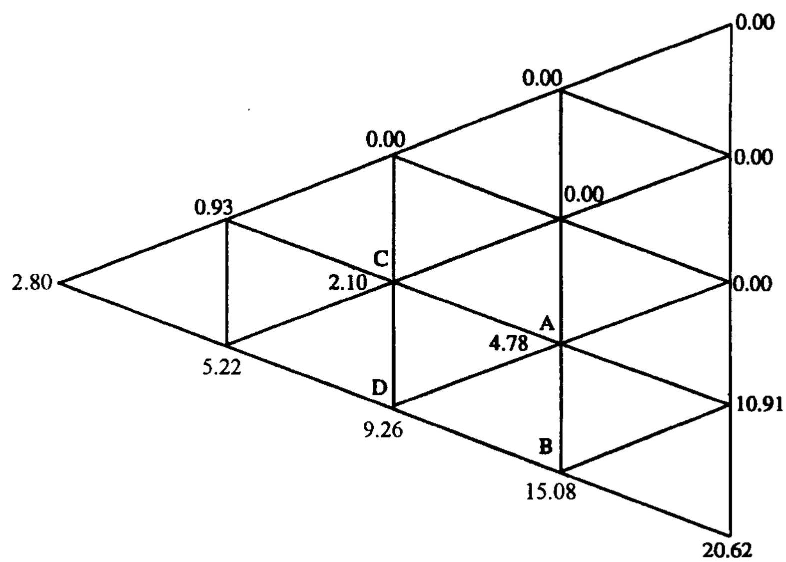

Fig. 4.5. Valuation of a European put option using the backward method.

We can use the prices shown in Figure 4.4 because all the parameters are the same in the two examples. As the option is at-the-money, its strike price is the same as its spot price \$100. Of the five-terminal prices shown in Figure 4.4, only the two smallest prices yield positive payoffs for the put option using the European put option payoff formula given in (2.2):

$$100 - 89.09 = \$10.91, \text{ and } 100 - 79.38 = \$20.62\,.$$

The two payoffs at maturity are shown in Figure 4.5. We can find the risk-neutral probability using (4.11) and (4.12)

$$\pi_u = \frac{a(r,n) - d(\sigma,\tau,n)}{u(\sigma,\tau,n) - d(\sigma,\tau,n)}$$
$$= \frac{\exp(0.5 \times 0.1/4) - 1.0594}{1.0594 - 0.9439} = 0.558 = 55.8\%$$

and

$$\pi_d = 1 - \pi_u$$
$$= 1 - 0.558 = 0.442 = 44.2\%\,.$$

From node B in Figure 4.5, the upward value is \$10.91 and the downward value is \$20.62 with probability 55.8% and 44.2%, respectively. Thus, we can find the expected value in the next period from point B in the risk-neutral case:

$$10.91 \times 0.558 + 20.62 \times 0.442 = \$15.202\,.$$

Discounting the expected value \$15.202 at the discounting factor $1/a(r,\tau,n) = 0.9917 = 99.17\%$ yields

$$15.202 \times 0.9917 = \$15.08$$

which is the value at the point B.

Following the same method, we can obtain the value at node A to be \$4.78. Using the same method to work one period backward from nodes A and B, we can obtain the values at nodes C and D to be \$2.10 and \$9.26. The values of other nonzero nodes are shown in Figure 4.5. Continuing the same method, we can reach the current node and its value is \$2.80. The value of the current node is exactly that of the European put option, which is about two cents greater than its corresponding value \$3.13 using the Black-Scholes formula.

4.3.2. Pricing American Options

The method to price American options is approximately the same as that to price European options as illustrated above in Example 4.4. The only difference between a European option and an American option is that the latter can be exercised optimally before maturity while the former cannot. Because of this important difference, we have to consider the values of earlier exercises before maturity in pricing American options. We will again show how to price American options using the backward method in an example.

Example 4.5. What is the price of an at-the-money American put option to expire in four months in a 4-period binomial model with all other parameters the same as in Example 4.4?

There are three steps to price American options in a binomial model. Firstly, we need to find the payoffs of the American put option at each node in the binomial tree illustrated in Figure 4.4. The payoffs can be easily found using (4.1), and they are shown in Figure 4.6. From Figure 4.6, we can readily observe that the payoffs at the two nodes H and G are \$15.90 and \$5.61, respectively, if the put option is exercised at the third period.

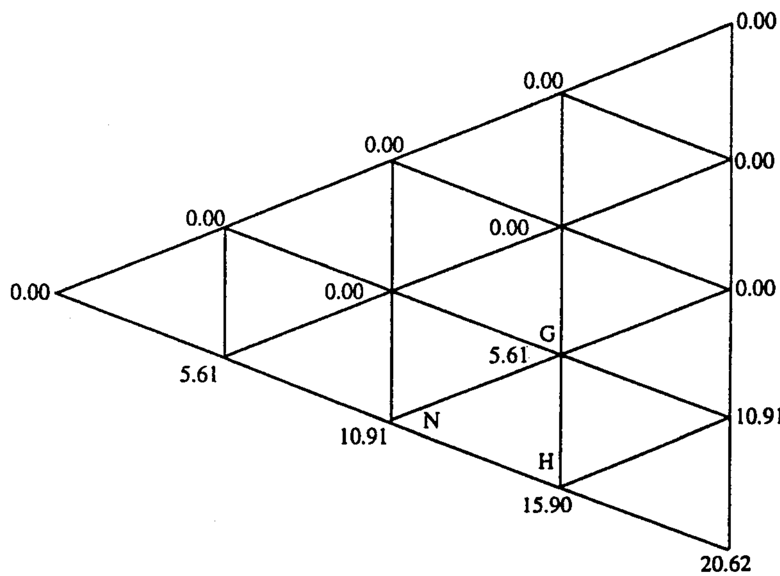

Fig. 4.6. Payoffs of an American put option in a 4-period model.

Secondly, we need to compare the payoff at each node if the option is exercised at that node with that if the option is held to the next period using the backward method shown in Example 4.3, and then choose the larger of the two payoffs. In our example, the payoff at node H is \$15.90 if the option is exercised at H, as shown in Figure 4.6, and the value at node B if the option is held to the next period is \$15.07. Therefore, we should choose the larger value \$15.90 as the value of the American option at that node. Similarly, we can find the value of the American put option at the other nonzero node in the third period to be \$5.61 because it is larger than its corresponding value \$4.78 if the option is held to the fourth period. The

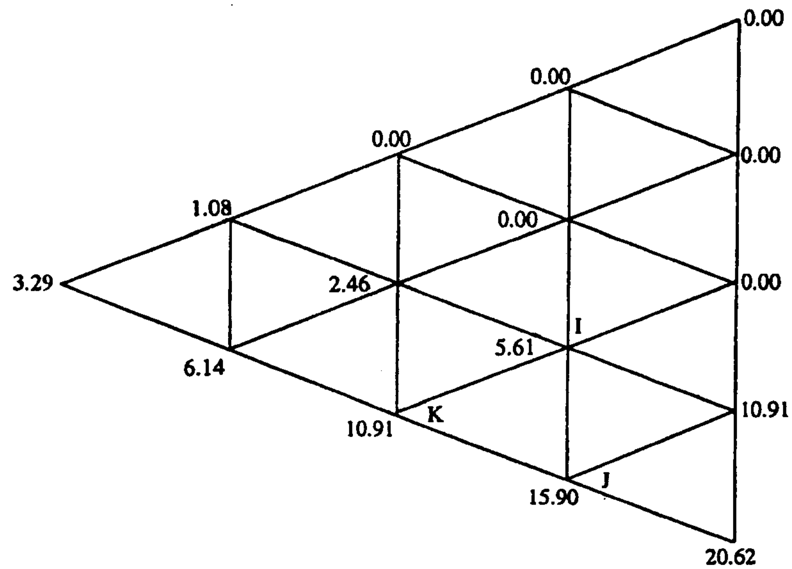

Fig. 4.7. Values of an American put option at various nodes in a 4-period model.

values at all nodes in the second last period are calculated in the same way. These values are shown in Figure 4.7.

Lastly, we find the value of another period back using the backward method, compare it with the payoff at each node, and choose the larger value as the value of the American option at that node. For example, the payoff of the option would be \$5.61 at node G if exercised shown in Fig. 4.6, yet its value would be \$4.78 at node A shown in Fig. 4.5 if held to the next period. Thus we choose the larger \$5.61 as the value shown in Fig. 4.7. Repeating the same method until we reach the current node, we find the value of the American put option. Similarly, we obtain the value at node J is \$15.90. Following the same procedure, we can find the value at node K as \$10.91 which is the larger of \$10.91 at node N in Fig. 4.6 and the discounted value \$10.08 of nodes I & J. In our example, values at all nodes are shown in Figure 4.7 and the value of the option is \$3.29, which is greater than the corresponding European option value \$2.80 shown in Fig. 4.5..

We found the value of an American put option with four periods in Example 4.5. As the time to maturity of the option is four months, each period corresponds to one month. The monthly period was used mainly for illustrative purpose. In practice, it is too infrequent. We can shorten the length of each period and find better value for the option. The method illustrated above can be used for arbitrary number of periods.

4.3.3. Number of Periods and Calculating Time

When the number of periods is increased, the underlying asset price becomes smoother, yet these smoother prices are obtained with a higher cost. As the number of periods becomes large, the time which the computer needs to calculate the option prices becomes extremely long. This is the major drawback of the binomial model. However, as long as the process is convergent, we may obtain reasonably accurate values for the options at moderate rather than very large number of periods.

4.3.4. Convergence From the Binomial Model to the Black-Scholes Model

We described the Black-Scholes model in Chapter 2 and the binomial model in the previous section of this chapter. Cox, Ross, and Rubinstein (1979) showed that under certain conditions, the multiple-binomial model will converge to the Black-Scholes model when the number of periods n approaches infinity. Specifically, if the upward and downward parameters are set as in (4.9) and (4.10), $ud = du = 1$ always hold, the multiple-binomial model will approach the Black-Scholes model as n approaches infinity. This convergent relationship indicates that the two models can yield very similar results when the number of periods becomes very large.

4.3.5. A Pricing Method for Both American and European Options Incorporating Liquidity

We have shown above that the binomial model approaches the Black-Scholes model when the number of periods becomes extremely large. This convergent relationship is intuitive because as the number of periods gets larger and the time to maturity remains the same, the price difference between each two consecutive nodes in each period becomes smaller. In other words, the price difference between each two consecutive nodes approaches zero or the underlying asset price approaches continuity when the number of periods approaches infinity. As the price approaches continuity, the option prices in the binomial model approach those in the Black-Scholes model.

In reality, asset prices are not distributed continuously because price quotations are in sixteenth, thirtysecond, or sixtyfourth of a dollar or one percent in most exchanges, and liquidity of most assets is less than perfect as we have argued in Chapter 3. If we constrain liquidity as an additional factor in pricing options, we cannot increase the number of periods unlimitedly. As liquidity can be understood as the average time between two consecutive transactions of the underlying market, lower liquidity implies smaller number of periods given the same time to maturity.

In Examples 4.4 and 4.5, the European and American put option prices are \$3.0667 and \$3.4069, respectively, with the number of periods $n = 120$. As there are about 120 days in four months, $n = 120$ corresponds to a daily period. Thus these values may be interpreted as the values of the European and American options when the underlying market has such a level of liquidity, when the average time span between two consecutive transactions is about one day.

4.4. AN ANALYTICAL APPROXIMATION

For clarity, we simply consider one, and probably the most popular analytical approximation in pricing American options, the "efficient analytical approximation of American option values" by Barone-Adesi and Whaley (1987). This method is also called quadratic approximation because the approximation uses one root of a standard quadratic equation. The basic idea is that both European and American option values follow the same partial differential equation (PDE) as given in (2.12). In other words, the difference between an American option value and its corresponding European option value also follows the same PDE. The PDE of the early exercise premium is given as follows:

$$\frac{1}{2}\sigma^2 S^2 \epsilon_{ss} - r\epsilon + gS\epsilon_s + \epsilon_t = 0\,, \tag{4.13}$$

where $\epsilon = c(S,\tau) - C(S,\tau)$ is the early exercise premium. $c(S,\tau)$ and $C(S,\tau)$ are the American and European option values, respectively; S, σ, r, g stand for the spot price, standard deviation, interest rate, and the cost of carrying the underlying commodity, respectively; ϵ_s and ϵ_{ss} represent the first- and second-order partial derivatives of the early exercise premium with respect to the underlying asset price; and $\tau = t^* - t$ is the time to maturity of the option.

Equation (4.13) is a second-order partial differential equation. Using the substitution $\epsilon(S,\tau) = \epsilon(S,X) = X(\tau)f(S,X)$ and $X(\tau) = 1 - e^{-r\tau}$, the partial differential equation given in (4.13) can be rearranged as follows after some algebraic simplifications:

$$S^2 f_{ss} + NSf_s - (M/X)f - (1-X)Mf_X = 0\,, \tag{4.14}$$

where $M = 2r/\sigma^2$, $N = 2b/\sigma^2$, f_s and f_{ss} represent the first- and second-order partial derivatives of $f(S,X)$ with respect to S, and f_X is the partial derivative of $f(S,X)$ with respect to X.

The focal point in the quadratic approximation is the assumption that the last term in (4.14) $(1 - X)f_X$ is equal to zero. This assumption is justified in the two polar cases: extremely short-term options ($\tau \to 0$) and extremely long-term options ($\tau \to +\infty$), because for $\tau \to 0$ (resp. $+\infty$), f_X approaches zero (resp. $X \to 1$), the term $(1 - X)f_X \to 0$. Dropping the last term in (4.14), it becomes a standard second-order PDE which can be solved conveniently. Let $f(S, X) = aS^q$, substituting it into (4.14) yields the following

$$aS^q[q^2 + (N-1)q - M/K] = 0$$

or

$$q^2 + (N-1)q - M/K = 0\,, \tag{4.15}$$

since the term aS^q is not zero and the term in bracket must be zero.

Solving the quadratic equation given above (because the first term is not equal to zero) yields

$$q_1 = \left[-(N-1) - \sqrt{(N-1)^2 + 4M/X}\right]/2 \tag{4.16a}$$

and

$$q_2 = \left[-(N-1) + \sqrt{(N-1)^2 + 4M/X}\right]/2\,. \tag{4.16b}$$

As q_1 is always negative for any combination of given parameters, it is not a reasonable solution because $f(S, X)$ will approach infinity as the spot price S approaches zero. With the second root q_2, the American call option price can be expressed as follows:

$$c(S, \tau) = C(S, \tau) + (1 - e^{-r\tau})a_2 S^{q_2}\,, \tag{4.17}$$

where a_2 is a parameter to be determined later.

Equation (4.17) is a formula for American call options. However, the parameter a_2 is not known. We will now illustrate a method to determine its value. There exists one critical commodity price S^* above which an American call option price is equal to $S - X$ and below which its price is given by Equation (4.17). Therefore

$$S^* - X = C(S^*, \tau) + (1 - e^{-r\tau})a_2(S^*)^{q_2}\,, \tag{4.18}$$

and the sensitivity of the American call option with respect to the underlying asset price for prices above S^* and that for prices below S^* should also be equal to each other

$$1 = e^{(b-r)\tau} N[d_1(S^*)] + (1 - e^{-r\tau}) a_2 q_2 (S^*)^{q_2 - 1}, \tag{4.19}$$

where b is the difference between the risk-free rate of return and the payout rate of the underlying asset.

Equations (4.18) and (4.19) are two independent equations for two variables a_2 and S^*, thus a_2 and S^* can be solved simultaneously. As both of these equations are nonlinear equations, we cannot find a closed-form solution for a_2. Solving (4.19) for a_2 and substituting it into (4.18) yields an equation in one unknown variable S^*:

$$S^* - K = C(S^*, \tau) + \{1 - e^{-r\tau} N[d_1(S^*)]\} S^*/q_2 . \tag{4.20}$$

With the solution from (4.20), we can express the approximated American call option price as follows

$$\begin{aligned} c(S, \tau) &= C(S, \tau) + A_2 \left(\frac{S}{S^*} \right)^{q_2}, \text{ when } S < S^* \\ &= S - K, \text{ when } S \geq S^*, \end{aligned} \tag{4.21}$$

where

$$A_2 = \left(\frac{S^*}{q_2} \right) \{1 - e^{-r\tau} N[d_1(S^*)]\} .$$

Similarly, we can express the approximated American put option price as follows

$$\begin{aligned} p(S, \tau) &= P(S, \tau) + A_1 \left(\frac{S}{S^{**}} \right)^{q_1}, \text{ when } S > S^{**} \\ &= K - S, \text{ when } S \leq S^{**}, \end{aligned} \tag{4.22}$$

where

$$A_1 = \left(\frac{S^{**}}{q_1} \right) \{1 - e^{-r\tau} N[-d_1(S^{**})]\} ,$$

S^{**} is the solution of the following nonlinear equation

$$K - S^* = P(S^{**}, \tau) + \{1 - e^{-r\tau} N[-d_1(S^{**}]\} S^{**}/q_1 ,$$

and q_2 is given in (4.16b), $P(S, \tau)$ is the pricing formula for a European put option with spot price S and time to maturity τ.

Example 4.6. Find the prices using the analytical approximation method of at-the-money American call options to expire in three months, half a year, and one year, respectively, given the spot underlying asset price \$100, interest rate 8%, volatility of the underlying asset 20%, yield on the underlying 12%.

Substituting $S = K = \$100$, $r = 0.08$, $g = 0.12$, $\sigma = 0.20$, $\tau = 0.25$, 0.50, and 1.00 into Equation (4.20) and solving for the critical price S^* yields

$S^* = \$114.5439, \$118.2539$, and $\$122.3524$ for $\tau = 0.25, 0.50$, and 1.00, respectively.

Substituting these critical prices into (4.21) yields the American call option prices

$$c(100, 0.25) = \$3.5249,\ c(100, 0.50) = \$4.7241, \text{ and } c(100, 1.00) = \$6.1750\,.$$

As we pointed out earlier, the focal point in the above quadratic approximation is to assume that the last term in (4.14), $(1 - X)f_X$, is equal to zero which could be justified for extremely short-term and long-term options. However, most American options have time to maturity between half a year and one year, and most long term options such as LEAPS (long-term equity anticpation securities) trading in the Chicago Board of Options Exchange and American Stock Exchange have time to maturity of at least two years. For American options with time to maturity of a few years, the above quadratic approximation is likely to generate significant errors. It could be extended to increase the accuracy, yet the discussion is very long and we choose not to include it here.

Example 4.7. Compare the American call option prices in Example 4.6 using the analytical approximation with the corresponding prices using the binomial tree method.

Using the binomial tree method introduced in Section 4.3 with the subperiod number $n = 800$ and other parameters given in Example 4.6, we can find the American call option prices to be \$3.5242, \$4.7091, and \$6.1211 for $\tau = 0.25$, 0.50, and 1.00, respectively. Comparing the prices with those in Example 4.6, we can readily find that the prices using the analytical approximation are higher than those using the binomial method, and the difference becomes larger with longer time to maturity. As a matter of fact, the differences as percentages of the corresponding call option prices using the binomial method are 0.02%, 0.32%, and 0.88% for $\tau = 0.25$, 0.50, and 1.00, respectively.

The results in Example 4.7 confirm our theoretical discussion above that the analytical approximation becomes less accurate for options with longer time to maturity.

4.5. SUMMARY

We introduced the popular binomial model in this chapter and priced American options using the binomial model. The only difference between pricing American and European options is that the larger of the exercise value and the value of holding the options one period further is used as the value at each node for American options, compared to the simple backward calculation starting from the payoff at maturity in the case of European options. It is shown that the binomial model converges the Black-Scholes model for European options when the number of periods approaches infinity. Although the binomial method is an efficient model to price American options, the computation time increases exponentially with the number of periods.

The binomial method is an efficient method to price not only standard American options but also all American-style exotic options. Unless otherwise specified,we will concentrate on European-style exotic options throughout the remaining chapters of this book, to show the characteristics of various exotic options rather than further explaining the corresponding American exotic options.

QUESTIONS AND EXERCISES

Questions

4.1. What is the difference between an American option and its corresponding European option?

4.2. What is the major difficulty in pricing American options?

4.3. What are the existing approaches to price American options?

4.4. How can liquidity be captured in the binomial model?

4.5. How can we make the results from the binomial and the Black-Scholes models comparable?

4.6. What is the major advantage of the binomial model over the Black-Scholes model?

4.7. What is the major shortcoming of the binomial model?

4.8. Under what conditions does the binomial model approach the Black-Scholes model?

4.9. Why is Barone-Adesi and Whaley's approximation to price American options called a quadratic approximation?

4.10. Is it true that Barone-Adesi and Whaley's quadratic approximation is more accurate for short-term options? Why?

Exercises

4.1. Find the perfect hedging ratio and the put and call option prices to expire in one year using the single-period binomial model if the spot stock price is \$100, the strike price \$100, the upward parameter $u = 1.15$, the downward parameter $d = 0.9$, the interest rate 8%.

4.2. Answer the same questions in Exercise 4.1 if d is changed to 0.75.

4.3. Answer the same questions in Exercise 4.1 if the interest rate is changed to 15%.

4.4. Find the perfect hedging ratio and the put and call option prices in the multiperiod binomial model with 4 periods and the time to maturity is one year if the underlying stock price is \$100, the strike price \$100, the upward parameter $u = 1.15$, the downward parameter $d = 0.9$, the interest rate 8%.

4.5. Answer the same questions in Exercise 4.4 if the number of periods is changed to 12 and other factors remain unchanged.

4.6. Answer the same questions in Exercise 4.4 if the number of periods is changed to 52 and other factors remain unchanged.

4.7. Find the price of the straddle including the call and put options in Exercises 4.4 and 4.6.

4.8. Find the price of the straddle including the call and put options in Exercises 4.4 and 4.5.

4.9. Find the price of the strangle including a call with strike price \$100 and a put with strike price \$105 and other parameters the same as in Exercise 4.4.

4.10. Find the price of the strangle including a call with strike price \$105 and a put with strike price \$100 and other parameters the same as in Exercise 4.4.

PART III: PATH-DEPENDENT OPTIONS

The payoff of a vanilla option depends only on the relative magnitude of its underlying asset price at maturity and its strike price, regardless of how the price of the underlying asset at maturity is reached from above, below, or in a zigzag way. Since the way the settlement price is reached represents the change of the value in the underlying asset, it should also be relevant to the option value written on the underlying asset. Path-dependent options are designed to capture how the settlement prices of the underlying assets are reached. There are several kinds of path-dependent options: Asian options, barrier options, lookback options, one-clique options, shout options, forward-start options, and others. These path-dependent options represent the most popular options in the OTC marketplace.

We will analyze most of these path-dependent options in this Part. First of all, we will analyze Asian options. After providing closed-form solutions for various geometric Asian options in Chapter 5, we will concentrate on how to approximated arithmetic averages with their corresponding geometric averages, and then find approximated closed-form formulas for arithmetic Asian options in Chapter 6. This approximation method will be used to obtain approximate pricing formulas for many other kinds of exotic options such as multiple spread options, basket options, and so on. Chapter 7 introduces the concept of flexible Asian options which allocates uneven weights to various observations in the average, determines closed-form solutions to flexible geometric Asian (FGA) options, and provides an analytical approximation for flexible arithmetic Asian options using the method developed in Chapter 6. Comparisons between the approximation and simulation results show that the approximation formula provides very good approximations with reasonable parameters. The approximation formula not only reduces computation time substantially but also makes it possible to express the Greeks in convenient expressions.

Chapter 8 introduces and prices forward-start options: options that do not become valid right after the buying of the contracts but after some specific time in the future within the option's maturity. Chapter 9 introduces and prices one-clique options, and Chapter 10 introduces and prices barrier options, or trigger options, which become a vanilla option or a fixed rebate, depending on whether the trigger is touched or not within the life of the option. Barrier options are getting more popular these days because they are cheaper than their corresponding vanilla options in general, and can better capture participants' specific expectation of the underlying asset movement. If we call the barrier options discussed in Chapter 10 standard or vanilla barrier options, Chapter 11 illustrates and prices other more nonstardard or second-generation barrier options such as Asian barrier options, forward-start barrier options, window-barrier options, double-barrier options, and so on.

Chapter 12 studies lookback options, options whose payoffs depend not only on the terminal underlying asset but also on extrema values: the maximum or minimum value of the underlying asset within the maturity of the option. Besides regular lookback options, we also study partial lookback options which are a percentage of the maximum or a multiple of the minimum values of the underlying asset price within the option maturity.

Chapter 5

ASIAN OPTIONS

5.1. INTRODUCTION

Asian options, or options based on some average underlying asset prices, indices, or rates, are one of the most popular path-dependent options. They are the natural development of vanilla options to capture path-dependence. Generally speaking, an Asian option is an option whose payoff depends on the average price of the underlying asset during a prespecified period within the option's lifetime and a prespecified observation frequency. As there are two kinds of averages — arithmetic and geometric, there are two kinds of Asian options — arithmetic and geometric Asian options. Aside from the path-dependent characteristic, Asian options are less susceptible to possible spot manipulation at settlement, and their payoffs are generally less volatile than vanilla options. As a result, they offer a cheaper way to hedge periodic cash flows and reduces costs for airlines and exporters. Because of these characteristics, Asian options have attracted much attention and their volume has grown rapidly in the OTC marketplace.

Asian options are also called average-price or average-rate options. They also include average-strike Asian options in which strike prices are some averages of the underlying asset prices rather than fixed as in vanilla options. Asian options can be used by corporations with reasonably predictable cash flows to hedge conveniently as a cheaper alternative to a string of vanilla options. Longstaff (1995) pointed out the efficiency of Asian interest-rate options in hedging average costs of funds and provided a closed-form solution for a cap (a string of call options with periodic exercises) using the simple interest-rate model of Vasicek (1977).

The characteristics of arithmetic and geometric averages certainly affect the properties of Asian options. Arithmetic averages are very different from their corresponding geometric averages. The most important difference between them is that geometric averages are lognormally distributed when the underlying asset prices are lognormally distributed whereas arithmetic averages are not lognormally distributed even when the underlying asset prices are. Because of this difference, a closed-from solution for a geometric Asian option is a straightforward extension of the Black-Scholes model. Yet, it is very difficult, if not impossible, to obtain similar results for arithmetic Asian options. We will analyze and price geometric Asian options in this chapter to obtain closed-form solutions and these solutions will be used to approximate arithmetic Asian options in the following chapter.

5.2. GEOMETRIC AND ARITHMETIC AVERAGES

The arithmetic average (AA) of n positive numbers $a_1, a_2, \ldots, a_n$, is defined as

$$AA(n) = \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad i = 1, 2, 3, \ldots, n, \tag{5.1}$$

where n is the number of observations and a_i is the ith observation.

The standard geometric average (GA) of n positive numbers is defined as follows:

$$GA(n) = \left(\prod_{i=1}^{n} a_i\right)^{1/n}, \qquad i = 1, 2, 3, \ldots, n, \tag{5.2}$$

where n is the number of observations and a_i is the ith observation.

The arithmetic average defined in (5.1) is often used in daily life and many other applications where average is concerned. Yet the geometric average is not as popular as its corresponding arithmetic average for it is not so often used. When $n = 2$, for example, the arithmetic average is simply $(2 + 4)/2 = 3$ for the two numbers 2 and 4, and the corresponding geometric average is $\sqrt{2 \times 4} = \sqrt{8} = 2\sqrt{2} = 2.828$ which is smaller than 3, the corresponding arithmetic average. The geometric average is generally smaller than its corresponding arithmetic average with the only exception when all observations are the same.

Suppose that the underlying asset price $S(\tau)$ follows the geometric Brownian motion given in (3.1) with the underlying asset payout rate g. Using the method described in Appendix of Chapter 2, we know that the underlying asset price at any time T between current time t and any time in the future t^* can be expressed

$$S(T) = S\exp\left[\left(r - g - \frac{1}{2}\sigma^2\right)T + \sigma z(T)\right], \tag{5.3}$$

where $t < T < t^*$, and t and t^* stand for the current time and the time to maturity of the option, respectively, and $z(T)$ is a standard Gauss-Wiener process.

The equation given in (5.3) includes the solution in the original Black-Scholes model as a special case when $g = 0$. Suppose that the n prices are taken from the geometric Brownian motion or from (5.3) with observation frequency h, or

$$a_i = S[\tau - (n-i)h] = S\exp\left\{\left(r - g - \frac{1}{2}\sigma^2\right)[\tau - (n-i)h] + \sigma z[\tau - (n-i)h]\right\}, \tag{5.4}$$

where $i = 1, 2, \ldots, n$, and $\tau = t^* - t$ is the time to maturity.

From (5.4), we can see that the averaging period starts with the first observation at $T = \tau - (n-1)h$ and stops at the last observation ($i = n$) at $T = \tau$. The averaging time period is thus from $\tau - (n-1)h$ to τ, or $(n-1)h$.

The payoff of a European-style option based on the geometric average of n prices of the underlying asset can be expressed as follows:

$$PFGA = \max[\omega GA(n) - \omega K, 0], \tag{5.5}$$

where K stands for the strike price of the option, ω is a binary indicator (1 for a call option and -1 for a put option), and $\max[.,.]$ is the same mathematical function as in (2.1) which gives the larger of the two numbers.

If we compare the payoff of a European geometric Asian option given in (5.5) with that of a European call and put options given in (2.1) and (2.2), we can readily obtain (5.5) by simply substituting the underlying asset price at maturity $S(\tau)$ with the geometric average $GA(n)$.

The geometric Asian option defined in (5.5) is very general as it includes standard geometric Asian options with averaging periods starting at the same time with the option, and forward-start or deferred-start geometric Asian options with averaging periods starting sometime in the future within the option's lifetime. This is because the beginning of the averaging period $\tau - (n-1)h$ in our definition can be either zero or nonzero depending upon the number of observations n and the observation frequency h.

5.3. PRICING GEOMETRIC ASIAN OPTIONS

In order to price the geometric Asian option with payoff given in (5.5), we have to know the distribution of the geometric average $GA(n)$, given (i) the time to maturity of the option τ; (ii) the observation frequency h; (iii) the number of observations n; and (iv) the distribution of the underlying asset price in (2.4) or (4.3). To obtain the distribution of $GA(n)$ is almost the same as to find a closed-form solution of a geometric Asian option. For a general formula, we need to consider two cases: first, the averaging period has not started and second, it has already started. Let $0 \leq j \leq n$ be the number of observations which have already been observed. When $j = 0$, the averaging period has not started, when $j = n$, the option is expired, when $1 \leq j < n$, the option is within the averaging period. Clearly, the uncertainty in the geometric average is reduced with more observations.

In order to obtain a distribution function for $GA(n)$, we have to know how various observations $a_i, i = 1, 2, \ldots, n$ are correlated among themselves. The covariance between any two overlapping variables from the standard Brownian motion is a standard result in stochastic calculus. As we need to use this result repeatedly in this book, we express it formally in the following proposition.

Proposition 5.1. The covariance of any two overlapping observations of the standard Gauss-Wiener process equals the smaller of the two corresponding time intervals. Mathematically,

$$\mathrm{Cov}[z(t_i), z(t_j)] = \min(t_i, t_j)\,,$$

where $z(t_i)$ and $z(t_j)$ are two observations from one standard Gauss-Wiener process at two overlapping time points t_i and t_j, and min $(.,.)$ is the mathematical function which gives the smaller of the two arguments.

Proof. See Malliaris and Brock (1982), page 37. □

Substituting the specification (5.4) into the geometric average definition (5.2) and using Proposition 5.1, we can have the following results:

Theorem 5.1. If the averaging numbers are specified as in (5.4), then the natural logarithm of $GA(n)/S$ or $\ln[GA(n)/S]$ is normally distributed with mean $(r - g - \sigma^2/2)T^{sa}_{\mu,n-j} + \ln B^{sa}(j)$ and variance $\sigma^2 T^{sa}_{n-j}$, where

$$B^{sa}(0) = 1,\ B^{sa}(j) = \left(\prod_{i=1}^{j} \frac{S[\tau - (n-j)h]}{S}\right)^{1/n} \quad \text{for} \quad 1 \leq j \leq n\,, \tag{5.6}$$

$$T^{sa}_{\mu,n-j} = \frac{n-j}{n}\left[\tau - \frac{h(n-j-1)}{2}\right] \tag{5.7}$$

$$T^{sa}_{n-j} = \tau\left(\frac{n-j}{n}\right)^2 - \frac{(n-j)(n-j-1)(4n-4j+1)}{6n^2}\,h\,, \tag{5.8}$$

n is the number of observations specified in the contract; h is the observation frequency or the time interval between two consecutive observations; j is the number of observations already passed; $B(j)$ is the geometric average of the gross returns[1] of those observations that have already passed; and τ is the time to maturity of the option.

Proof. Using Proposition 5.1, and the following two summations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \tag{5.9}$$

we could obtain Theorem 5.1 after a few steps of derivations and simplifications. □

The two functions $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} may be interpreted as the effective mean and volatility time functions, respectively, because they largely determine the effective mean and variance of the geometric average. The effective time functions $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} may be better understood if we compare them to the mean and variance of the log-return of the spot price in the Black-Scholes model, $(r - g - \sigma^2/2)\tau$ and $\sigma^2\tau$. It can be readily shown that both these effective time functions are always smaller than the actual time to maturity of the option τ, implying that the actual variance of the log-return of the geometric average is always smaller than that of the spot price at maturity $\sigma^2\tau$. A larger j implies smaller effective volatility time and smaller effective volatility, and therefore a smaller value of the option.

Example 5.1. What are the effective mean and volatility time values if there are 12 observations in the geometric average, observation frequency is monthly, the averaging period has not started, and the time to maturity of the option is one year? What are the effective mean and variance of the geometric average compared to those of the spot price at maturity?

[1] For any two prices $P(t_2)$ and $P(t_1)$ of one asset at time t_1 and t_2, $P(t_2)/P(t_1)$ is called the gross return and $[P(t_2) - P(t_1)]/P(t_1)$ is called the net return of this asset from t_1 to t_2. Net return is the net gain in price of the asset over the original asset price. Obviously, net return is always equal to gross return less one for any asset in the same time period. There is another useful relationship between these two returns: the logarithm of any gross return is approximately equal to the corresponding net return. This is because net returns are normally a small percentage and the relationship is simply the Taylor series expansion $\ln(1 + x) \cong x$, where x represents any net return.

Substituting $n = 12$, $h = 1/12$, $j = 0$, and $\tau = 1$ into (5.7) and (5.8) yields

$$\begin{aligned}
T^{sa}_{\mu,n-j} &= \frac{n-j}{n}\left[\tau - \frac{h(n-j-1)}{2}\right] \\
&= \frac{12-0}{12}\left[1 - \frac{(1/12)\times(12-0-1)}{2}\right] = 0.542 \text{ year}, \\
T^{sa}_{n-j} &= \tau\left(\frac{n-j}{n}\right)^2 - \frac{(n-j)(n-j-1)(4n-4j+1)}{6n^2}h \\
&= 1\times\left(\frac{12-0}{12}\right)^2 - \frac{(12-0)(12-0-1)(4\times 12-4\times 0+1)}{6\times 12^2}\times\frac{1}{12} \\
&= 0.376 \text{ year}.
\end{aligned}$$

Since the effective mean and variance of the geometric average are

$$(r-g-\sigma^2/2)T^{sa}_{\mu,n-j} \quad \text{and} \quad \sigma^2 T^{sa}_{n-j}, \quad \text{and} \quad (r-g-\sigma^2/2)\tau \quad \text{and} \quad \sigma^2\tau$$

for the spot underlying asset at maturity from Theorem 5.1 and $\ln[B^{sa}(0)] = 0$, the effective variance of the geometric average $0.376\sigma^2$ is significantly smaller than the variance of the underlying asset σ^2 for any volatility parameter chosen. The effective mean of the geometric average $0.542 \times (r-g-\sigma^2/2)$ is significantly smaller (resp. greater) than the mean of the spot price $(r-g-\sigma^2/2)$ if the drift μ is greater (resp. smaller) than $\sigma^2/2$.

Example 5.2. How would the results in Example 5.1 change if there are 253 observations in the geometric average, observation frequency is daily and other parameters remain unchanged?

Substituting $n = 253$, $h = 1/253$, $j = 0$, and $\tau = 1$ into (5.7) and (5.8) yields

$$\begin{aligned}
T^{sa}_{\mu,n-j} &= \frac{n-j}{n}\left[\tau - \frac{h(n-j-1)}{2}\right] \\
&= \frac{253-0}{253}\left[1 - \frac{(1/253)\times(253-0-1)}{2}\right] = 0.502 \text{ year} \\
T^{sa}_{n-j} &= \tau\left(\frac{n-j}{n}\right)^2 - \frac{(n-j)(n-j-1)(4n-4j+1)}{6n^2}h \\
&= 1\times\left(\frac{253-0}{253}\right)^2 - \frac{(253-0)(253-0-1)(4\times 253-4\times 0+1)}{6\times 253^2}\frac{1}{253} \\
&= 0.335 \text{ year}.
\end{aligned}$$

Example 5.3. What are the effective mean and variance time functions if observation frequency is continuous and averaging starts from the present time?

The number of observation n approaches infinity and the observation frequency h approaches zero if the observation frequency is continuous. Substituting $n \to \infty$ and $h \to 0$, $nh \to \tau$ (because averaging starts from the present) into (5.7) and (5.8) yields

$$T^{sa}_{\mu,n-j} \to \tau/2 \quad \text{and} \quad T^{sa}_{n-j} \to \tau/3\,.$$

The effective mean time and variance time decreased from 0.542 to 0.502 and from 0.376 to 0.335, respectively. Thus, the effective variance of the geometric average declines as the observation is more frequent.

With the distribution of the geometric average given in Theorem 5.1, we can readily obtain a pricing formula for geometric Asian options in closed-form:

Theorem 5.2. If the averaging numbers are specified in (5.4), then the price of a European geometric average option is given by the following formula:

$$C^{sa} = \omega S A^{sa}(j) e^{-gT^{sa}_{\mu,n-j}} N\left(\omega d^{sa}_{n-j} + \omega\sigma\sqrt{T^{sa}_{n-j}}\right) - \omega K e^{-r\tau} N(\omega d^{sa}_{n-j})\,, \tag{5.10}$$

where

$$A^{sa}(j) = e^{-r(\tau - T^{sa}_{\mu,n-j}) - \sigma^2(T^{sa}_{\mu,n-j} - T^{sa}_{n-j})/2} B^{sa}(j)\,,$$

$$d^{sa}_{n-j} = \left\{\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right) T^{sa}_{\mu,n-j} + \ln[B^{sa}(j)]\right\} \Big/ \left(\sigma\sqrt{T^{sa}_{n-j}}\right)\,,$$

ω is the same binary operator as in (5.5), and all other parameters are the same as in Theorem 5.1.

Proof. Using the distribution of $GA(n)$ given in Theorem 5.1, we can obtain the expected payoff of the geometric Asian option after integration. Discounting the expected payoff at the risk free rate of return r yields (5.10). □

We can easily verify that the Black-Scholes formula is a special case of (5.10). Since a vanilla option is an average option with only one observation, substituting $n = 1$ and $j = 0$ into (5.10) yields $T^{sa}_{\mu,n-j} = T^{sa}_{n-j} = \tau$, $A^{sa}(j) = B^{sa}(j) = 1$, thus

$$d^{sa}_{n-j} = [\ln(S/K) + (r - g - \sigma^2/2)\tau] / (\sigma\sqrt{\tau})\,,$$

and (5.10) collapses to the extended Black-Scholes formula given in (3.2) with the underlying asset payout rate g and the Black-Scholes formula is a special case of (3.2) when $g = 0$.

Formula (5.10) is essentially of the Black-Scholes type and it is of the same complexity level as the Black-Scholes formula. Formula (5.10) includes a memory variable $B^{sa}(j)$ keeping all observations already passed. In order to use (5.10), we only need to calculate the values of the two effective time functions and use them to calculate the argument d^{sa}_{n-j}. We will return to the topic on how to use (5.10) in Section 5.7.

Example 5.4. What are the prices of the call and put options with strike price \$400 to expire in one year, based on the geometric average of the monthly gold price, given the spot gold price is \$390 per ounce, interest rate 7%, yield on the gold zero, and volatility of gold return 20%?

Since the time to maturity, observation frequency, and the number of observations are the same as in Example 5.1, we can use the two effective time values in Example 5.1. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, and $T^{sa}_{n-j} = 0.376$ into (5.10) yields

$$\begin{aligned} d^{sa}_{n-j} &= \left\{\ln\left(\frac{390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.542 + \ln[1]\right\} \Big/ \\ &\quad \left(0.20 \times \sqrt{0.376}\right) = 0.0145\,, \\ C &= SA^{sa}(0)N\left(d^{sa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}}\right) - Ke^{-r\tau}N(d^{sa}_{n-j}) \\ &= 390 \times e^{-0.07(1-0.542)-0.20^2\times(0.542-0.376)/2} \\ &\quad + N(0.1372) - 400 \times e^{-0.07}N(0.0145) \\ &= 390 \times 0.9652 \times 0.5546 - 400 \times 0.9324 \times 0.5058 - \$20.117\,, \end{aligned}$$

and the corresponding geometric put option price can be found by substituting $\omega = -1$ and other parameters into (5.10)

$$\begin{aligned} P &= -SA^{sa}(0)N\left(-d^{sa}_{n-j}\sigma\sqrt{T^{sa}_{n-j}}\right) + Ke^{-r\tau}N(-d^{sa}_{n-j}) \\ &= -390 \times e^{-0.07(1-0.542)-0.20^2\times(0.542-0.376)/2} \\ &\quad + N(-0.1372) + 400 \times e^{-0.07}N(-0.0145) \\ &= -390 \times 0.9652 \times (1 - 0.5546) - 400 \times 0.9324 \times (1 - 0.5058) = \$16.637. \end{aligned}$$

Example 5.5. What are the prices of the call and put options in Example 5.3 if the observation frequency is daily and other parameters remain unchanged?

Since the time to maturity, observation frequency, and the number of observations are the same as in Example 5.2, we can use the two effective time values in Example 5.2. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.502$, and $T^{sa}_{n-j} = 0.335$ into (5.10) yields

$$
\begin{aligned}
d^{sa}_{n-j} &= \left\{\ln\left(\frac{390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.502 + \ln[1]\right\} \bigg/ \\
&\quad \left(0.20 \times \sqrt{0.335}\right) = -0.0019\,, \\
C &= SA^{sa}(0)N\left(d^{sa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}}\right) - Ke^{-r\tau}N(d^{sa}_{n-j}) \\
&= 390 \times e^{-0.07(1-0.502)-0.20^2\times(0.542-0.335)/2} \\
&\quad + N(0.1139) - 400 \times e^{-0.07}N(-0.0019) \\
&= \$18.519
\end{aligned}
$$

and the corresponding geometric put option price

$$
\begin{aligned}
P &= -SA^{sa}(0)N\left(-d^{sa}_{n-j}\sigma\sqrt{T^{sa}_{n-j}}\right) + Ke^{-r\tau}N(-d^{sa}_{n-j}) \\
&= -390 \times e^{-0.07(1-0.502)-0.20^2\times(0.502-0.335)/2} \\
&\quad + N(-0.1139) + 400 \times e^{-0.07}N(0.0019) \\
&= \$16.091\,.
\end{aligned}
$$

5.4. CONTINUOUS GEOMETRIC ASIAN OPTIONS

We defined arithmetic and geometric averages in discrete time in Section 5.2 and obtained a closed-form solution for Asian options based on the discrete geometric averages of the underlying asset prices in Section 5.3. In general, continuous averages are good approximations of discrete averages with very high observation frequency. In this section, we will turn to the concepts of continuous arithmetic and geometric averages and price options based on these averages.

Before we start our analysis, we need to establish a relationship between the number of observation n, the observation frequency h, and the averaging period T_{ap}. If we know any two of these three parameters, we can readily obtain the third using the following identity:

$$
T_{ap} = nh\,, \tag{5.11}
$$

which obviously indicates that the averaging period is zero for vanilla options with only one observation.

The continuous arithmetic average (CAA) of the underlying asset price $S(\tau)$ between any specific time in the future s and the time to maturity of the option t^* is defined as follows:

$$CAA(s,t^*) = \frac{1}{t^* - s}\int_s^{t^*} S(T)dT\,, \tag{5.12}$$

where $S(T)$ is given in (5.3).

As a matter of fact, (5.12) is not really a new definition. The identity (5.11) indicates that the number of observations n has to approach infinity when the observation frequency h approaches zero in the continuous case, given the averaging period $T_{ap} = t^* - s$ fixed. With some simple calculus manipulation, we can show that (5.12) is the limiting result of (5.1) when the observation frequency approaches zero and the number of observations approaches infinity.

Similarly, the continuous geometric average (CGA) of the underlying asset price $S(\tau)$ between any specific time in the future s and the time to maturity of the option t^* is defined as follows:

$$CGA(s,t^*) = \exp\left\{\frac{1}{t^* - s}\int_s^{t^*} \ln[S(T)]dT\right\}, \tag{5.13}$$

where $S(T)$ is given in (5.3).

We can also show that (5.13) is the limiting result of (5.2) when the observation frequency approaches zero and the number of observations approaches infinity, given the averaging period $T_{ap} = t^* - t$ fixed. Again, the continuous geometric average is not a new definition. It is the same geometric average when the observation frequency becomes infinitesimally small.

We can show that the continuous geometric average given in (5.13) is equal to the following if we substitute (5.3) into (5.13)

$$CGA(s,t^*) = S\exp\left\{\left(r - g - \frac{1}{2}\sigma^2\right)\frac{\tau}{2} + \frac{\sigma}{t^* - s}\int_s^{t^*} z(T)dT\right\}, \tag{5.14}$$

where $z(T)$ is the same standard Gauss-Wiener process as in (5.3).

Options can be written on the continuous geometric average given in (5.13) or (5.14). The payoff of such an option can be expressed as follows:

$$PFCGA = \max[\omega CGA - \omega K, 0]\,, \tag{5.15}$$

where K, ω, and $\max[.,.]$ are all the same as in (5.5).

We express the price of an Asian option based on the continuous geometric average in the following theorem.

Theorem 5.3. If averaging is continuous and the averaging period starts at t, the current time, the price of a European continuous geometric average option is given by

$$C^{csa} = \omega S e^{-(r\tau+\sigma^2/6)/2} e^{-g\tau/2} N\left(\omega d^{csa} + \omega\sigma\sqrt{\tau/3}\right) - \omega K e^{-r\tau} N(\omega d^{csa}), \tag{5.16}$$

where

$$d^{csa} = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}\right] \Big/ \left(\sigma\sqrt{\frac{\tau}{3}}\right).$$

Proof. As the continuous geometric average CGA given in (5.13) is a limiting case of the standard discrete geometric average in (5.2) when the observation frequency approaches zero and the number of observations approaches infinity, the pricing formula of the Asian options based on the CGA should also be the limiting case of the pricing formula (5.10). From Example 5.3, we know that the two effective time functions $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} approach $\tau/2$ and $\tau/3$, respectively as $h \to 0$ and $n \to \infty$. Substituting these limiting results into (5.10) yields (5.16). □

Example 5.6. What are the prices of the call and put options in Example 5.4 if the observation frequency is continuous and other parameters remain unchanged?

Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $\tau = 1$ into (5.16) yields

$$\begin{aligned}
d^{csa} &= \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}\right] \Big/ \left(\sigma\sqrt{\tau/3}\right) \\
&= \left[\ln\left(\frac{390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times \frac{1}{2}\right] \Big/ \\
&\qquad \left(0.20 \times \sqrt{1/3}\right) = -0.027434, \\
C &= S e^{-(r\tau+\sigma^2/6)/2} e^{-g\tau/2} N\left(d^{csa} + \sigma\sqrt{\tau/3}\right) - K e^{-r\tau} N(d^{csa}) \\
&= 390 \times e^{-(0.07\times 1+0.20^2/6)} N(0.112727) - 400 \times e^{-0.07} N(-0.027434) \\
&= \$18.440,
\end{aligned}$$

and the corresponding geometric put option price

$$\begin{aligned} P &= -SA^{sa}(0)N\left(-d_{n-j}^{sa} - \sigma\sqrt{T_{n-j}^{sa}}\right) + Ke^{-r\tau}N(-d_{n-j}^{sa}) \\ &= -390 \times e^{-(0.07\times 1+0.20^2/6)}N(-0.1127) \\ &\quad + 400 \times e^{-0.07}N(0.0274) \\ &= \$16.064\,. \end{aligned}$$

Comparing the results with Examples 5.4 and 5.5, we can readily find that the prices of the geometric Asian call and put options with daily observation \$18.519 and \$16.091 are very close to the two corresponding prices with continuous observation \$18.440 and \$16.064, or the differences are only 0.43% and 0.17% of the corresponding option prices with continuous observation. These results show that continuous observation is a very good approximation of daily observation. Table 5.1 lists the number of observations, observation frequency, and the differences between the geometric Asian call option prices with discrete and continuous observations, and the differences as percentages of the continuous geometric call option prices. From this table, we can see that differences between the semi-daily or quarter-daily and continuous observations are as small as 0.2% and 0.1%. Thus continuous observation provides a very good approximation for observation frequency more frequent than quarter-daily.

Table 5.1. Differences between prices of geometric Asian call options with discrete and continuous observations.

	Number of observation				
n	Frequency		Discrete	Diff	% Diff
12	0.083333	monthly	20.117	1.677	9.094360
52	0.019230	weekly	18.825	0.385	2.087852
253	0.003952	daily	18.519	0.079	0.428416
506	0.001976	semi-daily	18.479	0.039	0.211496
1012	0.000988	quarter-daily	18.459	0.019	0.103036
1518	0.000658	per 6-th of a day	18.453	0.013	0.070498
2024	0.000494	per 8-th of a day	18.449	0.009	0.048806
infinity	0	continuous	18.44	0	0

Formula (5.16) is much simpler than the corresponding formula (5.10). As observation frequency gets as frequent as hourly, formula (5.16) can provide almost the same price as (5.10). The limiting value of the effective variance time function $\tau/3$ in Theorem 5.3 explains the use of "the $1/\sqrt{3}$ rule" discussed by Levy and Turnbull (1992). However, the continuous formula (5.16) cannot replace the discrete formula (5.13) because the differences between prices with continuous observation and with less frequent observations can be too large to neglect, as shown in Table 5.1 with monthly or weekly observations.

The Asian call option prices become higher with more frequent observations as shown in Table 5.1. This is because the volatility of the average gets lower with more frequent observations (see Exercise 5.15).

5.5. GEOMETRIC-AVERAGE-STRIKE ASIAN OPTIONS

Theorems 5.2 and 5.3 provide closed-form formulas for European Asian options based on discrete geometric average prices and continuous geometric average prices with fixed strike prices respectively. There are Asian options with strike prices specified as some average prices of the underlying assets. In this section, we will price the Asian options with strike prices specified as the geometric averages of the underlying asset prices.

The payoff of an Asian option with strike price specified as the geometric average of the underlying asset price is given as follows:

$$PFCGA = \max[\omega S(\tau) - \omega GA(n), 0], \qquad (5.17)$$

where $GA(n)$ is given in (5.2), $S(\tau)$ in (5.3), and ω is the same as in (5.5).

In order to price Asian options with strike prices specified as the geometric averages of their underlying assets, we need to know the correlation coefficient between the log-return of the underlying asset and that of the geometric average defined in (5.2).

Theorem 5.4. The correlation coefficient between the log-return of the underlying asset and the log-return of the geometric average defined in (5.2) is

$$\rho = \frac{\{\sigma^2 + [r - g - \frac{1}{2}\sigma^2]\tau\}(\tau - \frac{n-1}{2}h) - [r - g - \frac{1}{2}\sigma^2]^2 \tau T^{sa}_{\mu,n-j}}{\sigma^2 \sqrt{\tau T^{sa}_{n-j}}}, \qquad (5.18)$$

where $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} are the effective mean time and variance time functions given in (5.7) and (5.8), respectively.

Proof. Using the definition of geometric averages in (5.2) and the covariance between any two overlapping observations of the same underlying geometric Brownian motion given in Proposition 5.1, we can obtain (5.18) after some simplifications. □

Example 5.7. What is the correlation coefficient between the log-return of the underlying asset and that of the geometric average if the interest rate is 7%, yield on the underlying asset is zero, volatility of the underlying asset is 20%, time to maturity is one year, and observation frequency is monthly?

We can use the results in Example 5.1, where $T^{sa}_{\mu,n-j} = 0.542$ and $T^{sa}_{n-j} = 0.370$, because the conditions of Example 5.1 are the same as in this example. Substituting $\tau = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.370$, $r = 0.07$, $g = 0.00$, $\sigma = 0.20$ into (5.18) yields $\rho = 0.883 = 88.3\%$.

Example 5.8. What is the correlation coefficient between the log-return of the underlying asset and that of the geometric average if averaging starts from the present and observation is continuous?

Substituting $\lim_{n\to\infty} T^{sa}_{n-j} = \tau/3$ and $\lim_{n\to\infty} T^{sa}_{\mu,n-j} = \tau/2$, and $(n-1)h \to \tau$ into (5.18) yields $\rho_c \cong \sqrt{3}/2 = 0.866 = 86.60\%$. This result indicates that the correlation coefficient with continuous observation is independent of any parameters which affect the correlation coefficient with discrete observation.

With the correlation coefficient given in (5.18), we can obtain the pricing formula for Asian options with payoffs specified in (5.17) in the following theorem.

Theorem 5.5. The price of a European-style Asian option with strike price specified as the geometric average of the underlying asset prices given in (5.2) is

$$AGESTK = \omega S\left[e^{-g\tau}N(\omega D_{g1}) - A^{sa}(j)e^{-gT^{sa}_{\mu,n-j}}N(\omega D_{g2})\right], \qquad (5.19)$$

where

$$D_{g2} = \frac{-\ln[B^{sa}(j)] + (r - g - \frac{1}{2}\sigma^2)(\tau - T^{sa}_{\mu,n-j}) + \sigma^2\left(\rho\sqrt{\tau T^{sa}_{n-j}} - 1\right)}{\sigma\sqrt{\tau_e}},$$

$$D_{g1} = D_{g2} + \sigma\sqrt{\tau_e}, \quad \tau_e = \tau - 2\rho\sqrt{\tau T^{sa}_{n-j}} + T^{sa}_{n-j},$$

$A^{sa}(j)$ and $B^{sa}(j)$ are the same as in (5.10), ρ is given in (5.18), and $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} are the same effective mean and variance functions given in (5.7) and (5.8).

Proof. We known that both $S(\tau)$ and $GA(n)$ are lognormally distributed and they are correlated with the correlation coefficient ρ given in (5.18).

Using the standard bivariate normal distribution (to be illustrated in greater detail at the beginning of Part IV) between $S(\tau)$ and $GA(n)$, we can obtain the expected payoff of $PFCGA$ given in (5.17). The pricing formula (5.19) can be readily obtained by discounting the expected payoff of $PFCGA$ given in (5.17) at the risk-free rate of return using the RNVR. □

Example 5.9. What are the prices of the Asian call and put options with the strike price specified as the geometric average of the underlying asset prices, given other information the same as in Example 5.4?

Because the time to maturity, observation frequency, and the number of observations are the same as in Examples 5.1 and 5.7, we can use the two effective time values in Example 5.1 and the correlation coefficient in Example 5.7. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, $\rho = 0.883$ into (5.19) yields

$$\begin{aligned}
\tau_e &= \tau - 2\rho\sqrt{\tau T^{sa}_{n-j}} + T^{sa}_{n-j} \\
&= 1 - 2 \times 0.883 \times \sqrt{1 \times 0.376} + 0.376 = 0.4583\,, \\
D_{g2} &= \frac{(0.07 - 0 - \frac{1}{2} \times 0.20^2)(1 - 0.542) + 0.20^2 \times (0.883 \times \sqrt{1 \times 0.376} - 1)}{0.20 \times \sqrt{0.4583}} \\
&= 0.042\,, \\
D_{g1} &= D_{g2} + \sigma\sqrt{\tau_e} = 0.034 + 0.20 \times \sqrt{0.4583} = 0.151\,,
\end{aligned}$$

the call average-strike option price is then

$$\begin{aligned}
C &= S\left[e^{-g\tau}N(D_{g1}) - A^{sa}(j)e^{-gT^{sa}_{\mu,n-j}}N(D_{g2})\right] \\
&= 390[N(0.034) - 0.9652 \times N(0.169)] = \$23.76;
\end{aligned}$$

and the corresponding put option price can be found by substituting $\omega = -1$ into (5.19)

$$\begin{aligned}
P &= -S\left[e^{-g\tau}N(-D_{g1}) - A^{sa}(j)e^{-gT^{sa}_{\mu,n-j}}N(-D_{g2})\right] \\
&= -390\,[N(-0.034) - 0.9652 \times N(-0.169)] = \$10.20\,.
\end{aligned}$$

Theorem 5.6. The price of a European-style Asian option with strike price specified as the continuous geometric average of the underlying asset prices given in (5.13) with averaging starting from the current time is

$$ACGESTK = \omega S\left[e^{-g\tau}N(\omega D_{cg1}) - e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(\omega D_{cg2})\right]\,, \tag{5.20}$$

where

$$D_{cg2} = \frac{(r - g - \sigma^2/2)\frac{\tau}{2} + \sigma^2(\frac{\tau}{2} - 1)}{\sigma\sqrt{\tau/3}} \quad \text{and} \quad D_{cg1} = D_{cg2} + \sigma\sqrt{\tau/3}\,.$$

Proof. Using the correlation coefficient between the log-returns of the underlying asset and the geometric average given in Example 5.7, and substituting $\lim_{n\to\infty} T^{sa}_{n-j} = \tau/3$ and $\lim_{n\to\infty} T^{sa}_{\mu,n-j} = \tau/2$ into $A^{sa}(j)$ yields $\lim_{n\to\infty} A^{sa}(j) = e^{-(r\tau+\sigma^2/6)/2}$. Substituting these three limiting values into (4.15) yields (4.16). □

Example 5.10. What are the prices of the call and put options with strike price as the continuous geometric average of the underlying asset prices and other parameters being the same as in Example 5.4?

Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, $\rho = 0.866$ into (5.20) yields

$$\begin{aligned} D_{cg2} &= \frac{(r - g - \sigma^2/2)\frac{\tau}{2} + \sigma^2(\frac{\tau}{2} - 1)}{\sigma\sqrt{\tau/3}} \\ &= \frac{(0.07 - 0 - 0.20^2/2)\frac{1}{2} + 0.20^2(\frac{1}{2} - 1)}{0.20 \times \sqrt{1/3}} = 0.0433\,, \\ D_{cg1} &= D_{cg2} + \sigma\sqrt{\tau/3} = 0.0433 + 0.20 \times \sqrt{1/3} = 0.1588\,, \end{aligned}$$

the price of the call option is

$$\begin{aligned} C &= S\left[e^{-g\tau}N(D_{cg1}) - e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(D_{cg2})\right] \\ &= 390\,[N(0.1588) - 0.9624 \times N(0.0433)] = \$25.45; \end{aligned}$$

and the corresponding put option price can be found by substituting $\omega = -1$ into (5.20)

$$\begin{aligned} P &= -S\left[e^{-g\tau}N(-D_{cg1}) + e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(-D_{cg2})\right] \\ &= -390\,[N(-0.1588) - N(-0.0433)] = \$10.79\,. \end{aligned}$$

5.6. ASIAN GREEKS

Sensitivities are used in most trading strategies. The Greeks of Asian options have very interesting characteristics which those of vanilla options do not possess. As an example, we will simply illustrate the delta of a geometric

Asian option. Taking the first-order partial derivative of (5.10) with respect to S yields the following after simplifications:[2]

$$Delta(GEASIAN) = \omega\left(\frac{n-j}{n}\right) A^{sa}(j)N\left(\omega d^{sa}_{n-j} + \omega\sigma\sqrt{T^{sa}_{n-j}}\right), \quad (5.21)$$

where all parameters are the same as in (5.10).

The delta formula in (5.21) clearly indicates that the delta is affected by the number of observations passed and the actual passed observations. Before averaging starts, the delta is also affected by averaging as the two effective time functions are affected by the total number of observations and the observation frequency. The delta of an Asian option based on continuous geometric average is also the limiting case of (5.21) as the number of observations approaches infinity and the observation frequency approaches zero. The limiting result can be given as follows:

$$Delta(CGEASIAN) = \omega e^{(-r\tau+\sigma^2/6)/2} N(\omega D_{cg1}), \quad (5.22)$$

where all parameters are the same as in (5.20).

Example 5.11. What are the deltas of the call and put options in Example 5.4?

Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, $\rho = 0.883$, $d^{sa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}} = 0.1372$ (see Example 5.4) into (5.21) yields

$$\begin{aligned}\text{delta of the call} &= \left(\frac{n-j}{n}\right) A^{sa}(j)N\left(d^{sa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}}\right)\\ &= \frac{12-0}{12} \times e^{-r(\tau-T^{sa}_{\mu,n-j})-\sigma^2(T^{sa}_{\mu,n-j}-T^{sa}_{n-j})/2} N(0.1372)\\ &= e^{-0.07(1-0.542)-0.20^2(0.542-0.376)/2} N(0.1372)\\ &= 0.5353 = 53.53\%,\end{aligned}$$

[2]Similar to the identity given in (3.30) of Chapter 3 (also see Exercise 3.4 of Chapter 3), we have the following identity for geometric Asian options

$$SA^{sa}(j)f\left(d^{sa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}}\right) = Ke^{-r\tau} f(d^{sa}_{n-j}),$$

which is used to simplify the delta expression and other geometric Asian Greeks. See Exercise 5.1.

$$\begin{aligned}\text{delta of the put} &= -\left(\frac{n-j}{n}\right)A^{sa}(j)N\left(-d^{sa}_{n-j}-\sigma\sqrt{T^{sa}_{n-j}}\right)\\ &= -\frac{12-0}{12}\times e^{-r(\tau-T^{sa}_{\mu,n-j})-\sigma^2(T^{sa}_{\mu,n-j}-T^{sa}_{n-j})/2}N(-0.1372)\\ &= e^{-0.07(1-0.542)-0.20^2(0.542-0.376)/2}N(-0.1372)\\ &= 0.4299 = 42.99\%\end{aligned}$$

Example 5.12. What are the deltas of the call and put options in Example 5.6 with continuous observation?

Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} \to \tau/2$, $T^{sa}_{n-j} \to \tau/3$, $\rho = 0.866$ into (5.22) yields

$$\begin{aligned}\text{delta of the call} &= e^{-(r\tau+\sigma^2/6)/2}N(D_{cg1})\\ &= e^{-(0.07\times 1+0.20^2/6)^2}N(0.1127)\\ &= 0.5244 = 52.44\%\,,\\ \text{delta of the put} &= -e^{-r\tau+\sigma^2/6)/2}N(-D_{cg1})\\ &= -e^{-0.07\times 0.20^2/6)/2}N(-0.1127)\\ &= 0.4380 = 43.8\%\,.\end{aligned}$$

Zhang (1995c) provided specific expressions for other Greeks of flexible geometric Asian options. They can be obtained by taking partial differential derivatives of the pricing formula. We skip these expressions in this book.

5.7. AN APPLICATION

In this section, we are going to provide an example to show how to apply geometric Asian options in practice. We argued that Asian options can provide a cheaper way to hedge the underlying asset with periodic cash flows. The example shows how Asian options can specifically hedge the foreign currency risk more efficiently than a string of standard options.

Example 5.13. The current US dollar/Japanese yen exchange rate is ¥ 85 per dollar. Because of its huge trade surplus with the US, many people still predict that the yen will appreciate further against the dollar. The prospective appreciation of the yen will create risks for American importers of Japanese products because they have to pay more for the same products. Suppose that an importer has to import Japanese products at the end of each month for one year. He has to buy twelve vanilla yen call options with the same strike price ¥85 per dollar to hedge the appreciation of the yen. In order words, if the dollar/yen rate falls below 85, he can exercise his power of these yen call options at ¥85 per dollar. What is the total cost of buying twelve consecutive yen call options with strike price ¥85 per dollar, given the US interest rate $r = 6\%$, Japanese interest rate $r_f = 3\%$, volatility of the dollar/yen exchange rate 18%?

Substituting $S = K = 1/85 = \$0.011765$, $r = 0.06$, $g = r_f = 0.03$, $\tau = 1/12$, $2/12$, $3/12$, $4/12$, $5/12$, $6/12$, $7/12$, $8/12$, $9/12$, $10/12$, $11/12$, and 1 into the extended vanilla option pricing formula in (3.2) yields the call option prices for ¥1 million (in the order of increasing time to maturity): \$257.87, \$372.23, \$462.62, \$540.45, \$610.16, \$674.02, \$733.40, \$789.18, \$841.98, \$892.25, \$940.34, and \$986.50. The total cost of these call options is \$8,101.

Instead of buying twelve consecutive call options, the importer can buy an Asian call option based on geometric average with monthly observation as we analyzed in Section 5.3. Given the same information as above, we can obtain the price of the geometric Asian option easily for ¥1 million: \$482.91.

The cost (\$482.91) is significantly smaller than the total cost of twelve consecutive call options (\$8,101). Therefore, Asian options can provide a cheaper way to hedge the underlying assets.

5.8. CONCLUSIONS

A significant portion of the materials in this chapter is based on Zhang (1994a). A few other authors, like Kemna and Vorst (1990), Turnbull and Wakeman (1991), among others, have obtained similar pricing formulas for geometric Asian options as given in Theorem 5.2. However, Theorem 5.2 is very general and intuitive as it is of the Black-Scholes type and includes passed observations as well as expected observations.

We concentrated on geometric Asian options in this Chapter and found closed-form solutions for Asian options based on standard discrete geometric averages, continuous geometric averages, and Asian options with strike prices specified as geometric averages. These formulas are of the Black-Scholes type and can be used very conveniently. Although these formulas can be used directly for geometric Asian options, geometric Asian options are still not as popular as their arithmetic Asian options. However, the study of geometric Asian options can serve at least three purposes: actual use of geometric Asian options, control variate for arithmetic Asian options in Monte Carlo simulations, and basis for approximating arithmetic Asian options, as we will show in the following chapter.

Asian options can also be written on moving averages or averages in process. These kinds of Asian options can easily be modified to American-style Asian options. They can be priced using the binomial model in Chapter 4. As most Asian options are of European-style, we leave this topic without further pursuing it.

QUESTIONS AND EXERCISES

Questions

5.1. What are path-dependent options?

5.2. Are Asian options always cheaper than their corresponding vanilla options?

5.3. What are geometric averages?

5.4. Are arithmetic averages always greater than their corresponding geometric averages? Why?

5.5. Are arithmetic Asian options always more expensive than their corresponding geometric Asian options? Why?

5.6. What does effective time value in (5.8) mean?

5.7. What is "the $1/\sqrt{3}$ rule"?

5.8. Are geometric Asian options always cheaper or more expensive with continuous observation than with discrete observation given other parameters unchanged?

5.9. What is the major advantage of the geometric Asian option pricing formula with continuous observation given in (5.10) over that given in (5.16)?

5.10. Is the correlation coefficient between the log-returns of the underlying asset and the geometric average always constant?

Exercises

5.1. Show the identity: $SA^{sa}(j)f(d_{n-j}^{sa} + \sigma\sqrt{T_{n-j}^{sa}}) = Ke^{-r\tau}f(d_{n-j}^{sa})$.

5.2. Show that the continuous geometric average given in (5.13) is the limiting case of the discrete geometric average (5.2) when n approaches infinity and the averaging period $t^* - s$ is fixed.

5.3. What are the effective mean and volatility time values if there are 52 observations in the geometric average, observation frequency is weekly, the averaging period has not started, and the time to maturity of the option is one year?

5.4. What are the effective mean and variance of the geometric average compared to those of the spot price at maturity in Exercise 5.3?

5.5. Answer the questions in Exercises 5.3 and 5.4 if the observation is bimonthly?

5.6. Find the prices of the geometric Asian call and put options with weekly observations to expire in half a year, with strike price $460, if the current underlying index is $450, interest rate is 8%, yield on the underlying index is 4%, volatility of the index is 25%?

5.7. Answer the same questions in Exercise 5.6 if the observation frequency is bimonthly and other parameters are the same as in Exercise 5.6?

5.8. Answer the same questions in Exercise 5.6 if the observation frequency is continuous and other parameters are the same as in Exercise 5.6?

5.9. Show that the correlation coefficient between the spot at maturity $S(\tau)$ and the geometric average are bivariate log-normally distributed with the correlation coefficient given in (5.18)

5.10. Find the correlation coefficient between the log-returns of the underlying asset and the geometric average with geometric strike prices with weekly observations, time to maturity 20 weeks, interest rate 7%, yield on the underlying asset 2%, and spot price $100.

5.11. Find the price of the European Asian options with geometric strike prices with weekly observations, time to maturity 20 weeks, interest rate 7%, yield on the underlying asset 2%, and spot price $100.

5.12. Answer the same questions in Exercise 5.10 with continuous observation and other parameters remain the same as in Exercise 5.10.

5.13. Find the deltas of the call and put options in Exercise 5.6.

5.14. Find the deltas of the call and put options in Exercise 5.8.

5.15. Show that the volatility of the geometric average gets lower with more frequent observations.

Chapter 6

APPROXIMATING ARITHMETIC ASIAN OPTIONS WITH CORRESPONDING GEOMETRIC ASIAN OPTIONS

6.1. INTRODUCTION

The majority of Asian options trading in the OTC marketplace are European-style options based on arithmetic average prices of the underlying assets. There exists one problem in pricing these options: their prices cannot be expressed in closed-forms under the same conditions of lognormality and risk-neutrality. This is because the arithmetic average is not lognormally distributed even when all the individual prices follow a lognormal process. However, closed-form solutions exist for European-style Asian options based on geometric average prices of the underlying assets, as we showed in Chapter 5, because the geometric averages are lognormally distributed if the individual prices are lognormally distributed. Closed-form solutions for options based on geometric average prices are often used as initial values to price arithmetic Asian options numerically employing the Monte Carlo control-variate method. This was first applied in financial economics by Boyle (1977).

Attempts have been made to approximate values of arithmetic Asian options using the closed-form solutions of geometric Asian options. Using a lognormal distribution to approximate the arithmetic average of lognormal variates, Turnbull and Wakeman (1991) provided an algorithm for pricing European-style arithmetic Asian options. The accuracy of their algorithm can be very high when the number of variates in the averaging period is either very large or very small, yet it becomes rather low when the number of variates is within a certain range. Levy (1992) approximated arithmetic Asian options using the otherwise identical geometric averages which have the same first two moments[1] with the corresponding arithmetic averages. Vorst (1992) established an approximation formula of the Black-Scholes type, yet his formula depends entirely on an inequality that is essentially an alteration of the fact that a geometric mean is always a lower bound for its corresponding arithmetic mean. Although the results can be reasonably accurate when the difference between an arithmetic mean and its corresponding geometric mean is very small, they become inaccurate when the difference is large. The existing studies are obtained either from an arbitrarily fixed number of equalized moments of an arithmetic average and its corresponding geometric average [the first two moments in the case of Levy (1992) and the first four moments in the case of Turnbull and Wakeman (1991)] or from an arbitrarily reduced effective strike price as in the case of Vorst (1992). The results to be shown in this chapter are firmly derived from mathematical approximation using Taylor's series expansion.

Yor (1992) derived formulas for the Laplace transform of an arithematic Asian Option, yet there have been no numerical studies of the inversion of this Laplace transform, and no simple analytical inversion has been found. Rogers and Shi (1995) provided a method for computing lower bounds on the price of an Asian option. Chalasani, Tha and Varikaoty (1997) improved Rogers and Shi (1995) by choosing alternative base random variables.

There are other studies on arithmetic Asian options. Ruttiens (1990) discussed how to price arithmetic Asian options using Monte Carlo simulation and Kemna and Vorst (1990) studied the European-style Asian options and priced them using the same method. The standard control variate is the otherwise identical geometric average option.

Little attention has been paid in literature to study what affects the difference between an arithmetic mean and its corresponding geometric mean. Without a thorough understanding of this difference, it would be hard to approximate mathematically arithmetic Asian options with geometric Asian options with known accuracy. Using a generalized mean measure which includes arithmetic means, harmonic means, quadratic means and geometric means as special cases, and the maximum and minimum observations as limiting cases, we will first study how the difference between an arithmetic

[1] Moments are statistical concepts. The ith moment of a random variable x is simply the weighted summation of x^i weighted with the density function or the probability distribution function of x. The ith central moment of the same random variable x is the weighted summation of $[x - E(x)]^i$ weighted with the density function or the probability distribution function of x, where $E(x)$ is the mean of x. Mean price or return, for example, is the first moment of price or return. Variance is actually a second-central moment. Two other popular statistical terms, skewness and kurtosis are determined by the third- and fourth-central moments, respectively.

means and its corresponding geometric mean is determined. Then we will approximate the arithmetic mean with its corresponding geometric mean. Finally, we will provide a closed-form formula of the Black-Scholes type for an arithmetic Asian option. Our examples show that the results are quite efficient and accurate.

Since the general mean used in this chapter includes both arithmetic and geometric means as special cases, and the maximum and minimum observations as limiting cases, the general mean has a potential to connect Asian options and lookback options, and possibly generate new forms of options between them.

6.2. THE GENERAL MEAN

All analyses in this and some other chapters are based on the general mean that is defined as follows:

$$M(\gamma|a) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^\gamma\right)^{1/\gamma} = \left(\frac{\sum a^\gamma}{n}\right)^{1/\gamma}, \tag{6.1}$$

where a_i's are all positive real numbers, $i = 1, 2, 3, \ldots, n, n$ represents the number of observations and γ is a real number that determines the characteristic of the general mean $M(\gamma|a)$. For simplicity, we use $M(\gamma)$ to represent $M(\gamma|a)$ unless it is necessary to use $M(\gamma|a)$ directly.

We can readily check Equation (6.1) in the trivial case when all the n positive numbers are equal to one another, or $a_i = \bar{a}$ for $i = 1, 2, 3, \ldots, n$. Substituting $a_i = \bar{a}$ into the general mean given in (6.1), we can easily obtain $M(\gamma|\bar{a}) = \bar{a}$, regardless of the parameter γ. We consider the nontrivial cases throughout this chapter for numbers not equal to each other, or the standard deviation of the given n numbers is greater than zero.

We first examine a few special cases of the general mean given in (6.1). When $\gamma = 1$, $M(1)$ is exactly the arithmetic mean $AA(a)$; when $\gamma = 2$, $M(2)$ becomes the quadratic mean or the root-mean square; when $\gamma = -1$, $M(-1)$ is the harmonic mean. As γ approaches zero, we have the following limiting result

$$\lim_{\gamma\to 0} M(\gamma) = \lim_{\gamma\to 0}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^\gamma\right)^{1/\gamma} = \left(\prod_{i=1}^{n} a_i\right)^{1/n}, \tag{6.2}$$

which is precisely the geometric mean $GA(a)$ defined in (5.2). Therefore, we define $M(0) = GA(a)$.

The following are two interesting limiting cases

$$\lim_{\gamma \to +\infty} M(\gamma) = \lim_{\gamma \to +\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i^{\gamma} \right)^{1/\gamma} = \max(a_1, a_2, \ldots, a_n) \tag{6.3}$$

and

$$\lim_{\gamma \to -\infty} M(\gamma) = \lim_{\gamma \to -\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i^{\gamma} \right)^{1/\gamma} = \min(a_1, a_2, \ldots, a_n), \tag{6.4}$$

where max(...) and min(...) are functions that give the largest and the smallest numbers of a_1 through a_n, respectively.

Table 6.1. Special and limiting cases of the general mean.

γ value	$M(\gamma)$	Results
$-\infty$	$\text{Min}(a)$	Degenerated to the Minimum Number
-1	$\frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$	Harmonic Mean
0	$G = \left(\prod_{i=1}^{n} a_i \right)^{1/n}$	Geometric Mean
1	$A = \frac{1}{n} \sum_{i=1}^{n} a_i$	Arithmetic Mean
2	$\left(\frac{1}{n} \sum_{i=1}^{n} a_i^2 \right)^{1/2}$	Quadratic Mean
$+\infty$	$\text{Max}(a)$	Degenerated to the Maximum Number

where Max(a) and Min(a) are functions which give the largest and the smallest numbers from a_1 through a_n respectively.

Table 6.1. shows these special and limiting cases of the general mean. We can observe that the two limiting cases when γ approaches infinity are two degenerated means in the sense that the general mean degenerates into one single special observation. As the minimum observation, the harmonic mean, the geometric mean, the arithmetic mean, the quadratic mean, and the maximum observation are obtained when the mean parameter $\gamma \to \infty$, $= -1$, $\to 0$, $= 1, 2$, and $\to +\infty$, respectively, we may consider the mean parameter γ as a weighting parameter in the general mean which allocates heavier weights to larger numbers. As the general mean is a rather complicated function, it is difficult for us to see clearly how it changes with

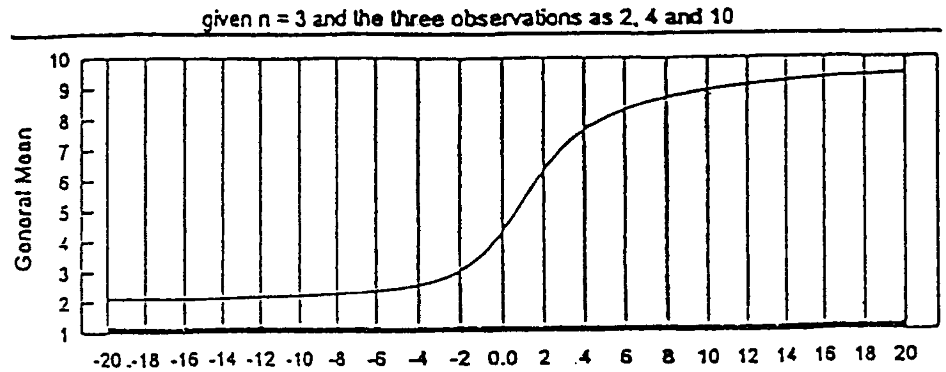

Fig. 6.1. The general mean with various weight parameters.

various γ values precisely. Figure 6.1 depicts $M(\gamma)$ for various γ values, given $n = 3$, $a_1 = 2$, $a_2 = 4$, and $a_3 = 10$. It clearly demonstrates that the general mean approaches the minimum value $\min(2, 3, 10) = 2$ as γ approaches negative infinity, and it approaches the maximum value $\max(2, 3, 10) = 10$ as γ approaches positive infinity.

Example 6.1. Find the harmonic, geometric, arithmetic, and quadratic means given $n = 3$ and $a = (2, 3, 10)$.

The harmonic average (HA) is

$$HA = \frac{3}{(1/a_1) + (1/a_2) + (1/a_3)} = \frac{3}{(1/2) + (1/3) + (1/10)} = 3.2143\,;$$

the geometric average (GA) is

$$GA = (a_1 \times a_2 \times a_3)^{1/3} = (2 \times 3 \times 10)^{1/3} = 3.9149\,;$$

the arithmetic average (AA) is

$$AA = \frac{a_1 + a_2 + a_3}{3} = \frac{2 + 3 + 10}{3} = 5\,;$$

and the quadratic average (QA) or the square-root mean is

$$QA = \left(\frac{a_1^2 + a_2^2 + a_3^2}{3}\right)^{1/2} = \left(\frac{2^2 + 3^2 + 10^2}{3}\right)^{1/2} = 6.0828\,.$$

The results in the above example show that the harmonic, geometric, arithmetic, and quadratic means with the parameter $\gamma = -1, 0, 1$, and 2 increase from 3.2143 to 3.9149, 5, and 6.0828, respectively.

One interesting observation about the harmonic mean is that its reciprocal is actually the arithmetic mean of the reciprocals of all the numbers under consideration. In our above example, the reciprocal of the harmonic mean is

$$\frac{1}{HA} = \frac{1}{3.2143} = 3.111 = AA\left(\frac{1}{a_1},\ \frac{1}{a_2},\ \frac{1}{a_3}\right) = \frac{1}{3}\left(\frac{1}{a_1},\ \frac{1}{a_2},\ \frac{1}{a_3}\right)$$
$$= \frac{1}{3}\left(\frac{1}{2},\ \frac{1}{3},\ \frac{1}{10}\right) = 3.111\,.$$

The above observation is useful because it can be used in calculating the average exchange rates related to US dollars. It is well known that most exchange rates are expressed in per US dollar basis in the US with the exception of British pound which is expressed in US dollar per pound. To compare the arithmetic average exchange rate of the US dollar/British pound with other exchange rates in dollar basis, we simply find the reciprocal of the harmonic mean of the dollar/pound exchange rate in the same time period.

6.3. PROPERTIES OF THE GENERAL MEAN

6.3.1. Monotonicity

We can see the characteristic of the general mean function from its first derivative with respect to the parameter γ,

$$M'(\gamma) = M(\gamma)\,\frac{\gamma \sum a^{\gamma} \ln a - \sum a^{\gamma} \ln(\sum a^{\gamma}/n)}{\gamma^2 \sum a^{\gamma}}\,, \tag{6.5}$$

where $\sum a^{\gamma} = \sum_{i=1}^{n} a_i^{\gamma}$ and $\sum a^{\gamma} \ln a = \sum_{i=1}^{n} a_i^{\gamma} \ln a_i$.

It can be shown that the first derivative of the general mean at $\gamma = 0$ is of the following value

$$M'(0) = \frac{GA(a)}{2}\, V(\ln a) > 0\,, \tag{6.6}$$

where $V(\ln a) = E[\ln a - AA(\ln a)]^2 = \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[\ln(a_i/a_j)]^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}[\ln(a_i/a_j)]^2$, which is the variance of the logarithm of the n given positive numbers with equal weights, and $GA(a)$ is the geometric mean of these n positive numbers.

From the statistical meaning of the variance of $V(\ln a)$, we could understand that it measures the degree of dispersion of these positive numbers. $Var[V(\ln a)]$ becomes zero when all the a_i's are the same.

It can also be shown that the second-order derivative of the general mean at zero is given as follows:

$$M''(0) = GA(a)\left[\frac{1}{2}V(\ln a)\right]^2 > 0\,, \tag{6.7}$$

which indicates that the general mean is always convex at $\gamma = 0$ for any set of positive numbers which are not equal to one another.

Using the two derivatives at $\gamma = 0$ given in (6.6) and (6.7), we obtain the following results.

Lemma 6.1. In the neighborhood of $\gamma = 0$, $M'(\gamma|a) > 0$ for $a > 0$ (or $a_i > 0$ for $i = 1, 2, \ldots,$ and n).

Proof. Using (6.7), we obtain $M'(\gamma) > M'(0) > 0$ for $\gamma > 0$. We can readily show for any $\gamma \neq 0$, $M(\gamma|a) = 1/M(-\gamma|\frac{1}{a})$ is always true. For any $\gamma < 0$, taking derivative with respect to γ to both sides of the identity $M(\gamma|a) = 1/M(-\gamma|\frac{1}{a})$ yields

$$M'(\gamma|a) = M'\left(-\gamma|\frac{1}{a}\right) \Big/ M^2\left(-\gamma|\frac{1}{a}\right)\,,$$

which is always positive because $-\gamma > 0$ and $1/a > 0$ for any $\gamma < 0$ and $a > 0$. □

Lemma 6.1 states that the general mean is an strictly increasing function of the parameter γ in the neighborhood of $\gamma = 0$. Actually, the general mean is a strictly increasing function for any real value of γ. The following theorem guarantees this result.

Theorem 6.1 (Schlomilch 1858): $M(\gamma|a) < M(s|a)$ for any $\gamma < s$.

Proof. The theorem is readily proven using Holder's inequality for any $0 < \gamma < s$. See Theorem 16 of Hardy, Littlewood, and Polya (1934). For $\gamma < s < 0$, the result can be obtained using the same method used in the second part of the proof in Lemma 6.1. □

The strictly increasing property of the general mean guarantees that we can safely regard γ as a weight parameter that allocates heavier weights to the larger numbers under consideration.

Proposition 6.1. An arithmetic mean is greater than or equal to its corresponding geometric mean.

Proof. If $a_i = \bar{a}$ for all i's $(i = 1, 2, 3, \ldots, n)$, then $AA(a) = GA(a) = \bar{a}$. Otherwise, $AA(a) = M(1|a) > M(0|a) = GA(a)$ using Lemma 6.1. □

Proposition 6.1 guarantees that the geometric mean is a lower bound of the arithmetic mean for any set of positive real numbers. The difference between any arithmetic mean and its geometric mean, in general, depends on how much the given numbers are different from one another, or on the standard deviation, skewness, kurtosis, and other higher moments of these numbers. We will explore how this difference is affected and find bounds for this difference in the following sections.

6.3.2. Brownian Motion

In the general case, we have to specify a_i as in (3.1). With the specification given in (3.1), $V(\ln a)$ becomes the variance of the n returns of the underlying asset. As all the a_i's are lognormally distributed and mutually correlated, $V(\ln a)$ is also stochastic. In principle, we can obtain the density function of $V(\ln a)$ using the joint lognormal distribution of the a_i's, but this calculation process is rather complicated and a compact expression is very unlikely. However, the first two moments of $V(\ln a)$ can be expressed rather conveniently in compact forms.

Proposition 6.2. If all the n observations follow the Brownian motion specified in (3.1), the first two moments of $V(\ln a)$ at $\gamma = 0$ defined in (6.6) can be expressed as:

$$E[V(\ln a)] = \frac{(n^2-1)h}{6}\left[\frac{1}{2}\left(r - g - \frac{1}{2}\sigma^2\right)^2 h + \frac{1}{n}\sigma^2\right] \tag{6.8a}$$

and

$$Var[V(\ln a)] = \frac{(n^2-1)(3n^2-2)}{15n^3}\left(r - g - \frac{1}{2}\sigma^2\right)^2 \sigma^2 h^3 . \tag{6.8b}$$

Proof. See Appendix. □

It is obvious that in the trivial case with only one observation, or $n = 1$, both the first two moments degenerate to zero. Alternatively, the above moments can be expressed in terms of the averaging period T_{ap} and the number of observations for $n > 1$

$$E[V(\ln a)] = \frac{(n^2-1)T_{ap}}{6n}\left[\frac{T_{ap}}{2n}\left(r - g - \frac{1}{2}\sigma^2\right) + \frac{1}{n}\sigma^2\right] \tag{6.9a}$$

and

$$Var[V(\ln a)] = \frac{(n^2-1)(3n^2-2)}{15n^6} T_{ap}^3 \left(r - g - \frac{1}{2}\sigma^2\right)^2 \sigma^2 . \tag{6.9b}$$

Example 6.2. Find the mean and variance of $V(\ln a)$ given in (6.9a) and (6.9b) if there are 12 observations in the arithmetic average, observation frequency is monthly, the averaging period has not started, time to maturity of the option is one year, interest rate is 6%, yield on the underlying market is zero, and volatility of the underlying asset is 20%.

Substituting $n = 12$, $h = 1/12$, $r = 0.06$, $g = 0$, $\sigma = 0.20$, and $\tau = 1$ into (6.8a) and (6.8b) yields

$$\begin{aligned} E[V(\ln a)] &= \frac{(n^2-1)h}{6}\left[\frac{1}{2}\left(r - g - \frac{1}{2}\sigma^2\right)^2 h + \frac{1}{n}\sigma^2\right] \\ &= \frac{(12^2-1)/12}{6}\left[\frac{1}{2}\left(0.06 - 0 - \frac{1}{2}\times 0.20^2\right)^2 \frac{1}{12} + \frac{1}{2}\times 0.20^2\right] \\ &= 0.00675 , \end{aligned}$$

and

$$\begin{aligned} Var[V(\ln a)] &= \frac{(n^2-1)(3n+2)}{12n^3}\left(r - g - \frac{1}{2}\sigma^2\right)^2 \sigma^2 h^3 \\ &= \frac{(12^2-1)(3\times 12-2)}{12\times 12^3}\left(0.06 - 0.02 - \frac{1}{2}\times 0.20^2\right)^2 \\ &\quad \times 0.20^2 \times \left(\frac{1}{12}\right)^3 = 0.19195 . \end{aligned}$$

Example 6.3. Find the mean and variance of $V(\ln a)$ given in (6.9a) and (6.9b) if volatility of the underlying asset is changed to 25% and other parameters remain unchanged.

Substituting $n = 12$, $h = 1/12$, $r = 0.06$, $g = 0$, $\sigma = 0.25$, and $\tau = 1$ into (6.8a) and (6.8b) yields

$$\begin{aligned} E[V(\ln a)] &= \frac{(n^2-1)h}{6}\left[\frac{1}{2}\left(r - g - \frac{1}{2}\sigma^2\right)^2 h + \frac{1}{n}\sigma^2\right] \\ &= \frac{(12^2-1)/12}{6}\left[\frac{1}{2}\left(0.06 - 0 - \frac{1}{2}\times 0.25^2\right)^2 \frac{1}{12} + \frac{1}{2}\times 0.25^2\right] \\ &= 0.01041 , \end{aligned}$$

and

$$
\begin{aligned}
Var[V(\ln a)] &= \frac{(n^2-1)(3n^2-2)}{15n^3}\left(r-g-\frac{1}{2}\sigma^2\right)^2\sigma^2h^3 \\
&= \frac{(12^2-1)(3\times 12^2-2)}{15\times 12^3}\left(0.06-0.02-\frac{1}{2}\times 0.25^2\right)^2 \\
&\quad \times 0.25^2\times\left(\frac{1}{12}\right)^3 = 0.29987\,.
\end{aligned}
$$

Comparing the results in Examples 6.2 and 6.3, we can readily find that both the mean and the variance increase significantly with higher volatility. The following corollary provides some general comparative statics results.

Corollary 6.1.

$$
\begin{aligned}
\frac{\partial}{\partial\mu}E[V(\ln a)] &= \frac{(n^2-1)h}{6}\left(\mu-\frac{1}{2}\sigma^2\right)h\,, \\
\frac{\partial}{\partial\sigma^2}E[V(\ln a)] &= \frac{(n^2-1)h}{6}\left[\frac{1}{n}-\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2\right)n\right] \\
&\geq 0, \quad \text{if} \quad \sigma^2 \geq 2\mu - 4/(nh) \\
&< 0, \quad \text{if otherwise,} \\
\frac{\partial}{\partial\mu}Var[V(\ln a)] &= \frac{2(n-1)^4(n+1)(3n^2-2)}{15n^3}\left(\mu-\frac{1}{2}\sigma^2\right)\sigma^2h^3\,, \\
\frac{\partial}{\partial\sigma^2}Var[V(\ln a)] &= \frac{(n-1)^4(n+1)(3n^2-2)}{15n^3}\left(\frac{3}{4}\sigma^4-2\mu\sigma^2+\mu^2\right)h^3\,, \text{ or} \\
&< 0, \quad \text{if } \sqrt{2\mu/3} < \sigma < \sqrt{2\mu}, \\
&\geq 0, \quad \text{if otherwise}\,.
\end{aligned}
$$

Proof. The derivatives are immediate from (6.9a) and (6.9b), and the sign of $\partial Var[V(\ln a)]/\partial\sigma^2$ is obtained by solving the inequality $\partial Var[V(\ln a)]/\partial\sigma^2 \geq 0$. □

Corollary 6.1 clearly indicates that both the mean $E[V(\ln a)]$ and the variance $Var[V(\ln a)]$ increase (decrease) with the instantaneous drift μ when $\sigma < (>)\sqrt{2\mu}$. Simple calculations show that $\sqrt{2\mu}$ = 10%, 14.14%, and 20% when $\mu = 0.5\%$, 1%, and 2%, respectively. As annual volatilities of most financial assets are around 10% under normal market conditions, both the mean $E[V(\ln a)]$ and the variance $Var[V(\ln a)]$ should increase with the drift when $\mu > 0.5\%$. Corollary 6.1 also indicates that the mean $E[V(\ln a)]$ increases with the variance of the underlying asset return σ^2 when

$$\sigma^2 > 2\mu - 4/(nh) = 2\mu - 4/(T_{ap} + h).$$

As the drift parameter μ is almost always a small percentage, the frequency h is often one month or less, and the averaging period is about one year, $2\mu - 4/(T_{ap}+h)$ is almost certainly negative. Thus the mean $E[V(\ln a)]$ almost certainly increases with the variance σ^2. The variance $Var[V(\ln a)]$ increases (resp. decreases) with the instantaneous standard deviation σ when $\sigma < \sqrt{2\mu/3}$ or $\sigma > \sqrt{2\mu}$ (resp. $\sqrt{2\mu/3} < \sigma < \sqrt{2\mu}$). Simple calculations show that for $\mu \geq 3\%$, the lower boundary $\sqrt{2\mu/3} \geq 14.1\%$, thus $\sigma < \sqrt{2\mu/3}$ is more likely to be satisfied as the normal market volatility may often be smaller than 14%. Therefore, the variance $Var[V(\ln a)]$ should increase with the instantaneous volatility when $\mu \geq 3\%$.

6.4. THE DIFFERENCE BETWEEN ARITHMETIC AND GEOMETRIC MEANS

Geometric means are often used to approximate arithmetic means with the same given data in statistical analysis. It is very useful if we could understand what determines the difference between the two means. The difference can be readily obtained if we use the well-known mean-value theory in calculus. As $M(\gamma)$ is a continuous function of γ, there must exist some ψ between 0 and 1 such that

$$AA(a) - GA(a) = M(1) - M(0) = (1-0)M'(\psi) = M'(\psi),$$

which states that the difference between an arithmetic mean and its corresponding geometric mean equals the first-order derivative of the general mean with respect to the weight parameter at some point $0 < \psi < 1$. Substituting (6.5) into the above expression yields the following relationship:

$$AA(a) - GA(a) = M(\psi)\,\frac{\psi \sum a^{\psi} \ln a - \sum a^{\psi} \ln(\sum a^{\psi}/n)}{\psi^2 \sum a^{\psi}}. \tag{6.10}$$

Equation (6.10) states that the difference between an arithmetic mean and its corresponding geometric mean equals the multiplication of the general mean with a weight parameter between 0 and 1 and a positive function of this parameter. Although the difference is in explicit form, the right-hand side is more complicated than the general mean expression such that we cannot see clearly how the difference is affected by distribution measures of the given data such as standard deviation. However, two approaches may generate useful results. One is to find a least up-bound and a tight low-bound for the difference, and the other is to approximate the right-hand side of (6.10) using Taylor's series expansion.

The least up-bound for the difference is more interesting. Unfortunately it cannot be easily expressed explicitly in terms of the n given numbers. We can, however, obtain an universal up-bound for the difference given in (6.10) through amplifying the first part [let $a_i = \max(a_1, a_2, \ldots, a_n) = a_{\max}$] and reducing the second part [let $a_i = \min(a_1, a_2, \ldots, a_n) = a_{\min}$] of the numerator in (6.10).

Proposition 6.3. The difference between an arithmetic mean and its corresponding geometric mean is always smaller than

$$UPB(a) = AA(a)\ln(a_{\max})\left[\left(\frac{a_{\max}}{a_{\min}}\right)^{\psi} - \frac{\ln(a_{\min})}{\ln(a_{\max})}\right] \Big/ \psi\,,$$

where $0 < \psi < 1$ satisfies (6.10).

Proof. Immediate from the above construction. □

6.5. APPROXIMATING ARITHMETIC MEANS WITH GEOMETRIC MEANS

The previous section analyzed the difference between an arithmetic mean and its corresponding geometric mean. It cannot be used directly in Asian option analysis. In this section, we try to express the difference in terms of the distribution of the given observations using Taylor's series expansion, and then obtain an approximation formula for arithmetic means in terms of their corresponding geometric means. As $M(\gamma)$ is a continuous function for $0 < \gamma < 1$ and $0 < \psi < 1$ in (6.10), using the derivative in (6.6) and Taylor's series expansion at $\gamma = 0$ yields

$$M(\psi) = M(0) + \psi M'(0) + O(\psi^2) = GA(a)[1 + \psi v + O(\psi^2)]\,, \qquad (6.11)$$

where $v = \frac{1}{2}\,Var[V(\ln a)] = \sum_{i=1}^{n}\sum_{j=1}^{n}[\ln(a_i/a_j)]^2/(2n^2)$.

The second term on the right-hand side of (6.10) can be approximated with ϕ given in (6.11) because it approaches v if we allow s to approach zero. This approximation could be somewhat justified by the fact that slopes of the general means for weight parameters between 0 and 1 may not be very different. This can be illustrated in Figures 6.1 and 6.2. Figure 6.2 is a more detailed graph than Figure 6.1 for $-0.2 < \gamma < 1$ for the simple data set

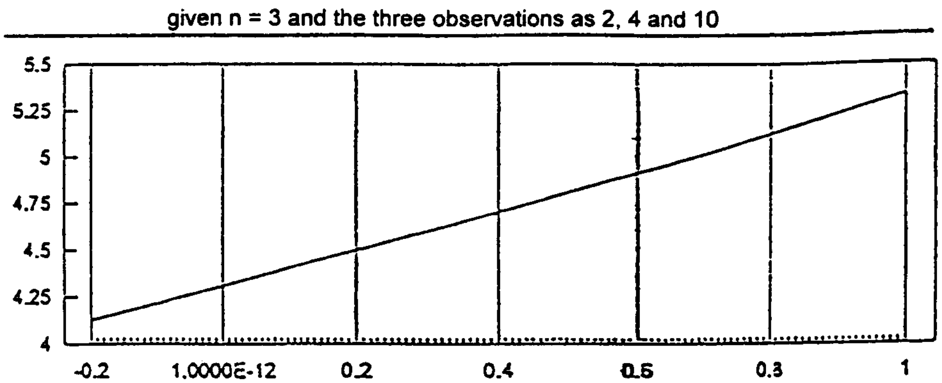

Fig. 6.2. The general mean with various weight parameter.

$n = 3$, $a_1 = 2$, $a_2 = 4$, and $a_3 = 10$. Using (6.11) and the approximation of the second term on the right-hand side of (6.10), we obtain the most important result of this chapter in the following theorem.

Theorem 6.2. The arithmetic average of the underlying asset prices following the Brownian motion specified in (3.1) can be approximated with its corresponding geometric mean as follows:

$$AA(a) \cong \kappa GA(a)\,, \tag{6.12}$$

where $\kappa = 1 + E(v + v^2) = 1 + \frac{1}{2}E[V(\ln a)] + \frac{1}{4}\{Var[V(\ln a)] + E[V(\ln a)]\}^2$ and $E[V(\ln a)]$ and $Var[V(\ln a)]$ are given in (6.8).

Proof. Immediate from the above discussions. □

Two points must be noted here. Firstly, we treated $v + \psi^2 v$ independent of $GA(a)$ in order to make the approximation feasible. Secondly, we enlarged s to 1 to offset the decline of the second term on the right-hand side of (6.10) as it is greater than v.

Although the procedure to derive (6.12) is long and complicated, the intuition behind it is rather straightforward. As both the standard arithmetic and geometric averages are two points of the general mean function which is continuous and twice differentiable, we could use the mean-value theory to find the difference between an arithmetic and its corresponding geometric means. Then we could use Taylor's series expansion to approximate the difference as a proportion of the geometric mean. The approximation given in (6.12) indicates that the standard arithmetic average can be approximated by its corresponding geometric average multiplied by a coefficient which is a function of the volatility of the underlying asset, observation frequency, the number of observations, and the risk-free rate of return.

In the trivial case of only one observation in the averaging period, $\kappa = 1$ as both $E[V(\ln a)] = Var[V(\ln a)] = 0$. Substituting $\kappa = 1$ into (6.12) yields $AA(a) = GA(a)$ which is obviously consistent with $AA(a) = GA(a)$ from definition. In the case of $\mu = \sigma = 0$, it is easy to check that all prices are constant starting from the current initial price S if they follow the Brownian motion specified in (3.1). It is straightforward to show that $\kappa = 1$ as $v = Var[V(\ln a)] = 0$ when all the observations are the same. Substituting $\kappa = 1$ into (6.12) again yields $AA(a) = GA(a)$ which is consistent with $AA(a) = GA(a) = S$ from definition (the arithmetic and geometric means are equal if all the prices are the same). We have the following comparative statics results.

Example 6.4. What is the value of the approximation coefficient given in (6.12) with the same information as in Example 6.2?

From Example 6.2, we know $E[V(\ln a)] = 0.00675$ and $Var[V(\ln a)] = 0.19195$. Substituting these values into (6.12) yields

$$\begin{aligned}\kappa &= 1 + E(v + v^2) = 1 + \frac{1}{2}E[V(\ln a)] + \frac{1}{4}\{Var[V(\ln a)] + E[V(\ln a)]\}^2 \\ &= 1 + \frac{1}{2} \times 0.00675 + \frac{1}{4}(0.19195 + 0.00675)^2 = 1.0132\,.\end{aligned}$$

Example 6.5. What is the value of the approximation coefficient given in (6.12) with the same information as in Example 6.3?

From Example 6.3, we know $E[V(\ln a)] = 0.01041$ and $Var[V(\ln a)] = 0.29987$. Substituting these values into (6.12) yields

$$\begin{aligned}\kappa &= 1 + E(v + v^2) = 1 + \frac{1}{2}E[V(\ln a)] + \frac{1}{4}\{Var[V(\ln a)] + E[V(\ln a)])\}^2 \\ &= 1 + \frac{1}{2} \times 0.01041 + \frac{1}{4}(0.29987 + 0.01041)^2 = 1.02927\,.\end{aligned}$$

Comparing the results in Examples 6.4 and 6.5, we can readily find that the approximation coefficient increases with the volatility parameter. This is consistent with our intuition that observations are more different from one another with higher volatility, thus the arithmetic mean is greater than its corresponding geometric mean.

Corollary 6.2. If the observation frequency and other factors are fixed, then κ increases (resp. decreases) with the drift when $\sigma < \sqrt{2\mu}$ (resp. $\sigma > \sqrt{2\mu}$).

Proof. Immediate from taking partial derivative of κ with respect to μ and using Corollary 6.1. □

Corollary 6.3. If the observation frequency and other factors are fixed, then κ

(1) increases with variance σ^2 when

$$D = \left\{\frac{2}{3}\mu + \frac{1}{8}h^2 1 + E[V(\ln a)]\frac{\delta_1}{\delta_2}\right\}^2 - \frac{4}{n}\{1 + E[V(\ln a)]\}\frac{\delta_1}{\delta_2} < 0,$$

where

$$\delta_1 = \frac{(n^2-1)h}{3}, \quad \delta_2 = \frac{(n^3-1)(3n^2-2)}{10}h^3;$$

(2) increases with variance σ^2 when $D \geq 0$ and

$$\sigma^2 < \frac{4}{3}\mu - \frac{1}{8}h^2\left\{1 + E[V(\ln a)]\frac{\delta_1}{\delta_2} - \sqrt{D}\right\}$$

$$\text{or} \quad \frac{4}{3}\mu - \frac{1}{8}h^2\left\{1 + E[V(\ln a)]\frac{\delta_1}{\delta_2} + \sqrt{D} < \sigma^2\right\};$$

and

(3) decreases when $D \geq 0$ and

$$\frac{4}{3}\mu - \frac{1}{8}h^2\left\{1 + E[V(\ln a)]\frac{\delta_1}{\delta_2}\right\} - \sqrt{D} < \sigma^2$$

$$< \frac{4}{3}\mu - \frac{1}{8}h^2\left\{1 + E[V(\ln a)]\frac{\delta_1}{\delta_2} + \sqrt{D}\right\}.$$

Proof. Solving the inequality $\partial\kappa/\partial\sigma^2 \geq 0$ using Corollary 6.1 yields the above results after simplifications. □

Corollary 6.3 indicates that κ may or may not change in the same direction with variance σ^2. Comparing Corollary 6.3 to Corollary 6.1, the range in which κ changes inversely with variance σ^2 is much smaller than the range in which $Var[V(\ln a)]$ changes inversely to σ^2.

In general, κ can be easily calculated when the parameters n, h, μ, and σ are given. With the value of κ, we know how much an arithmetic mean is larger than its corresponding geometric mean. Theorem 6.2 clearly shows that the difference between an arithmetic mean and its corresponding geometric mean depends on the degree of dispersion among the n positive numbers or the parameter $v = [V(\ln a)]/2$. When the asset prices follow the Brownian motion specified in (3.1), the difference between an arithmetic average price and its corresponding geometric average price of the underlying asset, on average, is determined by the first two moments of the distribution of $V(\ln a)$.

6.6. APPROXIMATING ARITHMETIC ASIAN OPTIONS WITH GEOMETRIC ASIAN OPTIONS

The payoff of a European-style option based on the general average price (AOPGMPF) of the underlying asset and a fixed strike price can be expressed as follows:

$$AOPGMPF = \max[\omega M(\gamma) - \omega K, 0]\,, \tag{6.13}$$

where K stands for the strike price of the option and ω is the same binary indicator (1 for a call option and -1 for a put option).

If $\gamma = 0$, $M(0) = GA(a)$, (6.13) becomes the same as the payoff of a geometric Asian option given in (5.5) of Chapter 5. If $\gamma = 1$, $M(1) = AA(a)$, it is the payoff of an arithmetic Asian option.

Theorem 6.3. The price of a European-style Asian option based on the arithmetic average of the underlying asset prices following the Brownian motion specified in (3.1) can be approximated to be

$$C^{aa} \cong \omega\kappa S e^{-gT^{sa}_{\mu,n-j}} A^{sa}(j) N\left(\omega d^{aa}_{n-j} + \omega\sigma\sqrt{T^{sa}_{n-j}}\right) - \omega K e^{-r\tau} N(\omega d^{aa}_{n-j}),$$

where (6.14)

$$d^{aa}_{n=j} = \left[\ln\left(\frac{\kappa S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right) T^{sa}_{\mu,n-j} + \ln B^{sa}(j)\right] \Big/ \left(\sigma\sqrt{T^{sa}_{n-j}}\right),$$

κ is given in (6.12), and all other parameters are the same as in Theorems 5.1 and 5.2 of Chapter 5.

Proof. Theorem 5.1 shows that $GA(a)$ is lognormally distributed and Theorem 6.2 states that $AA(a)$ is approximately lognormally distributed because the approximation coefficient κ is constant. Following a similar procedure to derive the results in Theorem 5.2, we can readily obtain (6.14). □

Obviously, Equation (6.14) can be easily obtained by substituting the current spot price S of the pricing formula for geometric Asian options in

Theorem 5.2 with κS. Before the averaging period starts, the parameter $B^{sa}(j)$ is always 1. As κ is always greater than one, it augments (resp. reduces) the Asian call (resp. put) option price through amplifying the effective current spot price, making the call (resp. put) option more in- (resp. out-of-) the-money. Thus, the arithmetic Asian call (resp. put) options are always more (resp. less) expensive than the corresponding geometric calls (resp. puts).

Example 6.6. What are the prices of the call and put options with strike price \$400 to expire in one year, and based on the arithmetic average of monthly gold prices, given that the spot gold price is \$390 per ounce, interest rate 7%, yield on the gold is zero, and volatility of gold return is 20%?

Since the time to maturity, the observation frequency, and the number of observations are the same as in Examples 5.1 and 6.4, we can use the two effective time values in Example 5.1 and the approximation coefficient in Example 6.4. Substituting S = \$390, K = \$400, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, and $\kappa = 1.0132$ into (6.14) yields

$$\begin{aligned} d^{aa}_{n-j} &= \left[\ln\left(\frac{\kappa S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right) T^{sa}_{\mu,n-j} + \ln B^{sa}(j)\right] \Big/ \left(\sigma\sqrt{T^{sa}_{n-j}}\right) \\ &= \left[\ln\left(\frac{1.0132 \times 390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.542 + \ln(1)\right] \Big/ \\ &\qquad \left(0.20 \times \sqrt{0.376}\right) = 0.3966\,, \end{aligned}$$

the call option price

$$\begin{aligned} C &= \kappa S e^{-gT^{sa}_{\mu,n-j}} A^{sa}(j) N\left(d^{aa}_{n-j} + \sigma\sqrt{T^{sa}_{n-j}}\right) - K e^{-r\tau} N(d^{aa}_{n-j}) \\ &= 1.0132 \times 390 \times e^{-0.07(1-0.542) - 0.20^2 \times (0.542-0.376)/2} N(0.4988) \\ &\quad - 400 \times e^{-0.07} N(0.3966) \\ &= \$31.472\,, \end{aligned}$$

and the corresponding geometric put option price can be found by substituting $\omega = -1$ and other parameters into (6.14)

$$\begin{aligned} P &= -\kappa S e^{-gT^{sa}_{\mu,n-j}} A^{sa}(j) N\left(-d^{aa}_{n-j} - \sigma\sqrt{T^{sa}_{n-j}}\right) + K e^{-r\tau} N(-d^{aa}_{n-j}) \\ &= -1.0132 \times 390 \times e^{-0.07(1-0.542) - 0.20^2 \times (0.542-0.376)/2} N(-0.4988) \\ &\quad + 400 \times e^{-0.07} N(-0.3966) \\ &= \$9.900\,. \end{aligned}$$

We may also partially interpret our approximation in (6.14) as reducing the effective strike price if we rewrite $\kappa S/K$ as $S/(K/\kappa)$. As κ is always greater than 1, K/κ is always smaller than K, therefore the effective strike price K is reduced to K/κ. The second explanation is somewhat similar to that of Vorst's (1992) approximation; however, there exists three important differences. The first difference is that Vost's effective strike price is obtained by subtracting the expected difference between the arithmetic and geometric means from the actual strike price, whereas our effective strike price is obtained by dividing the actual price by a number greater than 1. The second difference is that whereas Vorst's approximation is a one-moment approximation in the sense that the degree of effective strike price reduction depends only on the first moment of the difference of the two means, our approximation is a two-moment approximation. Lastly, Vorst's approximation is obtained by using the upper bound of the difference between the arithmetic and geometric Asian option prices arbitrarily as a correction of the difference between the expectation of the arithmetic average and the geometric average, ours is derived firmly from mathematical approximation.

6.7. CONTINUOUS ARITHMETIC ASIAN OPTIONS

We defined continuous arithmetic and geometric averages in Section 5.4 and provided closed-form solutions for continuous geometric Asian options. An exact formula for continuous arithmetic Asian options do not exist as for discrete arithmetic Asian options. However, we can find an approximated pricing formula for continuous arithmetic Asian options using the result in Theorem 6.2.

Proposition 6.4. The approximation coefficient κ given in Theorem 6.1 approaches

$$\kappa_c = 1 + \frac{1}{24}\left(r - g - \frac{1}{2}\sigma^2\right)^2 T_{ap}^2 + \frac{1}{576}\left(r - g - \frac{1}{2}\sigma^2\right)^4 T_{ap}^4 \qquad (6.15)$$

when the observation frequency approaches zero and the averaging period is fixed.

Proof. From (6.9), the number of observation n must approach infinity when the observation frequency approaches zero and the averaging period is fixed. Taking limits to both $E[V(\ln a)]$ and $Var[V(\ln a)]$ in (6.9) as n approaches infinity yields $E[V(\ln a)] \to (r - g - \sigma^2/2)^2 T_{ap}^2/12$ and $Var[V(\ln a)] \to 0$. Substituting these limiting results into the expression of the coefficient κ in Theorem 6.1 yields Proposition 6.4. □

Example 6.7. What is the continuous approximation coefficient given in (6.15) if observation is continuous, the averaging period has not started, and the time to maturity of the option is one year, interest rate is 6%, yield on the underlying market is zero, and volatility of the underlying asset is 20%?

Substituting $r = 0.06$, $g = 0$, $\sigma = 0.20$, and $T_{ap} = 1$ into (6.15) yields

$$K_c = 1 + \frac{1}{24}\left(r - g - \frac{1}{2}\sigma^2\right)^2 T_{ap}^2 + \frac{1}{576}\left(r - g - \frac{1}{2}\sigma^2\right)^4 T_{ap}^4$$

$$= 1 + \frac{1}{24} \times \left(0.06 - 0 - \frac{1}{2} \times 0.20^2\right)^2 \times 1^2$$

$$+ \frac{1}{576}\left(0.06 - 0 - \frac{1}{2} \times 0.20^2\right)^4 \times 1^4 = 1.00007\,.$$

The payoff of an Asian option based on the continuous arithmetic average of the underlying asset prices can be expressed as follows:

$$R = \max[\omega CAA(n) - \omega K, 0]\,, \tag{6.16}$$

where $CAA(n)$ is defined in (5.9) and ω is a binary indicator (1 for a call option and -1 for a put option).

Theorem 6.4. The price of an Asian option based on the continuous arithmetic average defined in (5.9) can be approximated as follows:

$$C^{caa} \cong \omega\kappa_c S e^{-g\tau/2} e^{-(r\tau+\sigma^2/6)/2} N\left(\omega d_{n-j}^{caa} + \omega\sigma\sqrt{\tau/3}\right)$$

$$- \omega K e^{-r\tau} N(\omega d_{n-j}^{caa})\,, \tag{6.17}$$

where

$$d_{n-j}^{caa} = \left[\ln\left(\frac{\kappa_c S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}\right] \Big/ \left(\sigma\sqrt{\tau/3}\right)\,,$$

κ_c is given in (6.15), and all other parameters are the same as in Theorems 5.1 and 5.2.

Proof. As the continuous arithmetic average in (5.12) is the limit of the discrete arithmetic in (5.1), the continuous arithmetic Asian option prices should be the limit of the discrete arithmetic Asian option prices of (6.14). The rest of the proof is the same as that of Theorem 5.4. □

Example 6.7. What are the prices of the call and put options with strike price \$400 to expire in one year and based on the continuous arithmetic average of gold prices, given that the spot gold price is \$390 per ounce, the interest rate 7%, yield on the gold is zero, and volatility of gold return is 20%?

Since the time to maturity, observation frequency, and the number of observations are the same as in Examples 5.3 and 6.6, we can use the two effective time values in Example 5.3 and the approximation coefficient in Example 6.6. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = \tau/2 = 0.50$, $T^{sa}_{n-j} = \tau/3 = 0.3333$, and $\kappa = 1.00007$ into (6.17) yields

$$d^{caa}_{n-j} = \left[\ln\left(\frac{\kappa_c S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\frac{\tau}{2}\right] \Big/ \left(\sigma\sqrt{\tau/3}\right)$$

$$= \left[\ln\left(\frac{1.00007 \times 390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.50\right] \Big/$$

$$+ \left(0.20 \times \sqrt{0.3333}\right) = -0.0268\,,$$

the call option price

$$C = \kappa_c S e^{-g\tau/2} e^{-(r\tau+\sigma^2/6)/2} N\left(d^{caa}_{n-j} + \sigma\sqrt{\tau/3}\right) - K e^{-r\tau} N(d^{caa}_{n-j})$$

$$= 1.00007 \times 390 \times e^{-(0.07+0.20^2/6)/2} N(-0.0268 + 0.1155)$$

$$- 400 \times e^{-0.07} N(-0.0268)$$

$$= \$20.749$$

and the corresponding geometric put option price can be found by substituting $\omega = -1$ and other parameters into (6.14)

$$P = -\kappa_c S e^{-g\tau/2} e^{-r\tau+\sigma^2/6)/2} N\left(-d^{caa}_{n-j} - \sigma\sqrt{\tau/3}\right) + K e^{-r\tau} N\left(-d^{caa}_{n-j}\right)$$

$$= -1.00007 \times 390 \times e^{-(0.07+0.20^2/6)^2} N(0.0268 - 0.1155)$$

$$+ 400 \times e^{-0.07} N(0.0268)$$

$$= \$7.427\,.$$

6.8. ARITHMETIC-AVERAGE-STRIKE ASIAN OPTIONS

We provided a closed-form formula for Asian options with geometric average strike prices in Chapter 5. We will find the approximated formula for Asian options with arithmetic average strike prices in this section. The results are given in the following theorem.

Theorem 6.5. The price of an Asian option whose strike price is the arithmetic average of the underlying asset prices following the Brownian motion specified in (5.4) can be approximated as:

$$ASC^{aa} \cong \omega S \left[\kappa e^{-g\tau} N(\omega D_{a1}) - A^{sa}(j) e^{-gT^{sa}_{\mu,n-j}} N(\omega D_{a2})\right],$$

where

$$D_{a2} = \frac{-\ln[\kappa B^{sa}(j)] + (r - g - \frac{1}{2}\sigma^2)(\tau - T^{sa}_{\mu,n-j}) + \sigma^2(\rho\sqrt{\tau T^{sa}_{n-j}} - 1)}{\sigma\sqrt{\tau_e}},$$

$$D_{a1} = D_{a2} + \sigma\sqrt{\tau_e}, \ \tau_e = \tau - 2\rho\sqrt{\tau T^{sa}_{n-j}} + T^{sa}_{n-j},$$

$A^{sa}(j)$ is the same as in (4.7), $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} are the same effective mean and variance functions given in Theorem 5.1, and all other parameters are the same as in Theorem 5.4.

Proof. Similar to the proofs of Theorems 5.4 and 6.3. □

Example 6.8. What are the Asian call and put option prices with strike price specified as the arithmetic average of the underlying asset prices, given other information the same as in Example 5.4?

Since the time to maturity, observation frequency, and the number of observations are the same as in Examples 5.1 and 5.7, we can use the two effective time values in Example 5.1 and the correlation coefficient in Example 5.7. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, $\rho = 0.883$ into (6.18) yields

$$\tau_e = \tau - 2\rho\sqrt{\tau T^{sa}_{n-j}} + T^{sa}_{n-j} = 1 - 2 \times 0.883 \times \sqrt{1 \times 0.376} + 0.376 = 0.4583,$$

$$D_{g2} = \frac{(0.07 - 0 - \frac{1}{2} \times 0.20^2)(1 - 0.542) + 0.20^2 \times (0.883 \times \sqrt{1 \times 0.376} - 1)}{0.20 \times \sqrt{0.4583}}$$

$$= 0.042,$$

$$D_{g1} = D_{g2} + \sigma\sqrt{\tau_e} = 0.034 + 0.20 \times \sqrt{0.4583} = 0.151,$$

the call average-strike option price

$$C = S\left[e^{-g\tau}N(D_{g1}) - A^{sa}(j)e^{-gT^{sa}_{\mu,n-j}}N(D_{g2})\right]$$
$$= 390[N(0.151) - 0.9652 \times N(0.042)] = \$23.76\,;$$

and the corresponding put option price can be found by substituting $\omega = -1$ into (6.18)

$$P = -S\left[e^{-g\tau}N(-D_{g1}) + A^{sa}(j)e^{-gT^{sa}_{\mu,n-j}}N(D_{g2})\right]$$
$$= -390[N(-0.151) + 0.9652 \times N(-0.042)] = \$10.20\,.$$

Theorem 6.6. The price of a European-style Asian option with strike price specified as the continuous arithmetic average of the underlying asset prices given in (5.3) is

$$ACAASTK = \omega S\left[e^{-g\tau}N(\omega D_{ca1}) - \kappa e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(\omega D_{ca2})\right]\,,$$

where

$$D_{ca2} = \frac{(r - g - \frac{1}{2}\sigma^2)\frac{\tau}{2} + \sigma^2(\frac{\tau}{2} - 1) - \ln\kappa}{\sigma\sqrt{\tau/3}}$$

and

$$D_{ca1} = D_{ca2} + \sigma\sqrt{\tau/3}\,. \tag{6.19}$$

Proof. Similar to the proofs of Theorems 5.5 and 6.3. □

Example 6.9. What are the Asian call and put options prices with strike price specified as the continuous arithmetic average of the underlying asset prices, given other information the same as in Example 5.4?

Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^{sa}_{\mu,n-j} = 0.542$, $T^{sa}_{n-j} = 0.376$, $\rho = 0.866$, and $\kappa = 1.00007$ into (6.19) yields

$$D_{ca2} = \frac{(r - g - \sigma^2/2)\frac{\tau}{2} + \sigma^2(\frac{\tau}{2} - 1) - \ln\kappa}{\sigma\sqrt{\tau/3}}$$
$$= \frac{(0.07 - 0.20^2/2)\frac{1}{2} + 0.20^2(\frac{1}{2} - 1) - 0.0007}{0.20 \times \sqrt{1/3}} = 0.0427\,,$$

$$D_{ca1} = D_{ca2} + \sigma\sqrt{\tau/3} = 0.0427 + 0.20 \times \sqrt{1/3} = 0.1582\,,$$

the price of the call option with strike price specified as the continuous arithmetic average of the underlying asset prices;

$$C = S\left[e^{-g\tau}N(D_{ca1}) - \kappa e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(D_{ca2})\right]$$
$$= 390[N(0.1582) - 1.00007 \times 0.9624 \times N(0.0427)] = \$11.399\,;$$

and the corresponding put option price can be found by substituting $\omega = -1$ into (6.19)

$$P = S\left[e^{-g\tau}N(D_{ca1}) - \kappa e^{-(r\tau+\sigma^2/6)/2}e^{-g\tau/2}N(D_{ca2})\right]$$
$$= 390[N(-0.1582) - 1.00007 \times 0.9624 \times N(-0.0427)] = \$18.069\,.$$

6.9. GENERAL MEANS AND LOOKBACK OPTIONS

The payoff of a European-style option whose strike price is based on the general average price of the underlying asset can be expressed as follows:

$$R = \max[\omega S(\tau) - \omega M(\gamma), 0]\,, \tag{6.20}$$

where ω is a binary indicator (1 for a call option and -1 for a put option).

If $\omega = 1(-1)$ and $\gamma \to -\infty$, (6.20) becomes the payoff of a call (put) option on the minimum of the underlying asset prices studied by Goldman, Sosin, and Gatto (1979); if $\omega = 1(-1)$ and $\gamma \to \infty$, (6.20) becomes the payoff for a call (put) option on the maximum of the underlying asset prices also studied by Goldman, Sosin, and Gatto (1979); if $\gamma = 0$, (6.20) becomes the payoff of an Asian option with strike price specified as the geometric average of the underlying asset prices; if $\gamma = 1$, (6.20) becomes the payoff of an Asian option with strike price specified as the arithmetic average of the underlying asset prices. Thus, the parameter γ of the general mean connects lookback options and Asian options with average strike prices. Although lookback options can minimize investors' regret, they are, in general, very expensive. With the general mean, we may somehow structure partial lookback options.

6.10. AN APPLICATION

We have given quite a few examples in the previous sections of this chapter. In this section, we will provide an additional example to show how to apply arithmetic Asian options in practice. We argued that Asian options can provide a cheaper way to hedge the underlying asset with periodic cash flows. This example will show how Asian options can specifically hedge the foreign currency risks more efficiently than a string of standard options.

Example 6.10. Instead of using a geometric Asian option as in Example 5.13, let us look at the effectiveness of hedging with a continuous arithmetic Asian option in this example.

Instead of buying twelve consecutive call options, the importer can buy an Asian call option based on arithmetic average with monthly observation as we analyzed in Section 5.3. Given the same information, we can obtain the price of the arithmetic Asian option easily for one-million Japanese yen: \$987.074, which is larger than \$482.91, the price for the geometric Asian option in Example 5.13. This increase in price is expected as arithmetic options are more expensive than geometric ones. Although the premium (\$987.074) for the arithmetic Asian option is significantly larger than that of the corresponding geometric Asian option (\$482.91), it is still much smaller than the total cost of twelve consecutive call options (\$8101) as we calculated in Example 5.13. Thus, it is cheaper to hedge with arithmetic Asian options.

6.11. CONCLUSIONS

The general mean measure used in this chapter includes arithmetic, harmonic and quadratic means as special cases and geometric means and cases, the maximum and the minimum observations as limiting cases. We have shown that the mean parameter can be considered as a weight parameter which allocates heavier weights to larger prices in average. Using this general mean measure, we first showed that the difference between an arithmetic mean and its corresponding geometric mean is determined by the dispersion of the prices under consideration. We then found an efficient approximation for arithmetic means with their corresponding geometric means. Finally, we provided closed-form approximated formulas for European-style Asian options based on arithmetic prices and arithmetic strike prices. These formulas are of the Black-Scholes type. Our numerical examples show that the results are quite efficient and accurate.

Whereas the existing studies are based on either an arbitrarily fixed number of moments [the first two moments in the case of Levy (1992) and the first four moments in the case of Turnbull and Wakeman (1991)] or on an arbitrarily reduced effective strike price as in the case of Vorst (1992), our results are firmly derived from mathematical approximation using Taylor's expansion series and a general mean measure which connects any arithmetic mean with its corresponding geometric mean. Besides the use of mathematical expansion in this paper, the general mean measure has the potential to connect Asian options with lookback options and possibly generates other forms of options between them.

APPENDIX

Outline to the proof of Proposition 6.2

Taking logarithm to both sides of (5.4) yields

$$\ln(a_i) = \ln S + (r - g - \sigma^2/2)[\tau - (n-i)h] + \sigma z[\tau - (n-i)h],$$
$$i = 1, 2, \ldots, n\,.$$

Thus

$$\ln(a_i/a_j) = (r - g - \sigma^2/2)(i-j)h + \sigma\{z[\tau - (n-i)h] - z[\tau - (n-j)h\},$$
$$j \neq i$$

and

$$\begin{aligned}[\ln(a_i/a_j)]^2 &= (r-g-\sigma^2/2)^2(i-j)^2h^2 + 2(r-g-\sigma^2/2)(i-j)\\ &\quad \times h\sigma\{z[\tau-(n-i)/h] - z[\tau-(n-j)h]\}\\ &\quad + \sigma^2\{z^2[\tau-(n-i)h] + z^2[\tau-(n-j)h]\\ &\quad - 2z[(\tau-(n-i)h]z[\tau-(n-i)h]\}\,,\\ E[\ln(a_i/a_j)]^2 &= (r-g-\sigma^2/2)^2(i-j)^2h^2\\ &\quad + \sigma^2\{[\tau-(n-i)h] + [\tau-(n-j)h]\\ &\quad - 2\min[\tau-(n-i)h, \tau-(n-j)h]\}\,.\end{aligned}$$

Using the simplification

$$\begin{aligned}&[\tau-(n-i)h] + [\tau-(n-j)h] - 2\min[\tau-(n-i)h,\ \tau-(n-j)h]\\ &\quad = h[\max(i,j) - \min(i,j)] = h|i-j|\end{aligned}$$

and summations of i^p ($i = 1, 2, \ldots, n$ and $p = 1, 2,$ and 3) yields (6.8a). Equation (6.8b) can be similarly obtained by simplifying $\sum_{i=1}^n \sum_{j=1}^n (i-j)^2|i-j| = \sum_{i=1}^n \sum_{j=1}^n |i-j|^3$ using summations of i^p ($i = 1, 2, \ldots, n$ and $p = 1, 2, 3$ and 4).

QUESTIONS AND EXERCISES

Questions

6.1. What is meant by a general mean?

6.2. What is a harmonic mean?

6.3. Is a quadratic mean always greater than a harmonic mean?

6.4. Is it possible that a harmonic mean is greater than the corresponding geometric mean?

6.5. Are arithmetic averages always greater than their corresponding geometric averages? Why?

6.6. Under what conditions are arithmetic means equal to their corresponding geometric means?

6.7. Why can we regard the parameter in the general mean function as a weight parameter which allocates more weights to larger observations?

6.8. Are the effective mean time and variance time the same for arithmetic and geometric Asian options?

6.9. Are arithmetic Asian options always cheaper or more expensive with continuous observation than with discrete observation given other parameters unchanged?

6.10. Why is the correlation coefficient between the log-returns of the underlying asset and the arithmetic average approximation in Theorem 6.1 always constant and the same as the correlation coefficient between the log-returns of the underlying asset and the geometric average?

Exercises

6.1. Find the harmonic, geometric, arithmetic, and quadratic means of four observations $a = (2, 3, 4, 5)$.

6.2. Find the mean and variance of $V(\ln a)$ given in (6.9a) and (6.9b) if there are 52 observations in the arithmetic average, observation frequency is weekly, the averaging period has not started, and the time to maturity of the option is half a year, interest rate is 9%, yield on the underlying market is 3%, and volatility of the underlying asset is 20%?

6.3. Answer the same questions in Exercise 6.2 if volatility is changed to 30% and other parameters remain unchanged.

6.4. Find the approximation coefficient in Theorem 6.2 with the same information as in Exercise 6.2.

6.5. Find the approximation coefficient in Theorem 6.2 with the same information as in Exercise 6.3.

6.6. Show that the continuous arithmetic average given in (5.12) is the limiting case of the discrete arithmetic average (5.1) when n approaches infinity and the averaging period $t^* - s$ is fixed.

6.7. Find the prices of the arithmetic Asian call and put options with weekly observations to expire in half a year and with strike price \$460 if the current underlying index is \$450, interest rate is 8%, yield on the underlying index is 4%, volatility of the index is 20%.

6.8. Answer the same questions in Exercise 6.7 if the observation frequency is bimonthly and other parameters are the same as in Exercise 6.7?

6.9. Answer the same questions in Exercise 6.7 if the observation frequency is continuous and other parameters are the same as in Exercise 6.7?

6.10. Show that the correlation coefficient between the log-returns of the underlying asset and the approximated arithmetic average in Theorem 6.2 is always constant and the same as the correlation coefficient between the log-returns of the underlying asset and the geometric average.

6.11. Find the correlation coefficient between the log-returns of the underlying asset and the arithmetic average with weekly observations, time to maturity 20 weeks, interest rate 7%, yield on the underlying asset 2%, and spot price \$100.

6.12. Find the prices of European-style Asian options with arithmetic strike prices with weekly observations, time to maturity 20 weeks, interest rate 7%, yield on the underlying asset 2%, volatility of the underlying asset 15%, and spot price \$100.

Chapter 7

FLEXIBLE ARITHMETIC ASIAN OPTIONS

7.1. INTRODUCTION

Despite the fact that geometric and arithmetic averages are very different, as shown in Chapters 5 and 6, they share at least one characteristic — equal weighting. In other words, all observations are equally important in both geometric and arithmetic averages as defined in (5.1) and (5.2). Whereas equal weighting is not problematic for many applications, it cannot represent many problems satisfactorily. For example, an exporter who has monthly cash inflows in a foreign currency knows that the cash flows in a few particular months are far greater than those in other months. With monthly exchange rates treated with equal weights, the foreign exchange risks cannot be appropriately hedged. With a flexibly weighted average on monthly exchange rates, the exporter may simply assign heavier (less) weights for those months with greater (less) cash flows and may very likely obtain better hedging performance than with an equally weighted average.

Another example is from technical analysis used by most traders in almost all markets. Moving averages[1] are used in technical analysis to represent and detect trends of prices or market indices. Moving averages with different weights clearly possess advantages over those with equal weights as heavier weights are allocated for more recent observations. This is because more recent observations can better present and therefore should be more useful to forecast the future. It is this intuition that stimulates us to structure Asian options based on weighted averages of the underlying asset prices. Zhang (1993) discussed these Asian options. Zhang (1994a) introduced the concept of flexible Asian options (FAOs) and priced them based on

[1] A moving average is an average with changing numbers in the average. For example, a 10-day daily moving average always includes ten observations and the oldest observation is always dropped when each new observation is added into the average every day. Thus, moving averages are averages with periodically updated observations.

geometric averages within a Black-Scholes environment. Zhang (1995b) provided approximated closed-form pricing formulas for flexible arithmetic Asian options. The FAOs are no longer esoteric imaginations; many banks have been trading these products for some time since early 1994 [see *Derivatives Week* III (16), April 25, 1994]. The simple reason behind the popularity of FAOs is that they provide additional flexibility.

It should be clear that by "flexible" we mean flexibility in giving weights to a series of observations, but not the flexible options trading in the Chicago Board Options Exchange in which strike prices, time to maturity, ways to settle options, and other factors can be customized in contrast to most vanilla options currently trading in most options exchanges.

The purpose of this chapter is to illustrate the concepts of Asian options based on flexible averages of the underlying asset prices, to price flexible geometric Asian options in closed-form, and to approximate Asian options based on flexible arithmetic averages using the method developed in Chapter 6.

7.2. FLEXIBLE WEIGHTED AVERAGES

The most general flexible average may be obtained with the following weighting scheme:

$$W(n,i) = \frac{q(i)}{\sum_{i=1}^{n} q(i)}, \tag{7.1}$$

where $q(i)$ can be any non-negative function of the ith observation and n is the number of observations under consideration.

The function $q(i)$ in (7.1) can either be a power function, logarithm function, exponential function, or any other functions. If we choose $q(i) = \epsilon^i$, where $|\epsilon| \leq 1$ and $i = 1, 2, \ldots, n$, then the weight function given in (7.1) will become an exponential weight function. If we choose $q(i) = i^\alpha$ (although α can be any real number, we restrict it to be non-negative for convenience), the weighting scheme given in (7.1) becomes

$$W(n,\alpha,i) = \frac{i^\alpha}{\sum_{i=1}^{n} i^\alpha}, \qquad i = 1, 2, \ldots, n, \tag{7.2}$$

which is precisely the general weighted moving average (GWMA) measure developed in Hutchinson and Zhang (1993).

The exponential weighting scheme when $q(i)$ is specified as ϵ^i may have some advantages over the scheme in (7.2). We choose the latter in most of the examples and analyses in this book in order to be consistent with the majority of existing weighting schemes in moving averages. Interested readers may wish to work out the corresponding formulas for the exponential weighting function and they may find them a little more compact than those using the weighting scheme given in (7.2).

Example 7.1. What is the weight distribution if there are five observations and we choose the weight function $q(i)$ in (7.1) as $q(i) = e^{-0.5i}$?

We can readily find the summation

$$\sum_{i=1}^{n} q(i) = e^{-1\times0.5} + e^{-2\times0.5} + e^{-3\times0.5} + e^{-4\times0.5} + e^{-5\times0.5} = 1.4150\,.$$

Substituting $i = 1, 2, 3, 4$, and 5 and $q(i) = e^{-0.5i}$ into (7.1) yields

$$W(5,1) = q(1)\Big/\sum_{i=1}^{n} q(i) = 0.6065/1.4150 = 0.4287 = 42.87\%\,,$$

$$W(5,2) = q(2)\Big/\sum_{i=1}^{n} q(i) = 0.3679/1.4150 = 0.2600 = 26.00\%\,,$$

$$W(5,3) = q(3)\Big/\sum_{i=1}^{n} q(i) = 0.2231/1.415 = 0.1577 = 15.77\%\,,$$

$$W(5,4) = q(4)\Big/\sum_{i=1}^{n} q(i) = 0.1353/1.4150 = 0.0956 = 9.56\%\,,$$

and

$$W(5,5) = q(5)\Big/\sum_{i=1}^{n} q(i) = 0.0821/1.4150 = 0.0580 = 5.80\%\,.$$

Example 7.2. What is the weight distribution if there are five observations and the weight parameter $\alpha = 0.5$?

We can readily find the summation

$$\sum_{i=1}^{n} i^{\alpha} = 1^{0.5} + 2^{0.5} + 3^{0.5} + 4^{0.5} + 5^{0.5} = 8.3823\,.$$

Substituting $i = 1$, 2, 3, 4, and 5, $\alpha = 0.5$ into (7.2) yields

$$W(5,0.5,1) = 1^{0.5}\Big/\sum_{i=1}^{n} i^{\alpha} = 1/8.3823 = 0.1193 = 11.93\%\,,$$

$$W(5,0.5,2) = 2^{0.5}\Big/\sum_{i=1}^{n} i^{\alpha} = 1.4142/8.3823 = 0.1687 = 16.87\%\,,$$

$$W(5,0.5,3) = 3^{0.5}\Big/\sum_{i=1}^{n} i^{\alpha} = 1.7321/8.3823 = 0.2066 = 20.66\%\,,$$

$$W(5, 0.5, 4) = 4^{0.5} \Big/ \sum_{i=1}^{n} i^{\alpha} = 2/8.3823 = 0.2386 = 23.86\%,,$$

and

$$W(5, 0.5, 5) = 5^{0.5} = 5^{0.5} \Big/ \sum_{i=1}^{n} i^{\alpha} = 2.2361/8.3823 = 0.2668 = 26.68\%.$$

With the weight scheme given in (7.1) or (7.2), we can construct the general weighted average (GWA) as follows:

$$GWA(n, \alpha) = \sum_{i=1}^{n} W(n, i) a_i, \qquad i = 1, 2, \ldots, n, \tag{7.3}$$

where a_i stands for the ith observation.

We can readily find the special cases of the GWA given in (7.2) and (7.3): when $\alpha = 0$, $W(n, \alpha, i) = 1/n$ for all i, which is precisely the simple average with equal weights. When $\alpha = 1$, $W(n, \alpha, i) = i / \sum_{i=1}^{n} i = 2i/[n(n+1)]$, which is exactly the linearly weighted average with linearly increasing weights. When α approaches infinity, weights given to all previous observations approach zero or $W(n, \alpha, i) \to 0$ for $i = 1, 2, \ldots, n-1$, and nearly all weights are centered at the most recent observation, or $W(n, \alpha, n) \to 1$, and therefore $GWA(n, +\infty) = P(n)$.

The obvious advantage of the GWA is its flexibility. The weight distribution is always fixed once the number of observations is chosen in the traditional average measures. However, weights can be adjusted through either choosing different weight functions $q(i)$ in (7.1) or adjusting the weight control parameter α in the special case (7.2).

7.3. A MEASURE OF INEQUALITY IN WEIGHTING

The weight control parameter α in (7.2) is easy to grasp. The larger the value of α, the heavier the weights are allocated to the most recent observations. Nevertheless, α is not very convenient as it ranges between zero and infinity. We can, however, use the following measure to represent inequality in the weight distribution:

$$\beta = 1 - 1/(1 + \alpha) = \alpha/(1 + \alpha),$$

where β measures the degree of inequality in the weight allocation and is a better measure because it is always between 0 and 1, or between 0 and 100 percent. For example, when $\alpha = 0$ (the equally weighted case), $\beta = \alpha/(1 + \alpha) = 0\%$; when $\alpha = 0.50$, $\beta = \alpha/(1 + \alpha) = 1/3 = 33.3\%$; when $\alpha = 1$

(the linearly weighted case), $\beta = \alpha/(1+\alpha) = 0.50 = 50\%$; when $\alpha = 2$, $\beta = \alpha/(1+\alpha) = 2/3 = 66.7\%$; and when $\alpha \to +\infty$ (the unweighted spot), $\beta = \alpha/(1+\alpha) \to 1 = 100\%$. Thus, a larger α always yields a larger β.

7.4. FLEXIBLE GEOMETRIC AND ARITHMETIC AVERAGES

The flexible geometric average (FGA) is obtained by extending the standard geometric average given in (5.2) as follows:

$$FGA(n) = \prod_{i=1}^{n} (a_i)^{w(i)} = (a_1)^{w(1)}(a_2)^{w(2)} \dots (a_n)^{w(n)}\,, \tag{7.4}$$

where n is the number of observations, S_i is the ith observation, and $w(i)$ can be either $W(n,i)$ given in (7.1) or $W(n,\alpha,i)$ given in (7.2), $i = 1,2,3,\dots,n$.

Similarly, the flexible arithmetic average (FGA) is obtained by extending the standard arithmetic average in (5.1) to:

$$FAA(n) = \sum_{i=1}^{n} w(i)a_i = w(1)a_1 + w(2)a_2 + \dots + w(n)a_n\,, \tag{7.5}$$

where all parameters are the same as in (7.4).

The flexible geometric average defined in (7.4) is actually a flexible arithmetic average of the log-returns of the underlying asset. Obviously, the flexible arithmetic average defined in (7.5) becomes the same as the standard equal-weight average when $\alpha = 0$ and $\omega(i) = 1/n$.

7.5. FLEXIBLE GEOMETRIC ASIAN OPTIONS

We may call Asian options based on flexible geometric averages of the underlying asset prices flexible geometric Asian options (FGAO). The payoff of a FGAO can be given as follows:

$$PFGA = \max[\omega FGA(n) - \omega K, 0]\,, \tag{7.6}$$

where $FGA(n)$ is given in (7.4), K stands for the strike price of the option, ω is a binary indicator (1 for a call option and -1 for a put option), and max $[.,.]$ is the same mathematical function as in (3.1) which gives the larger of the two numbers.

Suppose that the underlying asset price S follows the geometric Brownian motion as specified in (2.10) and that all observations are specified as in (5.3). Substituting (5.3) into the flexible geometric average defined in (7.4) and using Proposition 5.1, we can have the following results:

Theorem 7.1. If the averaging numbers are specified as in (5.3), then the natural logarithm of $FGA(n)/S$ or $\ln[FGA(n)/S]$ is normally distributed with mean $(r - g - \sigma 2/2)T^f_{\mu,n-j} + \ln B^f(j)$ and variance $\sigma^2 T^f_{n-j}$, where

$$B^f(0) = 1, B^f(j) = \prod_{i=1}^{j} = \prod_{i=1}^{j}\{S[\tau - (n-i)h]/S\}^{w(i)}, \quad \text{for} \quad i \le j \le n\,,$$

$$T^f_{\mu,n-j} = \sum_{i=j+1}^{n} w(i)[\tau - (n-i)h]\,, \tag{7.7}$$

$$T^f_{n-j} = \sum_{i=j+1}^{n} w^2(i)[\tau - (n-i)h] + 2\sum_{i=j+1}^{n-1}\sum_{k=i+1}^{n} w(i)w(k)[\tau - (n-k)h]\,, \tag{7.8}$$

$B^f(j)$ is the weighted geometric average of the gross returns of those observations that have already passed; τ is the time to maturity of the option, and other parameters are the same as in Theorem 5.1.

Proof. Similar to that of Theorem 5.1. □

The two functions $T^f_{\mu,n-j}$ and T^f_{n-j} may be interpreted as the effective mean and volatility time functions for flexible geometric Asian options, respectively. Actually they are extensions of the mean and volatility time functions of standard geometric Asian options in Chapter 5. It can be readily shown that both these effective time functions for flexible geometric Asian options degenerate to those for standard geometric Asian options when $\alpha = 0$. It can also be shown that these two effective time functions are always smaller than the actual time to maturity of the option τ, implying that the actual variance of the log-return of the flexible geometric average is also always smaller than that of the spot price at maturity $\sigma^2\tau$.

Example 7.3. What are the effective mean and volatility time values if there are 12 observations in the geometric average, observation frequency is monthly, the averaging period has not started, the time to maturity of the option is one year, and the weight parameter $\alpha = 0.50$?

Substituting $\alpha = 0.5$ and $n = 12$ into (7.2) yields

$$w(1) = 0.034, w(2) = 0.048, w(3) = 0.059, w(4) = 0.068,$$
$$w(5) = 0.076, w(6) = 0.084\,,$$
$$w(7) = 0.090, w(8) = 0.097, w(9) = 0.103, w(10) = 0.108,$$
$$w(11) = 0.113, w(12) = 0.118\,.$$

Substituting $n = 12$, $h = 1/12$, $j = 0$, $\tau = 1$, and the weight distribution given above into (7.7) and (7.8) yields

$$T^f_{\mu,n-j} = \sum_{i=j+1}^{n} w(i)[\tau - (n-i)h]$$

$$= \sum_{i=1}^{12} w(i)[1 - (12-i)/12] = 0.629\,,$$

$$T^f_{n-j} = \sum_{i=j+1}^{n} w^2(i)[\tau - (n-i)h] + 2\sum_{i=j+1}^{n-1}\sum_{k=i+1}^{n} w(i)w(k)[\tau - (n-k)h]$$

$$= \sum_{i=1}^{12} w^2(i)[1 - (12-i)/12] + 2\sum_{i=1}^{11}\sum_{k=i+1}^{12} w(i)w(k)[1 - (12-k)/12]$$

$$= 0.476\,.$$

If we compare the effective mean and variance time values in Examples 5.1 and 7.3, we find that both the effective mean and variance time values with $\alpha = 0.50$ increase significantly from 0.542 and 0.376 to 0.629 and 0.476, respectively. The increased effective variance time indicates that the call options on flexible geometric averages with $\alpha = 0.50$ should be more expensive than those on standard geometric averages because of the increased effective volatility.

With the distribution of the flexible geometric average given in Theorem 7.1, we can readily obtain a pricing formula for flexible geometric Asian options.

Theorem 7.2. If the averaging numbers are specified in (5.3), then the price of a European-style geometric Asian option is given by the following formula:

$$C^f_{ga} = \omega S A^{fa}(j) N\left(\omega d^{fa}_{n-i} + \omega\sigma\sqrt{T^f_{n-j}}\right) - \omega K e^{-r\tau} N\left(\omega d^{fa}_{n-j}\right) \qquad (7.9)$$

where

$$A^f(j) = e^{-r(\tau - T^f_{\mu,n-j}) - \sigma^2(T^f_{\mu,n-j} - T^f_{n-j})/2} B^f(j)\,,$$

$$d^{fa}_{n-j} = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)T^f_{\mu,n-j} + \ln B^f(j)\right] \Big/ \left(\sigma\sqrt{T^f_{n-j}}\right),$$

ω is the same binary operator as in (5.5), and all other parameters are the same as in Theorem 5.1.

Proof. Similar to that of Theorem 5.1. □

We can readily show that the pricing formula for flexible geometric Asian options in Theorem 7.2 becomes exactly the same as that for standard geometric Asian options in Theorem 5.2 when $\alpha = 0$ or when the weighting scheme is even, because $T_{\mu,n-j}$, T^f_{n-j}, $B^f(j)$, $A^{fa}(j)$, and d^{fa}_{n-j} become the same as $T^{sa}_{\mu,n-j}$, T^{sa}_{n-j}, $B^{sa}(j)$, $A^{sa}(j)$, and d^{sa}_{n-j} in Chapter 5, respectively.

There is one interesting property about the pricing formula in Theorem 7.2. If we choose the weight function given in (7.2), nearly all weights will be allocated to the most current observation if the weight parameter α goes extremely large, given the number of observations and observation frequency. Specifically, $W(n, \alpha, n) \to 1$ and $W(n, \alpha, i) \to 0$ for $i = 1, 2, 3, \ldots, n-1$,[2] given the observation frequency h and number of observations n. Substituting $W(n, \alpha, n) \to 1$ and $W(n, \alpha, i) \to 0$ for $i = 1, 2, 3, \ldots, n-1$ into the expressions of the two effective time functions yields $T^f_{\mu,n-j} \to \tau$ and $T^f_{n-j} \to \tau$. As the two effective time functions approach the time to maturity of the option, the pricing formula for flexible geometric Asian options in Theorem 7.2 approaches the extended Black-Scholes formula given in (3.2).

Example 7.4. What are the prices of the call and put options with strike price \$400 to expire in one year and based on the flexible geometric average of monthly gold prices with the weight parameter $\alpha = 0.5$, given that the spot gold price is \$390 per ounce, the interest rate 7%, yield on the gold is zero, and volatility of gold return is 20%?

Since the time to maturity, observation frequency, and the number of observations are the same as in Example 7.3, we can use the two effective time values in Example 7.3. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^f_{\mu,n-j} = 0.629$, and $T^f_{n-j} = 0.476$ into (7.9) yields

$$\begin{aligned} d^{fa}_{n-j} &= \left\{ \ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right) T^f_{\mu,n-j} + \ln[B^f(j)] \right\} \Big/ \left(\sigma\sqrt{T^f_{n-j}}\right) \\ &= \left[\ln\left(\frac{390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.629 + \ln(1) \right] \Big/ \\ &\quad \left(0.20 \times \sqrt{0.476}\right) = 0.0444, \end{aligned}$$

[2]The last term in the denominator of the weight function given in (7.2) n^α dominates all other terms i^α for $i = 1, 2, 3, \ldots, n-1$. Dividing both numerator and denominator of (7.2) yields $W(n, \alpha, n) \to 1$ and $W(n, \alpha, i) \to 0$ as $\alpha \to +\infty$.

the call option price:

$$\begin{aligned} C &= SA^f(j)N\left(d_{n-j}^{fa} + \sigma\sqrt{T_{n-j}^f}\right) - Ke^{-r\tau}N(d_{n-j}^{fa}) \\ &= 390 \times e^{-0.07(1-0.629)-0.20^2\times(0.629-0.476)/2}N(0.1824) \\ &\quad - 400 \times e^{-0.07}N(0.0444) \\ &= \$23.746\,, \end{aligned}$$

and the corresponding geometric put option price can be found by substituting $\omega = -1$ and other parameters into (7.9)

$$\begin{aligned} P &= -SA^{fa}(j)N\left(-d_{n-j}^{fa} - \sigma\sqrt{T_{n-i}^f}\right) + Ke^{-r\tau}N(-d_{n-j}^{fa}) \\ &= -390 \times e^{-0.07(1-0.629)-0.20^2\times(0.629-0.476)/2}N(-0.1824) \\ &\quad + 400 \times e^{-0.07}N(-0.0444) \\ &= \$17.867\,. \end{aligned}$$

7.6. APPROXIMATING FLEXIBLE ARITHMETIC AVERAGES WITH FLEXIBLE GEOMETRIC AVERAGES

Using a general mean measure, we found an efficient approximation for a standard equal-weighting arithmetic average using its corresponding geometric average and following Taylor's series expansion in Chapter 6. We will now approximate a flexible arithmetic average given in (7.5) so that we will be able to find a reasonably approximated closed-form solution to flexible arithmetic Asian options. We mainly follow Zhang (1995b) in the following sections.

When we extend the standard Asian options with equal weights to flexible arithmetic Asian options based on the flexible arithmetic average given in (7.5), we simply need to find the corresponding expressions for the mean and variance functions which are necessary to calculate the approximation coefficient in Theorem 6.1. We can obtain the mean function for a flexible arithmetic average:

$$E(v^f) = \nu^2h^2Var(i|\omega_i) + \sigma^2h\left[\sum_{i=1}^{n} i\omega_i(1-\omega_i) - 2\sum_{i=1}^{n} i\omega_i \sum_{l=i+1}^{n} \omega_l\right] \quad (7.10)$$

where

$$Var(i|\omega_i) = \sum_{i=1}^{n}(i - M^2)\omega_i,$$

$$M = \sum_{i=1}^{n} i\omega_i, \ \nu = (r - g - \sigma^2/2),$$

and $E(v^f)$ stands for the mean function with flexible weights ω_i, $i = 1, 2, \ldots, n$ given in (7.1) or (7.2).

The mean function given in (7.10) cannot, in general, be expressed in compact form for arbitrary weight distributions. However, it is easy to obtain the weighted summation to calculate the mean function. It can be readily shown that the mean function with flexible weights $E(v^f)$ given in (7.10) degenerates to the mean function with equal weights $E(v)$ given in (6.8a) when $\omega_i = 1/n$ for all observations $i = 1, 2, \ldots, n$.

Example 7.5. Find the mean function in (7.10) given $\tau = 1$ year, the weight parameter $\alpha = 0.5$, interest rate 7%, yield on the gold zero, volatility of gold return 20%, the number of observation 12, and observation frequency 1/12.

Substituting $\tau = 1$, $\alpha = 0.5$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $n = 12$, $h = 1/12$, and the weight distribution in Example 7.2 into (7.10) yields

$$\nu = 0.07 - 0 - 0.20^2/2 = 0.05\,,$$

$$M = \sum_{i=1}^{n} i\omega_i = \sum_{i=1}^{12} i\omega_i = 7.5464\,,$$

$$Var(i|\omega_i) = \sum_{i=1}^{n}(i - M^2)\omega_i = \sum_{i=1}^{12}(i - 7.5464)^2\omega_i = 10.3486\,,$$

$$E(v^f) = \nu^2 h^2 Var(i|\omega_i) + \sigma^2 h \left[\sum_{i=1}^{12} i\omega_i(1 - \omega_i) - 2\sum_{i=1}^{12} i\omega_i \sum_{l=i+1}^{12} \omega_l\right]$$

$$= 0.0063\,.$$

The variance function, however, cannot be extended to the flexible arithmetic average so easily as the mean function. The difficulty is that expected values of products of observations in the third and the fourth powers[3] have

[3] A product of four variables Z_1, Z_2, Z_3, and Z_4 of the form $Z_1^a Z_2^b Z_3^c Z_4^d$ is of power a+b+c+d. The numbers of the tri-variate and quad-normal distribution function values which have to be estimated are $n(n-1)(n-2)/6$ and $n(n-1)(n-2)(n-3)/24$, respectively.

to be derived, and these expected values are generally expressed in terms of tri-variate and quad-variate cumulative normal distribution functions. It is not convenient to estimate these distribution function values. To make the matter worse, the number of tri-variate and quad-variate normal distribution function values which have to be estimated increases dramatically with the number of observation n.[4]

We could avoid these tri-variate and quad-variate normal distribution function values by using some appropriate approximations which do not significantly affect the accuracy of the final results. The coefficients of the tri-variate and quad-variate normal cumulative function values are products of weights in the third and fourth powers, respectively. As all weights are between 0 and 1, weight products in the third and the fourth powers are generally much smaller than 1, and values of all cumulative functions are always smaller than or equal to 1. Therefore, products of weights and cumulative functions are generally much smaller than 1 and can be neglected without seriously affecting the accuracy levels. Neglecting weight products in the third and fourth powers, we can obtain the approximated variance function as follows:

$$Var(v^f) = 2\nu^2 h^2 Var(i|\omega_i)\left[E(v^f) - \frac{1}{2}\nu^2 h^2 Var(i|\omega_i)\right] + 4\sigma^2 hQ - \left[E(v^f)\right]^2, \tag{7.11}$$

where

$$Q = \sum_{i=1}^{n} i(i-M)^2\omega_i^2 + 2\sum_{i=1}^{n} i(i-M)\omega_i \sum_{l=i+1}^{n}(l-M)\omega_l,$$

and $E(v^f)$ is given in (7.10) and all other parameters are the same as in (7.10).

The value of the variance function given in (7.11) can be easily estimated given a weight distribution because the summations involved can be obtained readily with any computer for any number of observations n, volatility of the underlying asset return σ, time to maturity τ, interest rate r, and yield on the underlying asset.

Example 7.6. Find the approximated variance function in (7.11) given $\tau = 1$ year, the weight parameter $\alpha = 0.5$, interest rate 7%, yield on the gold

[4] If all the observations are of the same, then both the flexible geometric and arithmetic averages are the same as the equal value; when $\alpha \to +\infty$, both the flexible geometric and arithmetic averages degenerate to one same current observation. However, these are two trivial cases as they do not exist in general.

zero, volatility of gold return 20%, number of observation 12, and observation frequency is 1/12.

Substituting $\tau = 1$, $\alpha = 0.5$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $n = 12$, $h = 1/12$, $\nu = 0.05$, $M = 7.5464$, $Var(i|\omega_i) = 10.3486$, and $E(v^f) = 0.0063$ in Example 7.5, and the weight distribution in Example 7.2 into (7.11) yields

$$\begin{aligned} Q &= \sum_{i=1}^{n} i(i-M)^2\omega_i^2 + 2\sum_{i=1}^{n} i(i-M)\omega_i \sum_{l=i+1}^{n} (l-M)\omega_l \\ &= \sum_{i=1}^{n} i(i-M)^2\omega_i^2 + 2\sum_{i=1}^{12} i(i-M)\omega_i \sum_{l=i+1}^{12} (l-M)\omega_l = 11.2791 \end{aligned}$$

and

$$\begin{aligned} Var(v^f) &= 2\nu^2h^2Var(i|\omega_i)\left[E(v^f) - \frac{1}{2}\nu^2h^2Var(i|\omega_i)\right] + 4\sigma^2hQ - [E(v^f)]^2 \\ &= 2\times 0.05^2\left(\frac{1}{12}\right)^2\times 10.3486\left[0.0063 - \frac{1}{2}\times 0.05^2\left(\frac{1}{12}\right)^2\times 10.3486\right] \\ &\quad + 4\times 0.20^2\times\frac{1}{12}\times 11.2791 - 0.0063^2 \\ &= 0.1504\,. \end{aligned}$$

With the mean and variance functions in (7.10) and (7.11) and following the similar procedures to obtain Theorem 6.1, we could approximate the flexible arithmetic average given in (7.5) with its corresponding geometric average given in (7.4).

Theorem 7.3. The flexible arithmetic average (FAA) given in (7.5) can be approximated with its corresponding flexible geometric average (FGA) given in (7.4) as follows:

$$FAA(\tau) \cong \kappa^f FGA(\tau)\,, \tag{7.12}$$

where

$$\kappa^f = 1 + \frac{1}{2}E(v^f) + \frac{1}{4}\left\{[E(v^f)]^2 + Var(v^f)\right\}\,,$$

and $E(v^f)$ and $Var(v^f)$ are given in (7.10) and (7.11), respectively.

As the lognormalization factor κ^f is greater than 1 in general, (7.12) indicates that a flexible arithmetic average is greater than its corresponding geometric average. This can be understood as an extension of the fact that an equally weighted standard arithmetic average is always greater than its corresponding geometric average for nontrivial unequal observations. Thus, (7.12) also indicates that a flexible geometric average is a lower bound for its corresponding arithmetic average.

Example 7.7. Find the lognormalization factor κ^f in Theorem 7.3 given $\tau = 1$ year, the weight parameter $\alpha = 0.5$, interest rate 7%, yield on the gold zero, volatility of gold return 20%, the number of observation is 12, and observation frequency is 1/12.

Substituting $E(v^f) = 0.0063$ in Example 7.5 and $Var(v^f) = 0.1504$ in Example 7.6 into (7.12) yields

$$\begin{aligned}\kappa^f &= 1 + \frac{1}{2}E(v^f) + \frac{1}{4}\{[E(v^f)]^2 + Var(v^f)\}\\ &= 1 + \frac{1}{2} \times 0.0063 + \frac{1}{4} \times (+0.0063^2 + 0.1504) = 1.04076\,.\end{aligned}$$

7.7. FLEXIBLE ARITHMETIC ASIAN OPTIONS

Using the flexible arithmetic average in (7.5), we can express the payoff of a European-style Asian option based on the flexible arithmetic average of the underlying asset prices as

$$FAAOPPOF = \max[\omega FAA(\tau) - \omega K, 0]\,, \tag{7.13}$$

where ω is the same binary indicator (1 for a call option and -1 for a put option).

Using the approximation formula given in Theorem 7.3, we can obtain the price of a flexible arithmetic Asian option in the following theorem.

Theorem 7.4. The price of a flexible arithmetic Asian option can be approximated as follows

$$C_j^{fa} = \omega S\kappa^f e^{-gT^f_{\mu,n-j}} A^f(j) N\left(\omega d^{fa}_{n-j} + \omega\sigma\sqrt{T^f_{n-j}}\right) - \omega K e^{-r\tau}(\omega d^{fa}_{n-j})\,, \tag{7.14}$$

where

$$d^{fa}_{n-j} = \left[\ln\left(\frac{\kappa^f S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)T^f_{\mu,n-j} + \ln B^f(j)\right] \Big/ \left(\sigma\sqrt{T^f_{n-j}}\right),$$

all other parameters involved are the same as in Theorems (6.1) and (6.2).

The parameter d^{fa} in (7.14) is for "flexible arithmetic" Asian options compared to d^f which is the corresponding parameter for "flexible geometric" Asian options in (7.7). If we compare (7.14) and (7.7), we can readily find that the flexible arithmetic Asian option prices can be easily obtained by multiplying the current asset price S by the approximation factor κ^f. As any flexible arithmetic average is always greater than its corresponding geometric average, (7.14) indicates that a flexible arithmetic Asian call option is more expensive than its corresponding geometric call option.

Example 7.8. What are the prices of the call and put options with strike price \$400 to expire in one year based on the flexible arithmetic average of monthly gold prices with the weight parameter $\alpha = 0.5$, given the spot gold price is \$390 per ounce, interest rate 7%, yield on the gold is zero, and volatility of gold return is 20%?

Since the time to maturity, the observation frequency, the number of observations and the weight parameter are the same as in Examples 7.5, 7.6, and 7.7, we can use the effective time values in Example 7.3 and the lognormalization factor in Example 7.7. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T^f_{\mu,n-j} = 0.629$, $T^f_{n-j} = 0.476$, $\kappa^f = 1.04076$ into (7.14) yields

$$\begin{aligned} d^{fa}_{n-j} &= \left[\left(\frac{\kappa^f S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right) T^f_{\mu,n-j} + \ln B^f(j)\right] \Big/ \left(\sigma\sqrt{T^f_{n-j}}\right) \\ &= \left[\ln\left(\frac{1.04076 \times 390}{400}\right) + \left(0.07 - 0 - \frac{1}{2} \times 0.20^2\right) \times 0.629 + \ln(1)\right] \Big/ \\ &\quad \left(0.20\sqrt{0.476}\right) \\ &= 0.3119\,, \end{aligned}$$

the call option price is

$$\begin{aligned} C &= S\kappa^f e^{-gT^f_{\mu,n-j}} A^f(j) N\left(d^{fa}_{n-j} + \sigma\sqrt{T^f_{n-j}}\right) - Ke^{-r\tau} N(d^{fa}_{n-j}) \\ &= 390 \times 1.04076 \times e^{-0.07(1-0.629)-0.20^2\times(0.629-0.476)/2} \\ &\quad + N\left(0.3119 + 0.20\sqrt{0.476}\right) - 400 \times e^{-0.07\times 1} N(0.3119) \\ &= \$32.634. \end{aligned}$$

7.8. FLEXIBLE-AVERAGE-STRIKE ASIAN OPTIONS

Theorem 7.4 provides an approximated formula for a flexible arithmetic Asian option with fixed strike price using the closed-form formula of a flexible geometric Asian option. The payoff of an Asian option with strike prices specified as the flexible geometric average of the underlying asset prices can be given as follows:

$$PFCGA = \max[\omega S(\tau) - \omega FGA(n), 0]\,, \tag{7.15}$$

where $FGA(n)$ is given in (7.4), and ω is the same binary operator as in (7.14).

Following the same procedure, we can obtain closed-form and approximated closed-form formulas for Asian options with strike prices specified as flexible geometric and flexible arithmetic averages. In order to price Asian options with strike prices specified as the flexible averages of their underlying asset prices, we need to know the correlation coefficient between the log-return of the underlying asset and that of the flexible geometric average defined in (7.4).

Theorem 7.5. The correlation coefficient between the log-return of the underlying asset and that of the flexible geometric average defined in (7.4) is

$$\rho^f = \frac{[\sigma^2 + (r - g - \frac{1}{2}\sigma^2)^2](r - \frac{n-1}{2}h) - [r - g - \frac{1}{2}\sigma^2]^2 \tau T^f_{\mu,n-j}}{\sigma^2 \sqrt{\tau T^f_{n-j}}}, \quad (7.16)$$

where $T^f_{\mu,n-j}$ and T^f_{n-j} are the effective mean time and variance time functions given in (7.7) and (7.8), respectively.

Proof. Similar to the proof of Theorem 5.4. □

Example 7.9. What is the correlation coefficient between the log-return of the underlying asset and that of the flexible geometric average with the weight parameter $\alpha = 0.50$, interest rate is 7%, yield on the underlying asset zero, volatility of the underlying asset 20%, time to maturity one year, and observation frequency monthly?

We can use the results in Example 7.3 $T^f_{\mu,n-j} = 0.629$, $T^f_{nj} = 0.476$ because the conditions of Example 7.3 are the same as in this example. Substituting $\tau = 1$, $T^f_{\mu,n-j} = 0.629$, $T^f_{n-j} = 0.476$, $r = 0.07$, $g = 0.03$, $\sigma = 0.20$ into (7.16) yields $\rho^f = 0.7772 = 77.72\%$.

With the correlation coefficient defined in (7.16), we can obtain the pricing formula for Asian options with payoffs specified in (7.15) in the following theorem.

Theorem 7.6. The price of a European-style Asian option with strike price specified as the geometric average of the underlying asset prices given in (7.15) is

$$AGESTK = \omega S \left[e^{-g\tau} N\left(\omega D^f_{fg1}\right) - A^f(j) e^{-g T^f_{\mu,n-j}} N(\omega D^f_{fg2}) \right], \quad (7.17)$$

where

$$D_{fg2}^{f} = \frac{-\ln[B^{f}(j)] + (r - g - \frac{1}{2}\sigma^2)(\tau - T_{\mu,n-j}^{f}) + \sigma^2(\rho^{f}\sqrt{\tau T_{n-j}^{f}} - 1)}{\sigma\sqrt{\tau_e^{f}}},$$

$$D_{fg1}^{f} = D_{fg2}^{f} + \sigma\sqrt{\tau_e^{f}}, \ \tau_e^{f} = \tau - 2\rho^{f}\sqrt{\tau T_{n-j}^{f}} + T_{n-j}^{f},$$

$A^{f}(j)$ and $B^{f}(j)$ are the same as in (7.9), ρ^{f} is given in (7.16), and $T_{\mu,n-j}^{f}$ and T_{n-j}^{f} are the same effective mean and variance functions given in (7.7) and (7.8).

Proof. Similar to the proof of Theorem 5.5. □

Example 7.10. What are the Asian call and put option prices with strike price specified as the flexible geometric average of the underlying asset prices, given other information the same as in Example 7.9?

Since the time to maturity, observation frequency, the number of observations, and the weight parameter α are the same as in Examples 7.7, 7.8, and 7.9, we can use the two effective time values $T_{\mu,n-j}^{f} = 0.629$, $T_{n-j}^{f} = 0.476$ and the correlation coefficient $\rho^{f} = 0.7772 = 77.72\%$ in Example 7.9. Substituting $S = \$390$, $K = \$400$, $r = 0.07$, $g = 0$, $\sigma = 0.20$, $\omega = 1$, $T_{\mu,n-j}^{f} = 0.629$, $T_{n-j}^{f} = 0.476$, and the correlation coefficient $\rho^{f} = 0.7772$ into (7.17) yields

$$\begin{aligned}\tau_e^{f} &= \tau - 2\rho^{f}\sqrt{\tau T_{n-j}^{f}} + T_{n-j}^{f} \\ &= 1 - 2 \times 0.7772 \times \sqrt{1 \times 0.476} + 0.476 = 0.4036,\end{aligned}$$

$$D_{fg2}^{f} =$$

$$\frac{-\ln(1) + (0.70 - 0 - \frac{1}{2} \times 0.20^2)(1 - 0.629) + 0.20^2(0.7772\sqrt{1 \times 0.476} - 1)}{0.20\sqrt{0.4036}}$$

$$= -0.0000$$

$$D_{fg1}^{f} = -0.00001 + 0.20 \times \sqrt{0.4036} = 0.1271,$$

the call average-strike option price

$$\begin{aligned}C &= S\left[e^{-g\tau}N(D_{fg1}^{f}) - A^{f}(j)e_{\mu,n-j}^{-gT^{f}}N(D_{fg2}^{f})\right] \\ &= 390 \times [N(0.1271) - 0.9741 \times N(-0.00001)] = \$25.299;\end{aligned}$$

and the corresponding put option price can be found by substituting $\omega = -1$ into (7.17)

$$P = S\left[e^{-g\tau}N(-D^f_{fg1}) + A^f(j)e^{-gT^f_{\mu,n-j}}N(-D^f_{fg2})\right]$$
$$= -390 \times [N(-0.1271) + 0.9714 \times N(0.00001)] = \$14.145\,.$$

Theorem 7.7. The price of an Asian option with strike price specified as the flexible arithmetic average given in (7.5) of the underlying asset prices specified in (5.3) can be approximated as:

$$AGESTK = \omega S\left[\kappa^f e^{-g\tau}N(\omega D^f_{fa1}) - A^f(j)e^{-gT^f}_{\mu,n-j}N\left(\omega D^f_{fa2}\right)\right], \quad (7.18)$$

where

$$D^f_{fa2} = \frac{-\ln[\kappa^f B^f(j)] + (r - g - \frac{1}{2}\sigma^2)(\tau - T^f_{\mu,n-j}) + \sigma^2(\rho^f\sqrt{\tau T^f_{n-j}} - 1)}{\sigma\sqrt{\tau^f_e}},$$

$$D^f_{fa1} = D^f_{fa2} + \sigma\sqrt{\tau^f_e}\,,$$

and all parameters are the same as in Theorems (7.4) and (7.5).

Proof. Similar to the proof of Theorem 7.6. □

7.9. FLEXIBLE SENSITIVITIES

The flexible Asian options (FAOs) discussed in this chapter have very interesting characteristics that neither plain vanilla nor standard Asian options possess. These characteristics can be clearly expressed by various sensitivities such as deltas, gammas, vegas, thetas, and so on. We try to illustrate the flexibility of FAOs by analyzing their deltas. The delta of a FAO can be readily derived from (7.12):

$$Delta^{fa}_{n-j} = \kappa^f\,Delta^f_{n-j}$$
$$= \omega\kappa^f\left[1 - \sum_{i=1}^{j} w(i)\right] A^f(j)N\left(\omega d^{fa}_{n-j} + \omega\sigma\sqrt{T^f_{n-j}}\right), \quad (7.19)$$

where all parameters are the same as in (7.9).

The deltas of plain vanilla options in the Black-Scholes model are fixed given the parameters S, r, K, σ, and τ, and they are also fixed in standard Asian options once the number of observations and the observation frequency are given. However, the deltas and other sensitivities of FAOs depend on the weight distribution function given in (7.1) or (7.2). In other words, the deltas can change with various weight functions $q(i)$ given in (7.1) or various values for the weight control parameter α in (7.2). The second columns of Tables 7.1 and 7.2 provide the values of the deltas of flexible arithmetic Asian call (FAAC) options with the observation number n = 2, 4, 6, 8, 10, and 12 and the weight parameter α = 0, 0.25, 0.5, 1, 2, 10, 50, and 100, given the spot price S = \$100, the strike price K = \$100, the interest rate r = 10%, the volatility of the underlying asset σ = 10% and 15%, the time to maturity of the option τ = 1 year, and the observation frequency h = 1 month.

Tables 7.1 and 7.2 show that deltas of FAACs are different with different values for the weight control parameter α and they tend to be more different with higher volatilities. Thus, desirable hedging ratios may be obtained by adjusting and finding some appropriate weight functions. Therefore, FAOs may have interesting applications for hedging and other trading strategies.

7.10. "TREND" OPTIONS

The weighting scheme in (7.1) can be either ascending, descending, U-shaped, V-shaped, W-shaped, inverse U-shaped, inverse V-shaped, inverse W-shaped, or even "Z"-shaped ("Zigzag" shaped), depending upon the buyer's trend expectation of the underlying market. If the buyer of the option believes that the underlying market will follow a particular trend or pattern, it would be perfect to have a weight design that best fits his/her trend expectation, so that the flexible average would be greater with this weight design than with any other weight allocations. Since these options can take the best advantage of clients' trend expectations, we may simply call these options Z-shaped trend options, or simply trend options, because Z-shaped patterns include all possible patterns.

Most existing path-dependent options depend on either one or a few points on the path. For example, a barrier option is characterized by whether the barrier is broken, while the pattern in which the underlying asset price moves does not matter; the payoff of a lookback option is only affected by the maximum or the minimum points on the path; the payoff of a standard Asian option is affected by the prespecified equally weighted number of observations. We can call these path-dependent options point-path-dependent as they depend only on one or a few points on the path. As a trend option depends on the shape or the curve of the path and can take the best advantage of a participant's trend expectation, we may say that they are

Table 7.1. Comparisons between prices of approximated Flexible Asian Options (FAASIAN) simulatex prices of FAOs, and prices of Flexible Geometric Asian Options (FGASIAN) given spot price s = strike price = \$100, interest rate = 10%, time to maturity = 1 year, volatility of the underlying asset σ = 10%, observation frequency = 1 month.

Alpha	Delta	FAASIAN	Simulated PS	FGASIAN	FAASIAN-Simulated PS
Number of Observations $n = 2$					
0.00	0.852	10.21	10.25	10.21	−0.040
0.25	0.852	10.22	10.26	10.22	−0.040
0.50	0.852	10.22	10.26	10.22	−0.040
1.00	0.852	10.24	10.28	10.24	−0.040
2.00	0.853	10.27	10.30	10.27	−0.040
10.0	0.853	10.31	10.34	10.31	−0.034
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034
Number of Observations $n = 4$					
0.00	0.849	10.02	10.06	10.01	−0.043
0.25	0.849	10.04	10.09	10.04	−0.043
0.50	0.850	10.07	10.11	10.06	−0.044
1.00	0.850	10.11	10.16	10.11	−0.044
2.00	0.851	10.17	10.22	10.17	−0.043
10.0	0.853	10.30	10.33	10.30	−0.035
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034
Number of Observations $n = 6$					
0.00	0.846	9.84	9.88	9.81	−0.033
0.25	0.847	9.89	9.92	9.86	−0.035
0.50	0.848	9.93	9.96	9.90	−0.036
1.00	0.849	9.99	10.03	9.97	−0.040
2.00	0.850	10.08	10.13	10.07	−0.043
10.0	0.853	10.27	10.31	10.27	−0.038
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034
Number of Observations $n = 8$					
0.00	0.844	9.69	9.69	9.62	−0.004
0.25	0.854	9.75	9.75	9.68	−0.008
0.50	0.846	9.80	9.81	9.74	−0.014
1.00	0.847	9.88	9.91	9.84	−0.024
2.00	0.850	10.00	10.03	9.97	−0.037
10.0	0.852	10.24	10.28	10.24	−0.040
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034

Table 7.1. (*Continued*)

Alpha	Delta	FAASIAN	Simulated PS	FGASIAN	FAASIAN-Simulated PS
Number of Observations $n = 10$					
0.00	0.844	9.56	9.50	9.42	0.051
0.25	0.845	9.63	9.59	9.51	0.040
0.50	0.846	9.69	9.66	9.58	0.027
1.00	0.847	9.79	9.78	9.71	0.004
2.00	0.848	9.92	9.94	9.88	−0.024
10.0	0.852	10.22	10.26	10.21	−0.041
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034
Number of Observations $n = 12$					
0.00	0.844	9.46	9.32	9.22	0.138
0.25	0.845	9.54	9.42	9.33	0.117
0.50	0.846	9.61	9.51	9.42	0.093
1.00	0.847	9.71	9.66	9.58	0.050
2.00	0.848	9.85	9.85	9.78	−0.002
10.0	0.852	10.19	10.23	10.18	−0.043
50.0	0.853	10.31	10.34	10.31	−0.034
100	0.853	10.31	10.34	10.31	−0.034

curve-path-dependent or trend-dependent, and that they are certainly more path-dependent than most other path-dependent options.

As timing is always the most crucial point in all financial decisions, it is almost impossible to correctly forecast spot prices, indices, or exchange rates upon which plain vanilla options are based. However, it is somehow easier to foresee the trend of the underlying markets using both technical and fundamental analyses. Thus, trend options are bound to rise in popularity soon.

The purpose of this section is to illustrate the construction of a "trend" option. If an investor believes that the underlying asset is bearish, he/she would buy put and sell call options using common knowledge of vanilla options. However, as vanilla options are path-independent, he/she may still lose money even if his/her trend expectation is correct most of the time but the price jumps up shortly before or at maturity resulting from some unexpected information. The standard Asian options cannot improve the situation much because at most it may average out the high and the low due to the equal weights. An upward trend option also cannot improve the situation because it allocates heavier (resp. less) weights on lower (resp. higher)

Table 7.2. Comparisons between prices of approximated Flexible Asian Options (FAASIAN), simulated prices of FAOs, and prices of Flexible Geometric Asian Options (FGASIAN) given spot price 8 = strike price = \$100, interest rate = 10%, time to maturity = 1 year, volatility of the underlying asset σ = 15%, observation frequency = 1 month.

Alpha	Delta	FAASIAN	Simulated PS	FGASIAN	FAASIAN-Simulated PS
Number of Observations $n = 4$					
0.00	0.753	10.32	10.45	10.25	−0.122
0.25	0.755	10.44	10.56	10.37	−0.122
0.50	0.757	10.55	10.67	10.48	−0.122
1.00	0.760	10.74	10.86	10.69	−0.120
2.00	0.764	11.03	11.14	11.00	−0.110
10.0	0.770	11.61	11.65	11.61	−0.040
50.0	0.771	11.67	11.70	11.67	−0.028
100	0.771	11.67	11.70	11.67	−0.028
Number of Observations $n = 8$					
0.00	0.749	8.94	8.77	8.36	0.176
0.25	0.752	9.20	9.06	8.66	0.145
0.50	0.753	9.42	9.32	8.94	0.104
1.00	0.755	9.77	9.75	9.41	0.023
2.00	0.757	10.25	10.32	10.05	−0.073
10.0	0.768	11.36	11.44	11.35	−0.081
50.0	0.771	11.67	11.70	11.67	−0.028
100	0.771	11.67	11.70	11.67	−0.027
Number of Observations $n = 12$					
0.00	0.800	8.37	7.00	6.37	1.372
0.25	0.794	8.70	7.49	6.90	1.207
0.50	0.787	8.93	7.92	7.36	1.010
1.00	0.775	9.25	8.60	8.11	0.648
2.00	0.763	9.70	9.48	9.09	0.218
10.0	0.765	11.08	11.19	11.05	−0.107
50.0	0.771	11.66	11.69	11.66	−0.030
100	0.771	11.67	11.70	11.67	−0.027

expected prices. Only a downward trend option can take the best advantage because it allocates heavier (resp. less) weights on higher (resp. lower) expected prices so that the flexible average can be rather high. Thus, one can buy a downward trend call option when one believes the trend is to be downward. This sounds somewhat contradictory to existing knowledge to buy puts (resp. calls) on downward (resp. upward) trends, yet it is a reasonable product according to our above construction.

A symmetric U-shaped (resp. inverse U-shaped) trend option consists of one downward (resp. upward) and one upward (resp. downward) trend option with equal number of observations in each segment. A symmetric V-shaped (resp. inverse V-shaped) trend option is a special case of the U-shaped (resp. inverse U-shaped) trend option when the weight distribution is a linear function in each segment. A W-shaped (resp. inverse W-shaped) trend option includes two V-shaped (resp. inverse V-shaped) trend options. With the weight allocation described above, the pricing formula remains the same as it is the same for both upward and downward trend options.

The symmetric U-shaped and inverse U-shaped trend options can be extended to have different number of observations in each trend pattern. These asymmetric U-shaped and inverse-U-shaped trend options are the actual building blocks for all trend options. In general, a trend option may have one asymmetric U-shaped or inverse-U-shaped trend in it or it may have both, or several of them in permutation, depending upon how specific the trend is expected. The order of the permutation is very important here as a trend consisting of one asymmetric U-shaped trend followed by an asymmetric inverse U-shaped one is obviously very different from a trend consisting of one asymmetric inverse U-shaped trend followed by a symmetric U-shaped one.

7.11. CONCLUSIONS

Asian options have been popular in the OTC markets for several years. To date, most of these products have employed equally weighted averages, but there is no sound reason why alternative weighting schemes cannot be used. In this chapter we provided approximated closed-form solutions for flexible arithmetic Asian options which are more attractive to most traders than standard Asian options. These flexible Asian options permit flexible schemes in weighting various observations and thus provide additional flexibility for traders to build their specific expectations of the underlying market movement into the model. These flexible Asian options are of particular interest to traders who wish to assign greater (less) emphasis to the role played by the more (less) recently observed prices in the average.

We have compared the results from the approximated formulas with those from Monte Carlo simulations. Our comparisons indicate that the approximated formulas provide very accurate results with given parameters. They not only reduce the time to calculate prices but also provide convenient ways to find expressions for the sensitivities of flexible arithmetic Asian options with respect to various parameters.

QUESTIONS AND EXERCISES

Questions

7.1. What is meant by a flexible average?

7.2. What is a flexible geometric average?

7.3. What is a flexible arithmetic average?

7.4. Are flexible arithmetic averages always greater than their corresponding geometric averages? Why?

7.5. Are flexible arithmetic Asian options always more expensive than their corresponding geometric Asian options? Why?

7.6. Why is the measure of inequality introduced?

7.7. Why do we need flexible Asian options?

7.8. Does the closed-form pricing formula for Asian options based on flexible geometric averages include the Black-Scholes pricing formula as a special case? Why?

7.9. What is the most serious difficulty in approximating prices of Asian options based on flexible arithmetic averages?

7.10. Is the correlation coefficient between the log-returns of the underlying asset and the flexible geometric average always constant?

7.11. Is the correlation coefficient between the log-returns of the underlying asset and the flexible geometric average always the same as that between the log-returns of the underlying asset and the flexible arithmetic average?

Exercises

7.1. Find the weight distribution if the number of observation $n = 10$ and the weight function $q(i)$ is specified as $q(i) = 2^{-0.75i}$ in (7.1).

7.2. Find the weight distribution if the number of observation $n = 5$ and the weight parameter $\alpha = 1$ in (7.2).

7.3. Find the weight distribution if the number of observation $n = 5$ and the weight parameter $\alpha = 1.5$ in (7.2).

7.4. What are the effective mean and volatility time values if there are 8 observations in the geometric average, observation frequency is weekly, the averaging period has not started, the time to maturity of the option is 8 weeks, and the weight parameter $\alpha = 0.25$?

7.5. What are the effective mean and volatility time values if α is changed to 0.75 and other parameters remain unchanged?

7.6. Compare the differences between the results of Exercises 7.4 and 7.5 and try to find how the effective time values change with the weight parameter.

7.7. Find the prices of the geometric Asian call and put options with monthly observation to expire in half a year with strike price \$460 if the current underlying index is \$450, the interest rate is 8%, yield on the underlying index is 4%, volatility of the index is 25%, and the weight parameter $\alpha = 0.50$.

7.8. Answer the same questions in Exercise 7.7 if the observation frequency is bimonthly and other parameters are the same as in Exercise 7.7.

7.9. Show the correlation coefficient between the spot at maturity $S(\tau)$ and the flexible geometric average is bivariate log normally distributed with the correlation coefficient given in (7.16).

7.10. Find the correlation coefficient between the log-returns of the underlying asset and the geometric average with weekly observations, time to maturity 20 weeks, interest rate 7%, yield on the underlying asset 2%, spot price \$100, and the weight parameter $\alpha = 1.2$.

7.11. Find the prices of European Asian options with flexible geometric strike prices with weekly observations, time to maturity 6 weeks, interest rate 6.5%, yield on the underlying asset 2%, spot price \$100, and the weight parameter $\alpha = 0.20$.

7.12. Find the deltas of the call and put options in Exercise 7.7.

7.14. Find the deltas of the call and put options in Exercise 7.8.

7.15. Show the identity: $SA^f(j)f(d_{n-j}^{fa} + \sigma\sqrt{T_{n-j}^f}) = Ke^{-r\tau}f(d_{n-j}^f)$.

Chapter 8

FORWARD-START OPTIONS

8.1. INTRODUCTION

Vanilla options become effective immdiately after they are bought or sold. There are, however, some exotic options such as forward-start options which are only effective some time after they are bought or sold. Forward-start options are options which start at some prespecified time in the future with the strike price set to be the underlying asset price at the time when it starts. Alternatively, forward-start options are at-the-money options when they actually start, yet the strike price is not known at present. Forward-start options and their variations are normally used in interest-rate derivatives markets in the form of periodic caps and floors, because they provide a cheaper way to hedge or speculate interest rate derivatives or any other assets which are highly sensitive to interest rates. As standard caps (resp. floors) are strings of standard calls (resp. puts) with prespecified strike prices, these caps and floors can be very expensive when interest rate changes dramatically. Yet risk managers could use periodic caps where the strike rate of each individual call option is set at a certain spread above the previous interest reference. The purpose of this chapter is to show how to price future-start options and analyze their basic properties compared with vanilla options.

8.2. PRICING FORWARD-START OPTIONS

The payoff of a European-style forward-start option can be expressed as:

$$PFST = \max\{\omega[S(\tau) - S(\tau_1)], 0\}, \tag{8.1}$$

where $\tau_1 = t_1 - t$ is the time in the future when the option becomes valid; $\tau = t^* - t$ is the time to maturity of the option, $t < t_1 < t^*$; max (. , .) is a function that gives the larger of two numbers; and ω is a binary operator (1 to stand for a call and -1 for a put option).

Obviously, the forward-start option is at-the-money at time $t = t_1$. This is a very important characteristic of forward-start options.

It seems difficult to price forward-start options because the strike price $K = S(t_1)$ is not known at present. The uncertainty of $K = S(t_1)$ does create some difficulties, yet they can be readily removed. Since any forward-start options are at-the-money options at the time when the underlying asset prices are observed, we know their values after the observation time. Suppose for the time being that the underlying asset price at the observation time $S(t_1)$ is known, the value of a forward-start option at the observation time can be written by substituting $K = S(t_1)$ into (3.2):

$$FST = \omega S(t_1)\left[e^{-g(\tau-\tau_1)}N(\omega d_{1fst}) - e^{-r(\tau-\tau_1)}N(\omega d_{fst})\right], \tag{8.2}$$

where

$$\begin{aligned} d_{fst} &= \frac{\nu}{\sigma}\sqrt{\tau-\tau_1}, \\ d_{1fst} &= d_{fst} + \sigma\sqrt{\tau-\tau_1}, \\ \nu &= r - g - \sigma^2/2, \end{aligned}$$

ω is a binary operator (1 for a call option and -1 for a put option) and other parameters are the same as in (3.2).

Formula (8.2) is the same as the pricing formula for at-the-money vanilla options with the only exception that the time to maturity is the effective time $\tau - \tau_1$. Although we do not actually know the underlying asset price at the observation time $S(t_1)$, we know its distribution from (5.3). As $S(t_1)$ is lognormally distributed with mean $\nu\tau_1$ and variance $\sigma^2\tau_1$, we can readily find the expected value of $S(t_1)$ using the moment-generating function of the normal distribution:

$$E[S(t_1)] = Se^{(r-g)\tau_1}, \tag{8.3}$$

where S is the spot price of the underlying asset.

Substituting (8.3) into (8.2) yields the expected value of the option

$$E(FST) = \omega Se^{(r-g)\tau_1}\left[e^{-g(\tau-\tau_1)}N(\omega d_{1fst}) - e^{-r(\tau-\tau_1)}N(\omega d_{fst})\right], \tag{8.4}$$

where all parameters are the same as in (8.3).

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate r, we can obtain the pricing formula of a forward-start option by discounting the expected payoff in (8.4) by the risk-free rate r. Discounting $E(FST)$ by the continuous risk-free rate r yields the forward-start option price ($FSTOPP$):

$$FSTOPP = \omega S[e^{-g\tau} N(\omega d_{1fst}) - e^{-r(\tau-\tau_1)-g\tau_1} N(\omega d_{fst})] \,, \tag{8.5}$$

where

$$d_{fst} = \frac{\nu}{\sigma}\sqrt{\tau - \tau_1},$$
$$d_{1fst} = d_{fst} + \sigma\sqrt{\tau - \tau_1},$$
$$\nu = r - g - \sigma^2/2 \,,$$

all parameters are the same as in (8.2) and (8.4).

Formula (8.5) looks very much like the Black-Scholes formula as the two arguments d_{1fst} and d_{fst} in (8.5) become precisely the same as the two corresponding arguments in the extended Black-Scholes formula in (3.2) with $S = K$ and $\tau_1 = 0$; however, there exist significant differences when $\tau_1 > 0$. These differences can be better seen from their sensitivities to various parameters. We leave the comparisons to the next section.

Example 8.1. Find the prices of the forward-start call and put options if the time to maturity is one year, the spot underlying asset price is \$50, volatility is 15%, the interest rate is 10%, and the yield on the underlying asset is 5%, and the options start half a year in future.

Substituting $S = \$50$, $\tau = 1$, $\tau_1 = 0.50$, $r = 0.10$, $g = 0.05$, $\sigma = 0.15$, and $\omega = 1$ into (8.5) yields the call option price

$$\begin{aligned} d_{fst} &= \frac{(r - g - \sigma^2/2}{\sigma}\sqrt{\tau - \tau_1} \\ &= \frac{(0.10 - 0.05 - 0.15^2/2)}{0.15}\sqrt{1 - 0.50} = 0.1827, \\ d_{1fst} &= d_{fst} + \sigma\sqrt{\tau - \tau_1} \\ &= 0.1827 - 0.15 \times \sqrt{1 - 0.50} = 0.2887\,, \\ FSTOPP(\omega = 1) &= S\left[e^{-g\tau} N(D_{f_1}) - e^{-r(\tau-\tau_1)-g\tau_1} N(D_{f2})\right] \\ &= 50 \times \left[e^{-0.05\times 1} N(0.2887) \right. \\ &\quad \left. - e^{-0.10\times(1-0.50)-0.05\times 0.50} N(0.1827)\right] \\ &= \$2.629, \end{aligned}$$

and the price of the corresponding put option is

$$\begin{aligned} FSTOPP(\omega=-1) &= -S\left[e^{-g\tau}N(-d_{1fst}) - e^{-r(\tau-\tau_1)-g\tau_1}N(-d_{fst})\right] \\ &= -50 \times \left[e^{-0.05\times 1}N(-0.2887)\right. \\ &\quad \left. -e^{-0.10\times(1-0.50)-0.05\times 0.50}N(-0.1827)\right] \\ &= \$1.454. \end{aligned}$$

We have obtained a pricing formula for forward-start options before the starting time. As soon as the starting time is reached, the price of the underlying asset is observed. Then, a forward-start option degenerates into a vanilla option once the starting time is passed. Therefore, the price of a forward-start option can be given simply by substituting $\tau_1 = 0$ into (8.2):

$$C_{afst} = \omega S(\tau_1)[e^{-g\tau}N(\omega d_{fst} + \omega\sigma\sqrt{\tau}) - e^{-r\tau}N(\omega d_{fst})], \tag{8.6}$$

where

$$d_{fst} = [(r - g - \sigma^2/2)\sqrt{\tau}/\sigma].$$

Equation (8.6) is exactly the extended Black-Scholes formula for vanilla options given in (3.2) with the strike price the same as the underlying asset price at the time when the option starts $S(t_1)$.

8.3. SENSITIVITIES OF FORWARD-START OPTIONS

The delta of a forward-start option before the starting time can be obtained by differentiating (8.5) with respect to the current price S

$$Delta(FST) = \omega[e^{-g\tau}N(\omega d_{1fs}) - e^{-r(\tau-\tau_1)-g\tau_1}N(\omega d_{fst})], \tag{8.7}$$

where all parameters are the same as in (8.5).

The gamma of a forward-start option is always zero because the delta formula is not affected by the spot price S.

The vega of a forward-start option can be similarly obtained as follows[1]

$$Vega(FST) = Se^{-g\tau}\sqrt{\tau - \tau_1}\, f(d_{1fst}), \tag{8.8}$$

[1]The following identity is used to obtain the simplified vega expression:

$$f(D_{f2})/f(D_{f1}) = e^{(r-g)(\tau-\tau_1)}.$$

where $f(z) = e^{-z^2/2}/\sqrt{2\pi}$ is the density function of the standard normal distribution and other parameters are the same as in (8.5).

The theta of a forward-start option can be similarly obtained

$$Theta(FST) = \omega S[-ge^{-g\tau}N(\omega d_{1fst}) + re^{-r(\tau-\tau_1)-g\tau_1}N(\omega d_{fst})] + Se^{-g\tau}\frac{\sigma}{2\sqrt{\tau-\tau_1}}f(d_{1fst}), \tag{8.9}$$

where all parameters are the same as in (8.5).

The interesting thing about forward-start options is that their thetas are zero any time before the starting time because the time difference $\tau - \tau_1$ which affects the forward-start option price is not affected by the passing of time before the options starts.

Example 8.2. Find the deltas, vegas, and thetas of the forward-start call and put options in Example 8.1.

Substituting $S = \$50$, $\tau = 1$, $\tau_1 = 0.50$, $r = 0.10$, $g = 0.05$, $\sigma = 0.15$, and $\omega = 1$ and -1 into (8.7) yields the deltas of the forward-start call and put options

$$\begin{aligned} Delta(FST, \omega = 1) &= e^{-g\tau}N(d_{1fst}) - e^{-r(\tau-\tau_1)-g\tau_1}N(d_{fst}) \\ &= e^{-0.05\times 1}N(0.2887) \\ &\quad - e^{-0.10\times(1-0.50)-0.05\times 0.50}N(0.1827) \\ &= 5.26\% \end{aligned}$$

and

$$\begin{aligned} Delta(FST, \omega = -1) &= -e^{-g\tau}N(-d_{1fst}) + e^{-r(\tau-\tau_1)-g\tau_1}N(-d_{fst}) \\ &= -e^{-0.05\times 1}N(-0.2887) \\ &\quad + e^{-0.10\times(1-0.50)-0.05\times 0.50}N(-0.1827) \\ &= -2.91\%; . \end{aligned}$$

Substituting $S = \$50$, $\tau = 1$, $\tau 1 = 0.50$, $r = 0.10$, $g = 0.05$, and $\sigma = 0.15$ into (8.8) yields the vega of the forward-start call and put options

$$\begin{aligned} Vega(FST) &= Se^{-g\tau}\sqrt{\tau-\tau_1}\,f(d_{1fst}) \\ &= 50 \times e^{-0.05\times 1}\sqrt{1-0.50}f(0.2887) = 12.869; \end{aligned}$$

Substituting $S = \$50$, $\tau = 1$, $\tau_1 = 0.50$, $r = 0.10$, $g = 0.05$, $\sigma = 0.15$, and $\omega = 1$ and -1 into (8.9) yields the thetas of the forward-start call and put options

$$\begin{aligned}
Theta(FST, \omega = 1) &= S[ge^{-g\tau} N(d_{1fst}) + re^{-r(\tau-\tau_1)-g\tau_1} N(d_{fst})] \\
&\quad + Se^{-g\tau} \frac{\sigma}{2\sqrt{\tau - \tau_1}} f(d_{1fst}) \\
&= 50 \times \Big[e^{-0.05\times 1} N(0.2887) \\
&\quad - e^{-0.10\times(1-0.50)-0.05\times 0.50} N(0.1827) \\
&\quad + 50 \times e^{-0.05\times 1} \frac{0.15}{2\sqrt{1-0.50}} \times f(d_{1fst})\Big] \\
&= 3.1267,
\end{aligned}$$

and

$$\begin{aligned}
Theta(FST, \omega = -1) &= -Se[-ge^{-g\tau} N(-d_{1fst}) + re^{-r(\tau-\tau_1)-g\tau_1} N(d_{fst})] \\
&\quad + Se^{-g\tau} \frac{\sigma}{2\sqrt{\tau - \tau_1}} f(d_{1fst}) \\
&= -50 \times [-0.05 \times e^{-0.05\times 1} N(-0.2887) \\
&\quad - e^{-0.10\times(1-0.50)-0.05\times 0.50} N(-0.1827) \\
&\quad + 50 \times e^{-0.05\times 1} \frac{0.152}{2\sqrt{1-0.50}} \times f(d_{1fst})] \\
&= 0.8667
\end{aligned}$$

Comparing the delta, vega, and theta of a forward-start option with those of a vanilla option given in (3.32), (3.33), and (3.34), we can readily find that all the sensitivities of a forward-start option exhibit discontinuity. For example, the theta of a forward-start option jumps from zero to the regular theta of the corresponding vanilla option in (3.34) at the option starting time $t = t_1$ or at $\tau_1 = 0$. The theta of any vanilla option is always positive because the time value of the option is always positive.

8.4. SUMMARY AND CONCLUSIONS

Forward-start options are one special kind of path-dependent options, which depend upon the price of the underlying asset at the observation time. Actually, these options are path-independent both before and after the observation time. Forward-start options are at-the-money options at the observation time. We have provided closed-form solutions for forward-start options in this chapter. The pricing formula for forward-start options is precisely the same as the extended Black-Scholes formula after the observation time, but it is rather different before that. The sensitivities of forward-start options exhibit significant discontinuity around the observation time. Most noticably, the gammas of all forward-start options are zero before the observation time.

QUESTIONS AND EXERCISES

Questions

8.1. What are forward-start options?

8.2. How is a forward-start option different from its corresponding vanilla option?

8.3. Where are forward-start options normally used?

8.4. Under what condition will a forward-start option become the same as a vanilla option?

8.5. Why are the gammas of all forward-start options always zero?

8.6. What is the difficulty involved in pricing forward-start options?

8.7. What is the price of a forward-start option if the observation time is the same as the option maturity?

8.8. Show that the identity $f(d_{fst})/f(d_{1fst}) = Se^{(r-g)(\tau-\tau_1)}$ is always correct.

8.9. Show the expectation in (8.3) is correct.

8.10. Find the prices of the forward-start call and put options if the time to maturity is one year, the spot underlying asset price is \$100, volatility is 25%, interest rate is 8%, the yield on the underlying asset is 4%, and the starting time of the options is half a year.

8.11. Find the deltas, vegas, and thetas of the call and put options in Exercise 8.10.

8.12. Find the answer to Exercise 8.10 if the observation time is changed to one month and other parameters remain unchanged.

8.13. Find the deltas, vegas, and thetas of the call and put options in Exercise 8.12.

8.14. Find the answer to Exercise 8.10 if the volatility is changed to 10% and other paramters remain unchanged.

8.15. Find the deltas, vegas, and thetas of the call and put options in Exercise 8.14.

Chapter 9

ONE-CLIQUE OPTIONS

9.1. INTRODUCTION

Holders of vanilla options can only get payoffs as differences of the underlying asset prices at maturity and their corresponding strike prices; and holders of standard American options get payoffs as differences of the underlying asset prices at any time between the start and maturity of the options and their corresponding strike prices. One-clique options are somewhat between European and American options in the sense that their holders get payoffs as differences of the underlying asset prices at some prespecified time before the option's maturity and their strike prices. Thus one-clique options are somewhat similar to forward-start options in Chapter 8 in that one quantity, the strike price in the case of forward-start options and the underlying asset price in one-clique options, is prespecified at some time in the future before the option's maturity.

Due to this similarity, the analysis of one-clique options is similar to that of forward-start options. However, as we will show in this chapter, one-clique options are very different from forward-start options. We will first define one-clique options formally , then we try to price one-clique options and apply them in practice.

9.2. ONE-CLIQUE OPTIONS

The payoff of a one-clique option (POCOP) can be formally expressed as

$$POCOP = \max\{\omega[S(\tau) - K],\ \omega[S(\tau_1) - K], 0\}\,, \tag{9.1}$$

where $\tau_1 = t_1 - t$ is the clique time in the future; $\tau = t^* - t$ is the time to maturity of the option, $t < t_1 < t^*$; max $(.,.)$ is a function that gives the larger of two numbers; and ω is a binary operator (1 for a call option and -1 for a put option).

It is obvious that in the extreme case the payoff given in (9.1) becomes the same as that of vanilla call and put options given in (2.1) and (2.2) when the clique time is the same as the maturity time. The payoff expression given in (9.1) indicates that the payoff of a one-clique option is at least as large as that of the corresponding European option because the payoff gives the larger of the payoff of a European option and that of an American option exercised at the clique time. As a result, the price of a one-clique option should be higher than that of the corresponding European option, yet not higher than the corresponding American option because the clique time may not be the optimal exercising time of the American option.

9.3. PRICING ONE-CLIQUE OPTIONS

It seems difficult to price one-clique options because both the underlying asset prices at maturity and at the clique time are unknown at present. Yet this difficulty can be readily removed. Although both the underlying asset prices at maturity and at the clique time t_1 are uncertain, they are uncertain not independently in a Black-Scholes environment. Proposition 5.1 in Chapter 5 indicates that the covariance of any two overlapping observations of the standard Wiener process equals the smaller of the two corresponding time intervals. Thus, these two prices are correlated with a correlation coefficient $\rho = \sqrt{\tau_1/\tau}$. With this in mind, we can price all forward-start options within a Black-Scholes environment.

Assume that the underlying asset price follows the stochastic process given in (3.1). Let $x = \ln[S(\tau)/S]$ and $y = \ln[S(\tau_1)/S]$. It can be easily proven using the results in (5.3) that both x and y are normally distributed with means $\mu_x = (r - g - \sigma^2/2)\tau$ and $\mu_y = (r - g - \sigma^2/2)\tau_1$ and variances $\sigma_x^2 = \sigma^2\tau$ and $\sigma_y^2 = \sigma^2\tau_1$, respectively. It can also be shown that x and y are jointly normally distributed with the correlation coefficient $\rho = \sqrt{\tau_1/\tau}$. The joint density function can be expressed as follows:

$$f(x, y) = f(y)f(x|y)\,, \tag{9.2}$$

where

$$f(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\Big(-\frac{v^2}{2}\Big),$$

$$f(x|y) = \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\frac{(u-\rho v)^2}{2(1-\rho^2)}\right],$$

$$u = \frac{x - \mu_x}{\sigma_x} \quad \text{and} \quad v = \frac{y - \mu_y}{\sigma_y}.$$

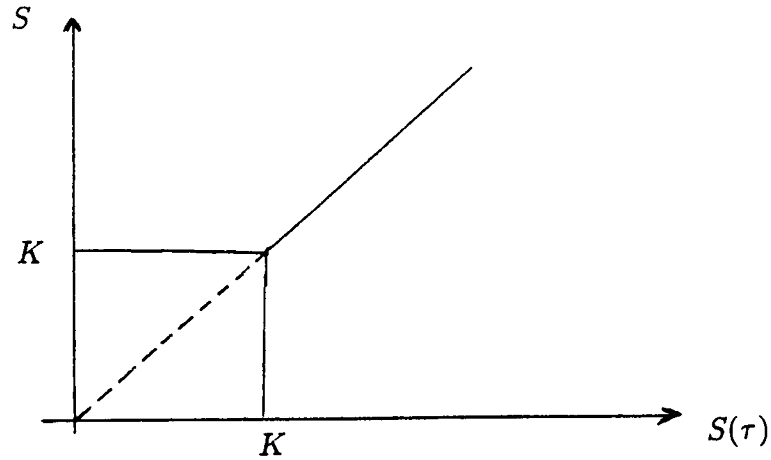

Fig. 9.1. Integration domain for a one-clique option.

Figure 9.1 depicts the integration domain of a one-clique call option. In the area below the forty-five degree line and greater than the strike price K, the underlying asset price at maturity is greater than that at the clique time. In the area above the forty-five degree line and greater than the strike price K, the underlying asset price at maturity is smaller than that at the clique time.

Using the bivariate normal distribution density function given in (9.2) and the integration domain in Figure 9.1, we can obtain the expected payoff of a European-style one-clique option (EXPOCOP) given in (9.1) through double integration:

$$\begin{aligned} EXPOCOP = {} & e^{(r-g)\tau} N_2(d_1, b_{11}, \rho_1) + e^{(r-g)\tau_1} N_2(d_{y1}, b_{12}, \rho_2) \\ & - K[N_2(d, b, \rho_1) + N_2(d_y, -b, \rho_2)]\,, \end{aligned} \tag{9.3}$$

where

$$d = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\tau\right] \Big/ (\sigma\sqrt{\tau}), \quad d_1 = d + \rho\sigma_x\,,$$

$$d_y = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1^2}{2}\right)\tau_1\right] \Big/ (\sigma\sqrt{\tau_1})\,, \quad d_{y1} = d_y + \rho\sigma_y\,,$$

$$b_{11} = b + \frac{\rho\sigma_x^2 - \sigma_x\sigma_y}{\sigma_a}, \quad b_{12} = -b + \frac{\rho\sigma^2 - \sigma_x\sigma_y}{\sigma_a}\,,$$

$$b = \frac{\mu_x - \mu_y}{\sigma_a}\,,$$

$$\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sigma_a}, \quad \rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\sigma_a},$$

$$\sigma_a = \sqrt{\sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2},$$

and $N_2(a, b, \theta)$ is the standard bivariate normal cumulative function with two upper boundaries a and b and the correlation coefficient θ.

Substituting $\rho = \sqrt{\tau_1/\tau}$, $\mu_x = (r - g - \sigma^2/2)\tau$, $\mu_y = (r - g - \sigma^2/2)\tau_1$, $\sigma_x^2 = \sigma^2\tau$, and $\sigma_y^2 = \sigma^2\tau_1$ into (9.3) yields $\sigma_a = \sigma\sqrt{\tau - \tau_1}$, $\rho_2 = 0$, and $\rho_1 = -\sqrt{1-\rho^2} = -\sqrt{1-(\tau_1/\tau)}$. Simplifying (9.3) using the identity $N_2(a, b, 0) = N(a)N(b)$ and discounting it at the risk-free rate of return yields the one-clique call option price (ONCQCP):

$$ONCQCP = S[e^{-g\tau}N_2(d_1, b, \rho_1) + e^{-r(\tau-\tau_1)-g\tau_1}N(d_{y1})N(b_{12})] \\ - Ke^{-r\tau}[N_2(d, b, \rho_1) + N(d_y)N(-b)], \tag{9.4}$$

where

$$d = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\tau\right] \Big/ (\sigma\sqrt{\tau}), \; d_1 = d + \sigma\sqrt{\tau_1},$$

$$d_y = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\tau_1\right] \Big/ (\sigma\sqrt{\tau}), \; d_{y1} = d_y + \sigma\tau_1/\sqrt{\tau},$$

$$b = \frac{r - g - \sigma^2/2}{\sigma}\sqrt{\tau - \tau_1}, \; b_{12} = -b - \sigma\sqrt{\frac{(\tau - \tau_1)\tau_1}{\tau}},$$

$$\rho_1 = -\sqrt{1-\rho^2} = -\sqrt{1-(\tau_1/\tau)}.$$

We can check one extreme case when the clique time τ_1 is the same as the maturity time τ. Substituting $\tau_1 = \tau$ into (9.4) yields $\rho_1 = b = b_{12} = 0$, $d_1 = d + \sigma\sqrt{\tau} = d_{y1}$, and $d = d_y$. Substituting $\rho_1 = b = b_{12} = 0$ into (9.4) yields[1]

$$N_2(d_1, b, \rho_1) = N(d_1)N(b) = N(d_1)N(0) = N(d_1)/2,$$

$$N(d_{y1})N(b_{12}) = N(d_{y1})N(0) = N(d_{y1})/2 = N(d_1)/2,$$

$$N_2(d, b, \rho_1) = N(d)N(b) = N(d)N(0) = N(d)/2,$$

and

$$N(d_y)N(-b) = N(d)/2.$$

Substituting the above expressions into (9.4) yields the extended Black-Scholes formula in (3.2).

[1]It can be readily shown using the bivariate density function given in (9.2) $N_2(a, b, 0) = N(a)N(b)$ if the correlation coefficient is zero.

Similarly, a one-clique put option price (ONCQPP) can be derived as follows:

$$\begin{aligned} ONCQPP = {} & Ke^{-r\tau}[N_2(-d, b, -\rho_1) + N(-d_y)N(-b)] \\ & - S\Big[e^{-g\tau}N(-d_y - \sigma\sqrt{\tau})N(-b) \\ & + e^{-r(\tau-\tau_1)-g\tau_1}N_2(-d - \sigma\tau_1/\sqrt{\tau}, b, -\rho_1)\Big] \end{aligned} \tag{9.5}$$

where all parameters are the same as in (9.4).

It is straightforward to check that (9.5) degenerate to the extended Black-Scholes pricing formula for put options when the clique time is equal to the maturity time.

9.4. EXAMPLES

We priced one-clique options in the previous section. We will now illustrate how to use the one-clique option pricing formula in this section with a few examples.

Example 9.1. The Standard & Poor 500 Index is \$535, the strike price \$540, the volatility of the underlying asset $\sigma = 15\%$, the interest rate $\sigma = 7\%$, the payout rate of the S&P Index 3.5%, what is the price of the one-clique call option if the clique time is four months and the time to maturity is half a year?

Substituting $S = \$535$, $K = \$540$, $\sigma = 0.15$, $r = 0.07$, $g = 0.035$, $\tau_1 = 1/12 = 0.0833$, and $\tau = 0.24$ into (9.4) yields

$$\rho_1 = -\sqrt{1 - (\tau_1/\tau)} = -\sqrt{1 - (0.083333/0.25)} = -0.5774,$$

$$d = \left[\ln\left(\frac{S}{K}\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\tau\right] \Big/ (\sigma\sqrt{\tau})$$

$$= \left[ln\left(\frac{535}{540}\right) + \left(0.07 - 0.035 - \frac{1}{2}0.15^2\right)0.25\right] \Big/ \left(0.15\sqrt{0.25}\right) = 0.0243,$$

$$d_1 = d + \sigma\sqrt{\tau_1} = 0.0243 + 0.15\sqrt{4/12} = 0.1109,$$

$$d_y = \left[\ln\left(SK\right) + \left(r - g - \frac{1}{2}\sigma^2\right)\tau_1\right] \Big/ (\sigma\sqrt{\tau_1}) = -0.0160,$$

$$d_{y1} = d_y + \sigma\tau_1/\sqrt{\tau} = 0.1547,$$

$$b = \frac{r - g - \sigma_2/2}{\sigma}\sqrt{\tau - \tau_1} = \frac{0.07 - 0.035 - 0.15^2/2}{0.15}\sqrt{1 - (4/12)} = 0.0646\,,$$

$$b_{12} = -b - \sigma\sqrt{\frac{(\tau - \tau_1)\tau_1}{\tau}} = -0.1000\,,$$

and thus the call option price is

$$\begin{aligned} ONCQCP &= S\left[e^{-g\tau}N_2(d_1, b, \rho_1) + e^{-r(\tau-\tau_1)-g\tau_1}N(d_{y1})N(b_{12})\right] \\ &\quad - Ke^{-r\tau}\left[N_2(d, b, \rho_1) + N(d_y)N(-b)\right] \\ &= 535\Big[e^{-0.035\times 0.5}N_2(0.1109, 0.0646, -0.5774) \\ &\quad + e^{-0.07\times(0.5-4/12)-0.035\times 4/12}N(0.1547)N(-0.10)\Big] \\ &\quad - 540 \times e^{-0.07\times 0.5} + \Big[N_2(0.0243, 0.0646, -0.5774) \\ &\quad + N(-0.0160)N(-0.0646)\Big] \\ &= \$14.149\,. \end{aligned}$$

Example 9.2. What is the price of the one-clique call options with the clique time five months and six months, respectively, and other parameters remain unchanged as in Example 9.1?

Following the same procedure as in Example 9.1, we can obtain the one-clique call option price to be \$18.058 when the clique time is five months, and the price is \$24.356 when the clique time is six months. We can readily find the price of the corresponding vanilla call option using the extended Black-Scholes formula given in (3.2) to be \$24.356. These results confirm our belief stated earlier in this chapter that the pricing formula for one-clique options becomes the same as the extended Black-Scholes formula when the clique time is the same as the option maturity time.

9.5. SUMMARY AND CONCLUSIONS

One-clique options are special path-dependent options. The important contribution of this chapter is that it has provided closed-form solutions for pricing these options by converting a bivariate problem into a univariate one within a Black-Scholes environment. The simplicity of these pricing formulas should enhance those options already in the market such as spread options and basket options, and facilitate further development of other options we have discussed in this chapter that do not yet exist.

QUESTIONS AND EXERCISES

9.1. What are one-clique options?

9.2. How are one-clique options different from vanilla options?

9.3. Under what conditions are they the same as vanilla options?

9.4. Why are one-clique options similar to forward-start options?

9.5. What is the most important difference between a forward-start option and a one-clique option?

9.6. Find the one-clique call option price with the clique time half a year and the time to maturity one year, give the spot underlying asset price is \$100, the strike price \$95, the volatility of the underlying asset $\sigma = 25\%$, the interest rate $r = 8\%$, the payout rate of the underlying asset 3%.

9.7. Find the price of the corresponding one-clique put option in Exercise 9.6.

9.8. What is the price of the one-clique call option in Exercise 9.6 if the clique time is changed to 8 months and other parameters remain unchanged?

9.9. Find the price of the corresponding one-clique put option in Exercise 9.8.

9.10. Show that the pricing formulas for one-clique options given in 9.4 and 9.5 include the extended Black-Scholes formula as a special case when the clique time is the same as the time to maturity.

Chapter 10

VANILLA BARRIER OPTIONS

10.1. INTRODUCTION

Barrier options are probably the oldest of all exotic options. It may be surprising to some people that barrier options have been traded sporadically in the US market since 1967, six years before the Chicago Board of Options Exchange (CBOE) came into being in 1973. Snyder (1969) described "down-and-out" options as "limited risk special options". Donaldson, Lufkin, and Jenrette started to use "down-and-out" options in the early 1970s (see *Fortune* November 1971, page 213). Hudson (1991) discussed how to use barrier options, especially up-and-out calls and puts. Benson and Daniel (1991) explained barrier options in general. These options were geared to the needs of sophisticated investors such as managers of hedge funds. They provided them with two things they could not obtain otherwise. One is that most "down-and-out" options were written on more volatile stocks and these options are significantly cheaper than the corresponding vanilla calls. The other is the increased convenience during a time when the trading volume of stock options was rather low. In other words, barrier options were created to provide risk managers with cheaper means to hedge their exposures without paying for the price ranges that they believed unlikely to occur.

Market for barrier options has continued to grow. It is estimated that it doubled in size every year since 1992. According to one source (RISK, April 1997, page 29), the estimated size of barrier options was over 2 trillion US dollars. Barrier options are becoming more and more popular simply because they give end-users greater flexibility to express a precise view.

Barrier options are actually conditional options, dependent on whether some barriers or triggers are breached within the lives of the options. They are therefore path-dependent. They are also called trigger options. There are two types of barrier options: knock-in and knockout barrier options, or simply knock-ins and knockouts. A knock-in is an option whose holder is entitled to receive a European option if the barrier is hit, and a rebate at expiration if otherwise. A knockout option is an option whose holder is entitled to receive a rebate as soon as the barrier is hit, and a European option if otherwise. As it makes a difference whether the settlement price is breached from above or below, there are down knock-ins and down knockouts, as well as up knock-ins and up knockouts, depending on whether the barrier is below or above the current underlying asset price. Therefore, it is easy to figure out that there are in total eight kinds of barrier options: down-in calls, up-in calls, down-out calls, up-out calls, down-in puts, up-in puts, down-out puts, and up-out puts. All these options are called standard or vanilla barrier options. The attractiveness of barrier options is that they are cheaper than their corresponding vanilla options, as the sum of the premiums of a knock-in and its corresponding knockout is always the same as the premium of their corresponding vanilla option if there are no rebates. Thus, we can say that both the payoff and the survival to the maturity date of a barrier option depend not only on the underlying asset price at maturity but also on whether the underlying asset sells at or goes through a predetermined barrier at any time during the life of the option.

Besides vanilla barrier options, there are many other kinds of barrier options: time-dependent barrier options, Asian barrier options or barrier options on the average of underlying asset prices, dual-barrier or double-barrier options, forward-start barrier options, window or limited-time barrier options, and so on. Although different kinds of barrier options possess different characteristics, they share one thing in common: their payoffs depend on whether one or more than one barriers are breached within the lives of the options. As the analysis of vanilla barrier options provides a foundation for other types of barrier options, we will concentrate on vanilla barrier options in this chapter, and explore other kinds of barrier options in the following one.

10.2. VANILLA BARRIER OPTIONS

A barrier option is also called a trigger option. It is thus named because its payoff depends critically on whether a prespecified barrier or trigger is touched during the life of the option. If the prespecified trigger is touched during the life of the option, the holder is entitled to receive a European option. Otherwise, he/she gets a rebate at the maturity of the option. This kind of barrier option is called a knock-in barrier option, or simply a knock-in. Figure 10.1 shows the situation when the barrier is reached within the life of the option. As soon as the barrier is touched, the option holder is entitled

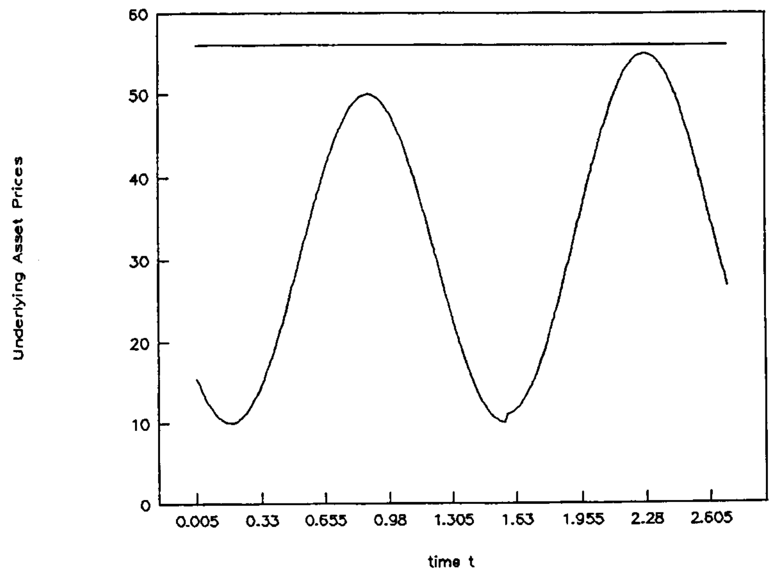

Fig. 10.1. A touched up barrier with the barrier h = 50.

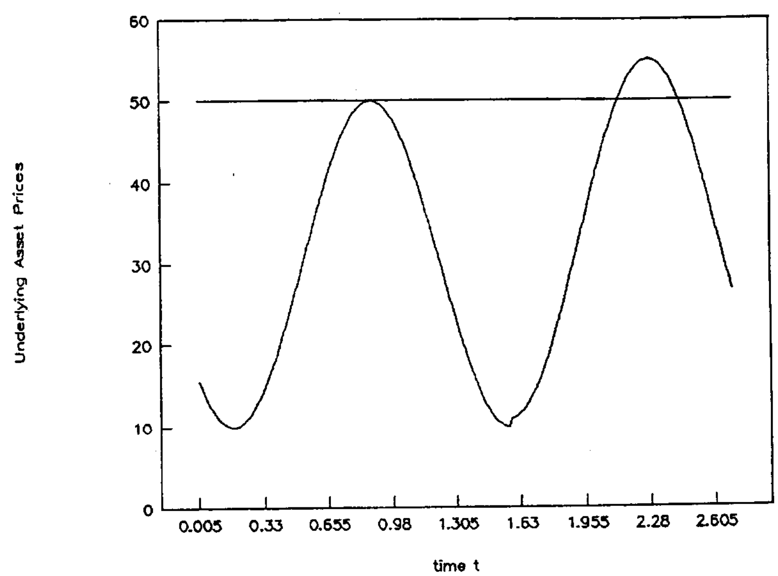

Fig. 10.2. An un-touched up barrier with the barrier H = 56.

to receive a European-style option. Figure 10.2 shows the situation when the barrier is not reached within the life of the option. Since the barrier is not hit, the holder can only receive a rebate at the maturity of the option.

Given the spot underlying asset price, the barrier can be placed either above or below it. If the barrier is below (resp. above) the spot price, the knock-in option is called a down (resp. up) knock-in option. The payoff of a down knock-in (PDI) option can be formally given as

$$PDI = \max\{[\omega S(t^*) - \omega K, 0] | S(t) > H \text{ and } S(T) \leq H, \text{ for some } t < T \leq t^*\}, \tag{10.1a}$$

or

$$PDI = Rm(\tau) \text{ if } S(t) > H \text{ and } S(T) > H, \text{ for all } t < T \leq t^*, \tag{10.1b}$$

where t and t^* stand for the current and expiration time of the option, respectively; H is the constant barrier or knock-in boundary of the option; K is the strike price of the option; ω is a binary operator (1 for a call option and -1 for a put option); the symbol "$A|B$" stands for A given B; and $Rm(\tau)$ stands for the rebate of the barrier option paid at maturity if the barrier is not touched.

The barrier option in (10.1) is called a down option simply because the current underlying asset price $S(t)$ is greater than the trigger H. Similarly, the payoff of an up knock-in (PUI) option can be given formally as

$$PUI = \max\{[\omega S(t^*) - \omega K, 0] | S(t) < H \text{ and } S(T) \geq H, \text{ for some } t < T \leq t^*\}, \tag{10.2a}$$

or

$$PUI = Rm(\tau) \text{ if } S(t) < H \text{ and } S(T) < H \text{ for all } t < T \leq t^*, \tag{10.2b}$$

where all parameters are the same as in (7.1).

Besides knock-in options, there are knockout options. Knockout barrier options are somewhat opposite to knock-in options because their payoff patterns are the direct opposite to those of knock-in options. Holders of a knockout option are entitled to receive a rebate if the barrier is touched within the life of the option (compared to a European option in the case of a knock-in option), and a European option if the barrier is never touched (compared to a rebate in the case of a knock-in option). The payoff of a down knockout option (PDO) is

$$PDO = R(T) \text{ if } S(t) > H \text{ and } S(T) \leq H, \text{ for some } t < T \leq t^*, \tag{10.3a}$$

or

$$PDO = \max\{[\omega S(t^*) - \omega K, 0] | S(t) > H \text{ and } S(T) > H, \text{ for all } t < T \leq t^*\}, \quad (10.3b)$$

where all parameters are the same as in (10.1) and (10.2) except the rebate function $R(T)$. $R(T)$ is most often an increasing function of time starting from zero, or $R'(T) > 0$ and $R(0) = 0$. We will specify the functional form of $R(T)$ later in this section.

Down knockout options are also sometimes called down-and-outers. The corresponding up knockout options can be called up-and-outers. The rebate we define in (10.3a) is called a non-deferred rebate, implying that the rebate is paid as soon as the barrier is reached. The rebate can also be deferred, that is, the rebate payment can be postponed until maturity. For deferrable rebates, we simply substitute $R(T)$ with $Rd(\tau)$ in (10.3a):

$$PDKO = Rd(\tau) \text{ if } S(t) > H \text{ and } S(T) \leq H, \text{ for some } t < T \leq t^*, \quad (10.3a')$$

where $Rd(\tau)$ is the rebate deferred to maturity and the other part of the down-outer is the same as in (10.3b).

$Rd(\tau)$ is often an increasing function of the time to maturity of the option, or $R'd(\omega) > 0$ and $Rd(0) = 0$. The specification of the functional form of $Rd(\tau)$ is not necessary for the derivation of the pricing formulas of barrier options.

An up knockout option is also called an up-and-out option or up-and-away option. The payoff of an up knockout option (PUO) is

$$PUO = R(T) \text{ if } S(t) < H \text{ and } S(T) \geq H, \text{ for some } t < T \leq t^*, \quad (10.4a)$$

or

$$PUO = \max\{[\omega S(t^*) - \omega K, 0] | S(t) < H \text{ and } S(T) < H, \text{ for all } t < T \leq t^*\}, \quad (10.4b)$$

where all parameters are the same as in (10.3).

Similarly, the rebate of an up-outer can also be deferred and (10.4a) can be changed to

$$PUKO = Rd(\tau) \text{ if } S(t) < H \text{ and } S(T) \geq H, \text{ for some } t < T \leq t^*, \quad (10.4a')$$

where $Rd(\tau)$ is the same deferred rebate as in (10.3a').

The rebate function $R(T)$ can be time-dependent. We may specify its functional form as follows:

$$R(T) = (\xi e^{\eta T} - 1)R\,, \tag{10.5}$$

where $\zeta \geq 1, R > 0$, and $\eta \geq 0$ are all constants, and $0 \leq T \leq \tau$.

The non-negative parameter η in (10.5) can be understood as the rate of increase of the rebate. When the rate of increase η is zero and the parameter $\zeta = 2$, the rebate function given in (10.5) becomes a constant R. In general, the rate of increase $\zeta > 0$, the parameter $\zeta = 1$, and the rebate function given in (10.5) obviously satisfies the conditions of a general rebate function as $R(0) = 0$ and $R'(T) = \eta\zeta Re^{\eta T} > 0$, implying that the rebate starts at $T = 0$ and increases strictly with time.

10.3. ABSORBING AND REFLECTING BARRIERS

Absorbing and reflecting barriers are popular in the study of stochastic processes and in physics as well. These two kinds of barriers are closely related to pricing barrier options we study in this chapter. They are almost always involved in solving partial differential equations (PDE), but we try not go into details of solving the related PDEs. However, it is necessary to introduce them briefly here because they are useful for us to better understand the necessary density functions for pricing barrier options.

10.3.1. Absorbing Barriers

An absorbing barrier is a barrier which upon touching, all particles vanish. Thus, absorbing barriers can also be called vanishing barriers. In other words, absorbing barriers function like a "Black hole" which can nullify anything attracted to them. Goldman, Sosin, and Shepp (1979) analyzed the optimal market timing using both absorbing and reflecting barriers. Using the method of images widely used in solving problems of heat conduction and diffusion, Cox and Miller (1965, p. 221) obtained the density function for the Brownian process with an absorbing barrier. Imaging the barrier as a mirror and placing an "image source" at $x = 2a$, the image of the origin in the mirror, Cox and Miller obtained the following density function

$$\begin{aligned} p(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}\Big\{&\exp\Big[-\frac{(x-vt)^2}{2\sigma^2 t}\Big] \\ &- e^{2av/\sigma^2}\exp\Big[-\frac{(x-2a-vt)^2}{2\sigma^2 t}\Big]\Big\} \text{ for } x < a\,, \end{aligned} \tag{10.6}$$

and $p(x,t) = 0$ for $x = a$ and all t, where $v = r - g - \sigma^2/2$ and a is the barrier.

The first term in (10.6) is the density function of a normal distribution with mean $v\tau$ and variance $\sigma^2 t$, and the second term is the density function of another normal distribution with mean $2a + vt$ and variance $\sigma^2 t$ multiplied by e^{2av/σ^2}. We can regard (10.6) as a superposition of a source of unit strength at the origin and a source of strength $-e^{2av/\sigma^2}$ at $2a$. We can also regard (10.6) as a superposition of a source of unit heat at the origin and a source of coldness e^{2av/σ^2} at $2a$.

10.3.2. Reflecting Barriers

A Brownian motion with a reflecting barrier is also called a Brownian motion reflected about some particular point. A Brownian motion $X(t)$ reflected about the line $x = b$ is given as follows

$$\begin{aligned}\tilde{X}(t) &= X(t) \text{ for } t < T_b\,, \\ &= 2b - X(t) \text{ for } t > T_b\,. \end{aligned} \tag{10.7}$$

The well-known result about the reflecting barrier is the reflection principle which states that for every sample path with $X(T) > b$ there are two

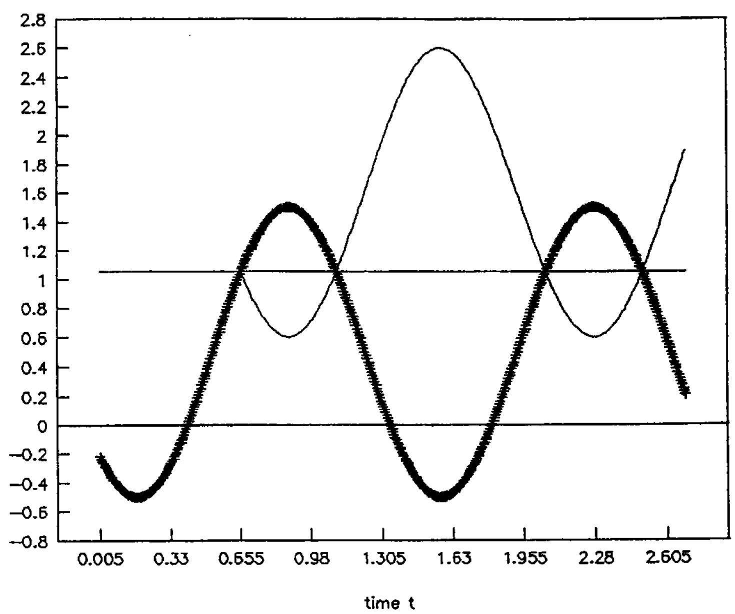

Fig. 10.3. Reflection principle with the reflection parameter b = 1.05.

sample paths $X(T)$ and $\tilde{X}(T)$ with the same probability of occurrence. Because of the symmetry with respect to b of a Brownian motion $X(t)$ starting at b, the "probability" of doing this is the same as the "probability" of traveling from b to the point $2b - X(t)$. The rationale behind this is that, for every path which crosses level b and is found at time t at a point below b, there is a "shadow path" $\tilde{X}(t)$ obtained from the reflection about the level b which exceeds this level at time t, and these two paths have the same "probability". The actual probability for the occurrence of any particular path is of course zero because the probability on one curve is always zero, so the above argument is only heuristic. Nevertheless, this argument leads us to the correct understanding of the reflection principle. Figure 10.3 depicts the reflection principle with a reflecting barrier b.

With the argument of the reflection principle shown in Figure 10.3, we can write the equation of the reflection principle as follows:

$$P[T_b < t,\ X(t) < b] = P[T_b < t,\ X(t) > b] = P[X(t) > b]\,, \tag{10.8}$$

where T_b stands for the time when the reflecting barrier b is first touched and P stands for probability.

We can use the reflection principle to find the first passage time conveniently. The solution of the density functions for the Brownian motion with a reflecting barrier can be found in Cox and Miller (1965, p. 224) and many other books on stochastic processes.

10.4. UNRESTRICTED AND RESTRICTED DENSITY FUNCTIONS

10.4.1. Unrestricted Distribution

Following the procedures in Appendix of Chapter 3, we can solve the partial differential equation given in (3.1) with the current underlying asset price $S(t) = S$ and the payout rate of the underlying asset g:

$$S(\tau) = S\ \exp[\upsilon\tau + \sigma w(\tau)]\,, \tag{10.9}$$

where $\tau = t^* - t$, t and t^* stand for the current time and the expiration time of the option, respectively, $\upsilon = r - g - \sigma^2/2$, and $w(\tau)$ is a standard Gauss-Weiner process.

Let $X_\tau = \ln[S(\tau)/S]$ be the log-return of the underlying asset. We can immediately find that the log-return X_τ is normally distributed with mean $\upsilon\tau$ and variance $\sigma^2\tau$. The density function of X_τ is readily obtained

$$f(x) = \frac{1}{\sigma\sqrt{2\pi\tau}}\ \exp\left[-\frac{(x-\upsilon\tau)^2}{2\sigma^2\tau}\right]. \tag{10.10}$$

The distribution in (10.10) is called the unrestricted distribution of the underlying asset return because no other conditions besides the initial condition S have been used. Actually, it is the density function that we have used so far to price vanilla options and all other exotic options in this book.

10.4.2. Restricted Distributions

From the specifications of the payoff of a barrier option, we know that in order to price it, we certainly need another density function conditioned on whether the barrier is reached during the life of the option. This density function is not often used in pricing exotic options, except in pricing barrier options. Before we can describe how this conditional density function can be derived, we need to introduce two variables often used in stochastic mathematics:

$$M_t^{t^*} = \max\ \{S(s)|s \in [t,\ t^*]\}\,, \tag{10.11}$$

and

$$m_t^{t^*} = \min\ \{S(s)|s \in [t,\ t^*]\}\,, \tag{10.12}$$

where $x \in X$ stands for that x belongs to X; $[t, t^*]$ stands for the set of real numbers starting from t and ending at t^* including t and t^*; max and min represent the functions giving the maximum and the minimum of a set of numbers, respectively.

The two variables given in (10.11) and (10.12) are actually the maximum and the minimum of all underlying asset prices within the life of the option. We need to transfer them in terms of log-returns:

$$Y_\tau = \ln(M_t^{t^*}/S)\,, \tag{10.13}$$

and

$$y_\tau = \ln(m_t^{t^*}/S)\,. \tag{10.14}$$

Let T_a stand for the time the underlying asset price first reaches an up barrier U. The following always hold:

$$P_r(T_a > \tau) = P_r(M_t^{t^*} < U) = P_r(Y\tau < a)\,, \tag{10.15}$$

and

$$P_r(T_a \leq \tau) = P_r(M_t^{t^*} \geq U) = P_r(Y\tau \geq a)\,, \tag{10.16}$$

where $P_r(.)$ stands for the probability when the condition "." is satisfied.

Equation (10.15) states that the barrier is never hit within the life of the option τ (because the first time the barrier is hit is after the expiration time of the option) and is equivalent to the fact that the maximum value of the underlying asset price within the life of the option is always below the barrier in a probabilistic sense. Equation (10.16) is the complement of (10.15), which states that the barrier is touched within the life of the option τ (because the first time the barrier is hit is within the maturity of the option) and is probabilistically equivalent to the fact that the maximum value of the underlying asset price within the life of the option is at least the barrier.

With (10.15) and (10.16) and some well-known stochastic results of the Brownian motion, we can find the conditional density function immediately. The log-return of the underlying asset is certainly related to the maximum of the underlying asset. The joint-cumulative distribution between the log-return of the underlying asset and the transferred maximum given in (10.13) is given as follows [see Harrison (1985), p. 13 for a proof] for $x : y$ and $y \geq 0$:

$$F(X_\tau \leq x,\ Y_\tau \leq y) = N\left(\frac{x - \upsilon\tau}{\sigma\sqrt{\tau}}\right) - e^{2y\upsilon/\sigma^2} N\left(\frac{x - 2y - \upsilon\tau}{\sigma\sqrt{\tau}}\right), \quad (10.17)$$

where $N(.)$ is the cumulative function of a standard normal distribution.

The joint-cumulative function in (10.17) is equivalent to the following

$$F(X_\tau \leq x, Y_\tau < y) = N\left(\frac{x - \upsilon\tau}{\sigma\sqrt{\tau}}\right) - e^{2y\upsilon/\sigma^2} N\left(\frac{x - 2y - \upsilon\tau}{\sigma\sqrt{\tau}}\right), \quad (10.18)$$

because the probability of one variable on the line of $Y_\tau = y$ is zero.[1]

Equations (10.15) and (10.18) together imply that (10.18) is the cumulative function of the log-return of the underlying asset conditional on the fact that the barrier is never touched within the life of the option. Since $y = a = ln(U/S)$ is known in our particular application to find the conditional density, there is only one variable — the log-return of the underlying asset given in (10.18). Differentiating (10.18) with respect to x yields the density function of the log-return of the underlying asset conditional on the fact that the barrier U is never touched within the life of the option:

$$\phi(x|Y_\tau < a) = f(x) - e^{2a\upsilon\tau/\sigma^2} f(x - 2a), \quad (10.19)$$

[1]For any continuous distribution of an univaritate random variable, the probability at one point is always zero. A nonzero probability is always obtained for a specified interval in which the variable is confined. Similarly, the probability on one line is always zero for any bivariate continuous distribution. A nonzero probability is always obtained for a specified area in which the two variables are confined.

or

$$\phi(x|Y_\tau < a) = f(x) - \left(\frac{U}{S}\right)^{2\upsilon/\sigma^2} f(x-2a) \text{ for } x < a\,, \tag{10.20a}$$

and

$$\phi(x|Y_\tau < a) = 0 \text{ for } x \geq a\,, \tag{10.20b}$$

where $f(x)$ is the unrestricted density function of the log-return of the underlying asset given in (10.10).

The restricted density function given in (10.19) or (10.20) is exactly the same as the solution to the Brownian motion with an absorbing barrier $a > 0$ given in (10.6). The coefficient $e^{2a\upsilon/\sigma^2}$ can be interpreted as the amount of coldness at $x = 2a$ compared to the unit heat at the origin $x = 0$. The second part given in (10.20b) is zero because it is outside the range in consideration.

The complement of being always below the barrier is not always being above or at the barrier, because it is possible that the barrier is reached and the price ends up below. The density function that the barrier is touched can be obtained from the following identity

$$\phi(x|Y_\tau \geq a) + \phi(x|Y_\tau < a) = f(x)\,, \tag{10.21}$$

which expresses that the summation of the probability when the barrier is touched and the probability when the barrier is never touched within the life of the option is the same as the unrestricted density given in (10.10). Thus, the density function of the final asset price conditioned on the barrier being reached $\phi(x|Y_\tau \geq a)$ can be readily found by subtracting the density function of the final asset price which is always below the barrier $\phi(x|Y_\tau < a)$ from the unrestricted density function $f(x)$ given in (10.10) or from (10.21) directly

$$\phi(x|Y_\tau \geq a) = e^{2a\upsilon/\sigma^2} f(x-2a) = \left(\frac{U}{S}\right)^{2\upsilon/\sigma^2} f(x-2a) \text{ for } x < a\,, \tag{10.22a}$$

$$\phi(x|Y_\tau \geq a) = f(x) \text{ for } x \geq a\,, \tag{10.22b}$$

where all parameters are the same as in (10.19) and (10.20).

The restricted density function in (10.20) has two parts because the conditional density function given in (10.20) has two parts, one is zero when $x > a$ and one is a positive term when $x \leq a$. The two parts of the restricted density function in (10.22) obviously exhibit discontinuity in the density function of the return of the underlying asset given the condition that the barrier is touched.

Similar to the joint-cumulative distribution function given in (10.17), the joint-cumulative distribution function between the log-return of the underlying asset and the log-return of the minimum value y_τ is given as follows [see Harrison (1985), p. 13] for a down-barrier with $y \leq x$ and $y \leq 0$:

$$F(X_\tau \geq x,\ y_\tau \geq y) = N\left(\frac{-x + v\tau}{\sigma\sqrt{\tau}}\right) - e^{2yv/\sigma^2} N\left(\frac{-x + 2y + v\tau}{\sigma\sqrt{\tau}}\right), \quad (10.23)$$

where all parameters and functions are the same as in (10.17) and (10.18).

Using the joint-cumulative function in (10.23), we can obtain the density function of the log-return of the underlying asset given that the down-barrier is never touched within the life of the option (see Appendix at the end of the book for the proof):

$$\phi(x|y_\tau > b) = f(x) - \left(\frac{L}{S}\right)^{2v/\sigma^2} f(x - 2b) \text{ for } x > b\,, \quad (10.24a)$$

and

$$\phi(x|y_\tau > b) = 0 \text{ for } x \leq b\,, \quad (10.24b)$$

where $b = \ln(L/S) < 0$ and L stands for a down-barrier $L < S$. The density function of the log-return of the underlying asset conditioned on the fact that the down-barrier is never touched in (10.24) is exactly the same in functional form as the corresponding conditional density function for the up-barrier in (10.20). We can obtain one from the other simply by substituting the up-barrier $U > S$ with the down-barrier $L < S$, or vice-versa.

Using a similar identity as given in (10.21)

$$\phi(x|Y_\tau > b) + \phi(x|Y_\tau \leq b) = f(x)\,, \quad (10.25)$$

we can obtain the restricted density function of the underlying asset log-return under the condition that the down-barrier is touched within the time span τ or the option lifetime:

$$\phi(x|Y_\tau \leq b) = e^{bv/\sigma^2} f(x - 2b) = \left(\frac{L}{S}\right)^{2v/\sigma^2} f(x - 2b) \text{ for } x > b\,, \quad (10.26a)$$

and

$$\phi(x|Y_\tau \geq b) = f(x) \text{ for } x \leq b\,, \quad (10.26b)$$

where all parameters are the same as in (10.24) and (10.25).

Careful observation of the restricted density function in (10.22) for an up-barrier and the corresponding density function in (10.26) for a down-barrier reveals that they are exactly the same in functional form if we substitute the up-barrier U and the down-barrier L with a barrier parameter

H. The identity of the functional form reflects the "symmetry" between an up-barrier and its corresponding down-barrier. More careful observation of the two restricted density functions shows that the ranges in which the density functions are effective are different for an up-barrier and a down-barrier. This results directly from the difference between an up-barrier and a down-barrier. The difference in the effective ranges determines the integration domains of an up-barrier and a down-barrier options. We will examine this in the following section.

Although the conditional density functions obtained so far in this section are the same as the results of solving a relevant partial differential equation given in (10.6), the intuition behind the results is better revealed in this section.

10.4.3. Distribution of the First Passage Time

The first passage time to a particular barrier level is of critical importance in pricing "out" barrier options as it is used to determine the discounting time. The first passage time to a particular point is the first time that this particular point is first reached. The joint probability that $x = y = a > 0$ for an up-barrier can be obtained using (10.15) and (10.17)

$$\begin{aligned} P(X_\tau \le a,\ Y_\tau \le a) &= P(X_\tau \le a,\ T_a > \tau) \\ &= N\left(\frac{a - \upsilon\tau}{\sigma\sqrt{\tau}}\right) - e^{2a\upsilon/\sigma^2} N\left(\frac{-a - \upsilon\tau}{\sigma\sqrt{\tau}}\right). \end{aligned} \tag{10.27}$$

If the drift term $\upsilon = r - g - \sigma^2/2 \ge 0$, the density function of the first passage time from zero to the transferred barrier point $a = \ln(U/S) > 0$ can be obtained by differentiating (10.27) with respect to the time to maturity

$$\begin{aligned} h(T|a > 0) &= \left[-\frac{\partial}{\partial\tau} F(X_\tau \le a,\ Y_\tau \le a) \right]\Bigg|_{\tau=T} \\ &= \frac{a}{\sigma\sqrt{2\pi T^3}} \exp\left[-\frac{(a - \upsilon T)^2}{2\sigma^2 T} \right]. \end{aligned} \tag{10.28}$$

The distribution of the first passage time in (10.28) is also called the inverse Gaussian distribution. If the drift υ is negative, then the first passage time has an improper distribution, so we only consider non-negative υ in all our analysis.

Similarly, the density function of the first passage time from zero to the transferred barrier point $b = \ln(L/S) < 0$ for a down-barrier can be obtained by differentiating (10.23) with respect to the time to maturity given $x = y = b$:

$$h(T|b<0) = \left[-\frac{\partial}{\partial_\tau} F(X_\tau \geq b,\ y_\tau \geq b) \right] \Big|_{\tau=T}$$

$$= \frac{-b}{\sigma\sqrt{2\pi^2 T^3}} \exp\left[-\frac{(b - vT)^2}{2\sigma^2 T} \right]. \qquad (10.29)$$

The first passage time distribution given in (10.29) for a down-barrier is almost the same as that for an up-barrier with the only difference in sign. We can write the two density functions compactly in one expression:

$$h(T) = \frac{\theta \ln(S/H)}{\sigma\sqrt{2\pi T^3}} \exp\left\{ -\frac{[\ln(H/S) - vT]^2}{2\sigma^2 T} \right\}, \qquad (10.30)$$

where θ is a binary operator (1 for an up-barrier $H = U > S$, and -1 for a down-barrier $H = L < S$).

The mean and variance of the first passage time can be found using the density function given in (10.30):

$$E(T|H) = \frac{\theta \ln(S/H)}{v},$$

and

$$Var(T|H) = \frac{\theta \sigma^2 \ln(S/H)}{2v^3}.$$

The density function in (10.30) looks somewhat similar to that of the normal distribution given in (10.10), yet it is rather different from the normal density function because the time cannot be negative in (10.30). In order to familiarize ourselves with this distribution function, we depict the density functions for various sets of parameters in Figure 10.4, given the spot price $S = \$100$, $v = 0.03$, the volatility $\sigma = 20\%$, and the barrier $H = \$93$, \$95, and \$104, respectively. From Figure 10.4, we can observe that the density function is skewed more to the left for a down-barrier, because $\ln(H/S) < 0$ the density function reaches the peak more quickly for a down-barrier than for an up-barrier. We can also observe from Figure 10.4 that for a down- (resp. up-) barrier, the deeper the barrier is below (resp. above) the spot price, the more (resp. less) the density function is skewed to the left. These observations will help us when we price out-barrier options in the following section.

The first passage time can also be obtained using the reflection principle given in (10.8). For simplicity, we only consider a Brownian motion with

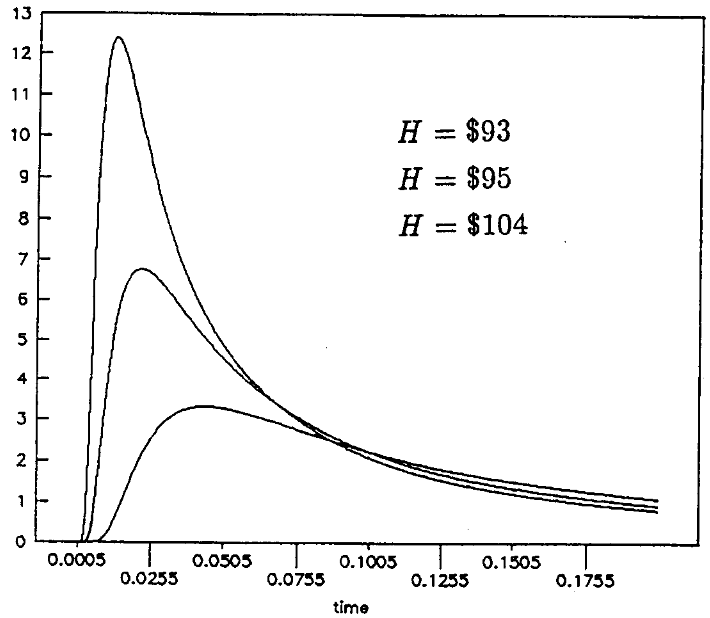

Fig. 10.4. First passage time distribution given S = \$100, H = 93, 95, and 104.

zero drift and unit volatility. The probability that the first passage time is not greater than T can be obtained readily by using the reflection principle given in (10.8):

$$P(T_b < T) = 2P[X(t) > a] = 2\int_{\sigma/\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 2\left[1 - N\left(\frac{a}{\sqrt{T}}\right)\right] = 2N\left(\frac{-a}{\sqrt{T}}\right),$$

and the density function of the first passage time T can be readily obtained by taking the partial derivative of the above probability $P(T_b < T)$ with respect to T and the result is exactly the same as that given in (10.30).

10.5. PRICING STANDARD BARRIER OPTIONS

To the author's best knowledge, Merton (1973) was the first researcher who studied the pricing of barrier options. He priced "down-and-outers" by solving a transferred stochastic differential equation with boundary conditions. The study of barrier options was then absent from financial literature for about a decade. Bergman (1983) developed a framework for pricing path-contingent claims such as barrier options, and Cox and Rubinstein (1985) provided a pricing formula for down-and-out barrier options and used the result to evaluate bonds with embedded characters. A series of papers published in *RISK* in 1991 significantly helped to popularize barrier options in the professional world. Besides the two short articles by Hudson (1991), and Benson and Daniel (1991) mentioned at the beginning of this chapter, Rubinstein and Reiner (1991) provided detailed results for all eight types of standard barrier options, assuming that the underlying asset follows a lognormal process as given in (3.1). Boyle and Lau (1994) priced barrier options with the binomial method. We will try to price all eight types of standard barrier options within a Black-Scholes environment in a more general setting and express them in a more compact forms.

Expected payoffs of "in" and "out" barrier options can be calculated in the same way as in vanilla options with the only exception that the unrestricted density function given in (10.10) is replaced by the restricted density functions given in (10.22) and (10.26). Using the risk-neutral evaluation relationship discussed in Chapter 2, we can obtain barrier option prices by discounting their expected payoffs at the risk-free rate of return.

For convenience in pricing all standard barrier options, we repeat the extended Black-Scholes pricing formula given in (3.2):

$$\begin{aligned} C_{bs}(S,\ K) &= \omega S e^{-g\tau} N[\omega d_{1bs}(S,\ K)] \\ &\quad - \omega K e^{-r\tau} N[\omega d_{bs}(S,\ K)], \end{aligned} \tag{10.31}$$

where

$$d_{bs}(S,\ K) = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln(S/K) + \upsilon\tau}{\sigma\sqrt{\tau}},$$

$$d_{1bs}(S,\ K) = d_{bs}(S,\ K) + \sigma\sqrt{\tau},$$

ω is the binary operator (1 for a call option and -1 for a put option), and other parameters are the same as in (3.2).

10.5.1. The Relative Magnitudes of Strike Price and Barrier

Because of the discontinuity in the restricted density functions given in (10.22) and (10.26) at the barrier and the discontinuity at the kinked point K in both the call and put option payoffs shown in Figures 2.1 and 2.2, we need to distinguish the two relative magnitudes of the strike price K and the barrier level H in all vanilla barrier options. For instance, if the strike price is greater than the barrier in a down-in call barrier option without any rebate, the payoff of down-in call is simply the integration of the payoff function of a vanilla call option given in (2.1) with the restricted density function given in (10.26a) for all possible underlying asset prices starting from the strike price K to infinity. However, if the strike price is smaller than the barrier, the payoff of the down-in call barrier option includes two parts: the integration of the payoff function of a vanilla call option with the restricted density function given in (10.26a) for all possible underlying asset prices starting from the barrier $H = L$ to infinity, and the integration of the same payoff function with the density function given in (10.26b) for all possible underlying asset prices starting from the strike price K to the barrier $H = L$, because the density function given in (10.26) divides at the barrier $H = L$.

The relative magnitude of strike and barrier is often represented by stealth. Stealth is defined as the difference between the strike price and barrier expressed as a percentage of the spot rate.

Figure 10.5 depicts the above argument clearly with the spot price $S =$ \$95, the down barrier $L =$ \$90, and the strike price $K =$ \$87. The stealth in this example is $-3/95 = -3.15\%$. The restricted density function is $f(x)$ below the barrier, and $\phi(x) = (L/S)^{2v/\sigma^2} f(x - 2b)$ above the barrier. For simplicity, we will not repeat the comparative magnitudes between the strike price and the barrier for all eight types of vanilla barrier options in the remaining part of this section and will concentrate instead on a few types of vanilla barrier options only.

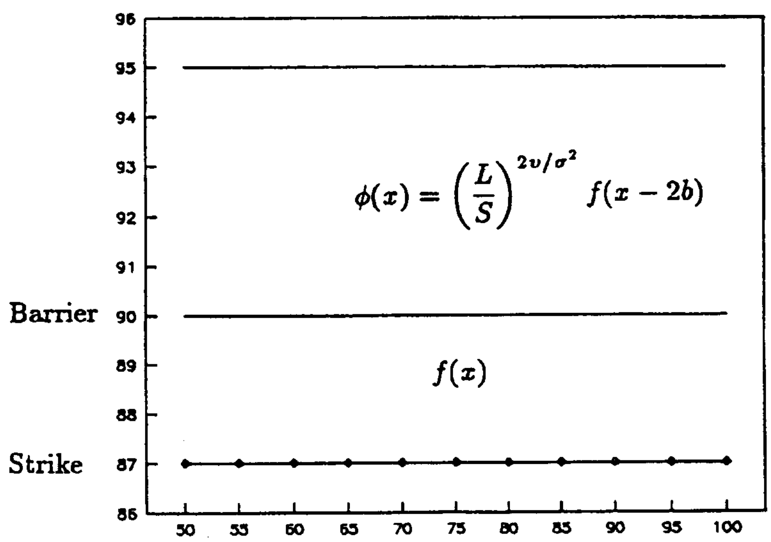

Fig. 10.5. Intergration ranges for a down knock-in call option.

10.5.2. Knock-In Options

Down-In Barrier Call Options

We will first price down knock-in options in this section and then extend the results to up knock-in options. Equation (10.2) indicates that there are two parts in the payoff of a down-in barrier option, one including the payoff of the corresponding vanilla option if the barrier is reached any time within the life of the option, and the other the rebate if the barrier is never reached within the life of the option. As argued earlier in this section, the expected payoff of a down-in barrier call option may include two parts because the integration may have to be divided into two parts as shown in Figure 10.5 when the strike price is lower than the down barrier. For simplicity, we first consider the simple case of $K > H$. The expected payoff of the vanilla option can be obtained following the same steps as in Section 3.4 using the restricted density function in (10.26a) rather than the unrestricted density function in (10.10):

$$E[PUKI|S(t) < H \text{ and } S(T) \geq H, \text{ for some } t < T \leq t^*]$$
$$= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ S\left(\frac{H}{S}\right)^2 e^{(r-g)\tau} N\left[d_{1bs}\left(\frac{H^2}{S}, K\right)\right] - KN\left[d_{bs}\left(\frac{H^2}{S}, K\right)\right]\right\}, \tag{10.32}$$

where d_{bs} is the same as in (10.31).

The value of the down-in call option ($VDIC$) without any rebate if the barrier is reached and $K > H$ is readily obtained by discounting its expected payoff given in (10.32) at the risk-free rate of return:

$$VDIC = \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ \left(\frac{H^2}{S}\right) e^{-g\tau} N\left[d_{1bs}\left(\frac{H^2}{S}\right), K\right] - Ke^{-r\tau} N\left[d_{bs}\left(\frac{H^2}{S}, K\right)\right]\right\},$$

or

$$VDIC = \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S}, K\right), \tag{10.33}$$

where C_{bs} is the extended Black-Scholes formula given in (10.31).

The pricing formula given in (10.33) looks new to us. This kind of formula will appear in the pricing formulas for all eight kinds of vanilla barrier options in this chapter and the exotic barrier options in the following chapter. As a matter of fact, the log-return of the first term in $C_{bs}(H^2/S,\ K)$, $\ln(H^2/S^2) = 2a$ is actually the "image source" of the origin in the barrier mirror, and the term $\ln[H^2/(SK)]$ is actually the reflection of the term $\ln(K/S)$ because

$$\ln\left(\frac{H^2}{SK}\right) = 2\ln\left(\frac{H}{S}\right) - \ln\left(\frac{K}{S}\right) = 2a - \ln\left(\frac{K}{S}\right).$$

The coefficient $(H/S)^{2v/\sigma^2}$ represents the degree of coldness at the image $2a$ compared to the unit of heat at the origin or a source of negative strength at the image $2a$ compared to a source of unit strength at the origin as explained in (10.6). Since the standard Black-Scholes call option pricing formula $C_{bs}(S,\ K)$ actually starts from S above K or from the origin to $\ln(K/S)$, the pricing formula $C_{bs}(H^2/S,\ K)$ starts from the "image source" $2a$ of the origin above the reflection of the strike K or starts from $\ln(H^2/S^2) = 2a$ above $\ln[H^2/(SK)] = 2a - \ln(K/S)$. Therefore, the pricing formula in (10.33) can be understood as a pricing formula starting from the "image source" discounted with the strength factor of the "image source".

Formula (10.33) gives the value of a down-in call option without any rebate when the strike price is greater than the barrier. When the strike price K is lower than the barrier H = L, we have to divide the whole integration range (K, ∞) into (K, H) and (H, ∞) because the corresponding density functions are different in the two subranges (K, H) and (H, ∞), as shown in Figure 10.5. For the range (H, ∞), we can obtain the value of the option in this up portion ($VDNUP$) following the same procedure as in deriving (10.33) using the density function given in (10.26a)

$$\begin{aligned} VDNUP &= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{\left(\frac{H^2}{S}\right) e^{-g\tau} N\left[d_{1bs}\left(\frac{H^2}{S},\ H\right)\right] \right. \\ &\quad \left. - Ke^{-r\tau} N\left[d_{bs}\left(\frac{H^2}{S},\ H\right)\right]\right\} \\ &= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{C_{bs}\left(\frac{H^2}{S},\ H\right) + (H-K)e^{-r\tau} N[d_{bs}(H, S)]\right\}, \end{aligned} \tag{10.34}$$

where C_{bs} is the extended Black-Scholes formula given in (10.31) and other parameters are the same as in (10.33).

Since the range (K, H) is equivalent to the difference of the two ranges $(-\infty, H)$ and $(-\infty, K)$, we can obtain the value of the down-in call option $(VDNIC)$ for the range (K, H) following a similar procedure as in deriving the Black-Scholes formula in Chapter 2 using the unrestricted density function given in (10.26b):

$$VDNIC = P_{bs}(S,\ K) - P_{bs}(S, H) + (H - K)e^{-r\tau}N[-d_{bs}(S,\ H)]\,, \quad (10.35)$$

where $P_{bs}(S,\ K)$ is the vanilla put option price given in (10.31) when $\omega = -1$ and $d_{bs}(S,\ K)$ is the same as in (10.31).

The value of the down-in call option $(DNIC)$ without any rebate is therefore the sum of the values of the options given in (10.34) and (10.35). From our above analysis, the pricing formula of a down-in option depends on whether $K > H$ or $K < H$. In order to obtain a general formula to cover both situations, we need to adopt one digital number $B_{H>K}$ which equals one when $H > K$, and zero if otherwise. With the digital number, we can express the price of a down-in barrier call option (DINC) without rebate:

$$\begin{aligned} DINC = \left(\frac{H}{S}\right)^{2v/\sigma^2} & \Bigg(C_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right] \\ & + [\max(H,\ K) - K]e^{-r\tau}N\left\{d_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right]\right\}\Bigg) \\ & + \Big\{P_{bs}(S,\ K) - P_{bs}(S,\ H) \\ & + (H - K)e^{-r\tau}N[-d_{bs}(S,\ H)]\Big\}B_{H>K}\,, \end{aligned} \quad (10.36)$$

where $\max(H,\ K)$ is the function which gives the larger of the two numbers H and K, and other parameters are the same as in (10.34) and (10.35).

It is straightforward to check that when $K > H$, the pricing formula given in (10.36) becomes the same as (10.33) because $\max(H,\ K) = K$, $B_{H>K} = 0$, the second term in the first brace and the second brace both become zero. We can also check that when $K < H, \max(H,\ K) = H,\ B_{H>K} = 1$, the pricing formula (10.36) is exactly the sum of the two pricing formulas given in (10.34) and (10.35) and the sum represents the value of the down-in call option when there is no rebate.

Before we start to find the present value of the rebate when the down barrier is never touched, let's take some examples to find down-in call option prices without any rebate.

Example 10.1. Find the prices of the down-in barrier call options with strike prices $K = \$98$ and $\$92$ to mature in half a year, given the spot price $S = \$100$, the down barrier $L = H = \$95$, interest rate $r = 8\%$, the yield of the underlying asset $g = 3\%$, the volatility of the underlying asset 20%.

Substituting $S = \$100$, $K = \$98$, $H = \$95$, $\omega = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.36) yields

$$v = r - g - \sigma^2/2 = 0.08 - 0.03 - 0.20^2/2 = 0.03\,,$$

$$H^2/S = 95^2/100 = 90.25\,,$$

$$\max(H,\ K) = \max(95,\ 98) = 98\,,$$

$$d_{bs}\left(\frac{H^2}{S},\ K\right) = \frac{\ln[(H^2/S)/K] + v\tau}{\sigma\sqrt{\tau}} = -0.4765\,,$$

$$d_{1bs}\left(\frac{H^2}{S},\ K\right) = d_{bs}\left(\frac{H^2}{S},\ K\right) + \sigma\sqrt{\tau}$$

$$= -0.4765 + 0.20\sqrt{50} = -0.3351\,.$$

Since $K = \$98 > \$95 = H$, the call option price is $B_{H>K} = 0$. We can find the down-in call price from (10.36) as follows:

$$DINC(K = 98)$$

$$= \left(\frac{H}{S}\right)^{2v\tau/\sigma^2} C_{bs}\left(\frac{H^2}{S},\ K\right)$$

$$= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{\frac{H^2}{S} e^{-g\tau} N\left[d_{1bs}\left(\frac{H^2}{S},\ K\right)\right] - Ke^{-r\tau} N\left[d_{bs}\left(\frac{H^2}{S},\ K\right)\right]\right\}$$

$$= 0.95^{2\times 0.03/0.20^2}\left[90.25e^{-0.03\times 0.5}N(-0.3351) - 98e^{-0.08\times 0.5}N(-0.4765)\right]$$

$$= \$2.731.$$

When the strike $K = \$92$, $\max(H, K) = \max(95,\ 92) = \$95$, $B_{BH>K} = 1$, all terms in (10.36) are nonzero. Substituting $S = \$100$, $K = \$92$, $H = \$95, \max(H, K) = \$95$, $B_{H>K} = 1$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and

$\tau = 0.50$ into (10.36) yields

$$d_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right] = \frac{\ln[(H^2/S)/H] + v\tau}{\sigma\sqrt{\tau}} = -0.2566\,,$$

$$d_{1bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right] = d_{bs}\left(\frac{H^2}{S},\ K\right) + \sigma\sqrt{\tau} = -0.1152\,,$$

$$P_{bs}(S,\ K) = -S[\mathrm{e}^{-g\tau}N[-d_{1bs}(S,\ K)] + K\mathrm{e}^{-r\tau}N[-d_{bs}(S,\ K)] = 1.5801\,,$$

$$P_{bs}(S,\ H) = -S\mathrm{e}^{-g\tau}N[-d_{1bs}(S,\ H)] + H\mathrm{e}^{-r\tau}N[-d_{bs}(S,\ H)] = 2.4896\,,$$

$$C_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right] = \left(\frac{95^2}{100}\right)\mathrm{e}^{0.03\times 0.5}N\left[d_{1bs}\left(\frac{95^2}{100},\ 95\right)\right]$$
$$- 95\mathrm{e}^{-0.08\times 0.5}N\left[d_{bs}\left(\frac{95^2}{100},\ 95\right)\right] = 3.9816\,.$$

Thus the down-in call price is

$$DINC = \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2}\left[3.9816 + (95-92)\mathrm{e}^{-0.08\times 0.5}N[-0.2566]\right]$$
$$+ \left[1.5801 - 2.4896 + (95-92)\mathrm{e}^{-0.08\times 0.5}N(-0.4689)\right] = \$4.863\,.$$

The value of the rebate at the option maturity can be obtained by integrating the restricted density function in (10.24a) from the down barrier $H = L$ to infinity:

$$E\{PUKI|S(t) < H \text{ and } S(T) < H\ \forall\, t < T \le t^*\}$$
$$= Rm(\tau)\left\{N[d_{bs}(S,H)] - \left(\frac{H}{S}\right)^{2v\tau/\sigma^2} N[d_{bs}(H,S)]\right\}, \quad (10.37)$$

where d_{bs} is the same as in (10.31).

The present value of the rebate ($RBDI$) is readily obtained by discounting (10.37) at the risk-free rate r:

$$RBDI = \mathrm{e}^{-r\tau}Rm(\tau)\left\{N[d_{bs}(S,\ H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2} N[d_{bs}(H,\ S)]\right\}. \quad (10.38)$$

The price of a down-in call option ($PDIC$) can now be expressed using (10.36) and (10.38):

$$PDIC = DINC + RBDI\,, \quad (10.39)$$

where $DINC$ and $RBDI$ are given in (10.36) and (10.38), respectively.

Example 10.2. Find the present value of the rebate when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the call options in Example 10.1.

Substituting $Rm(0.5) = 1.5$, $S =$ \$100, $H =$ \$95, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.38) yields

$$d_{bs}(S,\ H) = \frac{\ln(S/H) + \upsilon\tau}{\sigma\sqrt{\tau}} = \frac{\ln(100/95) + 0.03 \times 0.5}{0.20\sqrt{0.50}} = 0.4688\,,$$

$$d_{bs}(H, S) = \frac{\ln(H/S) + \upsilon\tau}{\sigma\sqrt{\tau}} = \frac{\ln(95/100) + 0.03 \times 0.5}{0.20\sqrt{0.50}} = -0.2566\,,$$

$$RBDI = 1.5\mathrm{e}^{-0.08\times 0.5}\left[N(0.4688) - \left(\frac{95}{100}\right)^{2\times 0.03/0.2^2} N(-0.2566)\right]$$

$$= \$0.449\,.$$

Example 10.3. Find the prices of the down-in barrier call options when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the options in Examples 10.1 and 10.2?

We can simply use the results from Examples 10.1 and 10.2. As the present value of the rebate given in (10.38) is the same for down-in call options with different strike prices, the down in call option prices can be readily found by adding up the call values in Example 10.1 and the present value of the rebate in Example 10.2:

The down in call option price with strike price $K =$ \$98

$$= DINC(K = 98) + RBDI = 2.731 + 0.449 = \$3.18\,,$$

and the down-in call option price with strike price $K=$ \$92

$$= DINC(K = 92) + RBDI = 4.862 + 0.449 = \$5.312\,.$$

Up-In Barrier Call Options

So far we have priced down-in call barrier options in this section. Using the restricted density function in (10.22) for an up-barrier instead of that given in (10.26) for a down-barrier, we can obtain the pricing formula of an up-in barrier call option (UINC) without rebate following the similar procedure as in deriving (10.36):

$$UINC = \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ P_{bs}\left(\frac{H^2}{S},\ K\right) - P_{bs}\left(\frac{H^2}{S},\ H\right) \right.$$

$$\left. + (H - K)\mathrm{e}^{-r\tau} N[-d_{bs}(H,\ S)] \right\} B_{H>K}$$

$$+ C_{bs}[S,\ \max(H,\ K)]$$

$$+ [\max(H,\ K) - K]\mathrm{e}^{-r\tau} N\{d_{bs}[S, \max(H,\ K)]\}\,, \qquad (10.40)$$

where all parameters and intermediate functions are the same as in (10.36).

Comparing (10.36) with (10.40), we can readily find some "symmetries" between the two formulas. The first observation is that the binary number $B_{H>K}$ is multiplied to the first brace in (10.40) rather than to the second brace as in (10.36). The second is that whereas the first brace in (10.40) includes the difference between two call option prices and the difference between the barrier and the strike price, these differences are included in the second brace in (10.36). We will use such symmetries to simplify the pricing formulas of the other kinds of vanilla barrier options.

Example 10.4. Find the prices of the up-in barrier call options with strike prices K = \$102 and \$108 to mature in half a year, given the spot price S = \$100, the up barrier $U = H$ = \$105, interest rate r = 8%, the yield of the underlying asset g = 3%, the volatility of the underlying asset 20%.

Substituting S = \$100, K = \$108, H = \$105, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.40) yields

$$v = r - g - \sigma^2/2 = 0.03,\ \ B_{H>K} = 0\,,$$

$$UINC = C_{bs}(100,\ 108) = 100\mathrm{e}^{-0.03\times0.5} N[d_{1bs}(100,\ 108)]$$

$$- 108\mathrm{e}^{-0.08\times0.5} N[d_{bs}(100,\ 108)] = \$3.454\,;$$

and substituting S = \$100, K = \$102, H = \$105, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.40) yields $B_{H>K} = 1$,

$$105^2/S = 110.25,\ \max(H,\ K) = \max(105,\ 102) = 105\,,$$

$$5d_{bs}\left(\frac{H^2}{S},\ K\right) = \frac{\ln[(H^2/S)/K] + \upsilon\tau}{\sigma\sqrt{\tau}} = -0.0368\,,$$

$$d_{1bs}\left(\frac{H^2}{S},\ K\right) = d_{bs}\left(\frac{H^2}{S},\ K\right) + \sigma\sqrt{\tau} = 0.1046\,,$$

$$\begin{aligned}
C_{bs}(100,\ 105) &= 100\mathrm{e}^{-0.03\times0.5}N[d_{1bs}(100,\ 105)] \\
&\quad - 105\mathrm{e}^{-0.08\times0.5}N[d_{bs}(100,\ 105)] \\
&= \$4.513\,, \\
P_{bs}(110.25,\ 102) &= -110.25\mathrm{e}^{-0.03\times0.5}N[-d_{1bs}(100,\ 102)] \\
&\quad - 102\mathrm{e}^{-0.08\times0.5}N[-d_{bs}(100,\ 102)] = \$1.989\,, \\
P_{bs}(110.25,\ 105) &= -110.25\mathrm{e}^{-0.03\times0.5}N[-d_{1bs}(100,\ 105)] \\
&\quad - 105\mathrm{e}^{-0.08\times0.5}N[-d_{bs}(100,\ 105)] \\
&= \$2.826\,, \\
UINC &= \left\{C_{bs}(100,\ 105) + (105-102)\mathrm{e}^{-r\tau}N[d_{bs}(100,\ 105)]\right\} \\
&\quad +1.05\mathrm{e}^2 \times 0.03/0.20^2\left\{P_{bs}(110.25,\ 102)\right. \\
&\quad \left.-P_{bs}(110.25,\ 105) + (105-102)\mathrm{e}^{-r\tau}N[-d_{bs}(105,\ 100)]\right\} \\
&= 4.513 + 3\mathrm{e}^{-0.08\times0.5}N(-0.2389) + 1.0759[1.989 - 2.826 \\
&\quad + 3\mathrm{e}^{-0.08\times0.5}N(-0.4511)] = \$5.792\,.
\end{aligned}$$

The present value of the rebate of an up-in barrier call option ($RBUI$) can be similarly obtained as in deriving (10.38) using the restricted density function given in (10.20a):

$$RBUI = \mathrm{e}^{-r\tau}Rm(\tau)\left\{N[-d_{bs}(S,\ H)] - \left(\frac{H}{S}\right)^{2\upsilon/\sigma^2} N[-dbs(H,\ S)]\right\}. \tag{10.41}$$

The price of an up-in call option (PUIC) can now be expressed readily using (10.40) and (10.41):

$$PUIC = PUINC + RBUI\,, \tag{10.42}$$

where $PUIC$ and $RBUI$ are given in (10.40) and (10.41), respectively.

Example 10.5. Find the present value of the rebate when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the call options in Example 10.4.

Substituting $Rm(0.5) = 1.5$, S = \$100, H = \$105, $\tau = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.41) yields

$$d_{bs}(S,\ H) = \frac{\ln(S/H) + \upsilon\tau}{\sigma\sqrt{\tau}} = \frac{\ln(100/105) + 0.03 \times 0.5}{0.20\sqrt{0.50}} = -0.2389\,,$$

$$d_{bs}(H,\ S) = \frac{\ln(H/S) + \upsilon\tau}{\sigma\sqrt{\tau}} = \frac{\ln(105/100) + 0.03 \times 0.5}{0.20\sqrt{0.50}} = 0.4511\,,$$

$$\begin{aligned} RBUI &= 1.5e^{-0.08\times0.5}\left[N(0.2389) - \frac{105}{100}^{2\times0.03/0.2^2} N(-0.4511)\right] \\ &= \$0.351\,. \end{aligned}$$

Example 10.6. Find the prices of the up-in barrier call options when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the options in Examples 10.4 and 10.5?

We can simply use the results from Examples 10.4 and 10.5. The up-in call option prices can be readily found by adding up the call values in Example 10.4 and the present value of the rebate in Example 10.5:

The down-in call option price with strike price $K = \$102$

$$= UINC(K = 102) + RBUI = 5.792 + 0.351 = \$6.143\,,$$

and the down-in call option price with strike price $K = \$108$

$$= UINC(K = 108) + RBUI = 3.454 + 0.351 = \$3.805\,.$$

Down-In Barrier Put Options

We have thus far priced knock-in call barrier options only. Using the restricted density function in (10.26) for a down-barrier as in pricing down-in call options, we can readily obtain the pricing formula of a down-in put option by integrating the payoff function of a vanilla put option given in (2.2). However, we do not have to go through these steps, for we can obtain the pricing formula of a down-in put option more conveniently using the "symmetry" between a down-in put option and an up-in call option. Since the open integration domain for a down-in put option is always from $-\infty$ to min (H, K), and that for an up-in call option is always from $\max(H, K)$

to $+\infty$, we can obtain the pricing formula of a down-in barrier put option (DINP) without rebate by substituting (i) the prices of all the call options with those of the corresponding put options in (10.40), (ii) max (H, K) in (10.40) with min (H, K), (iii) the digital number $B_{H>K}$ in (10.40) with the digital number $B_{K>H}$, (iv) the argument d_{bs} with $-d_{bs}$ as in (10.31) to find the vanilla put option pricing formula using the call option pricing formula, and (v) a positive sign in (10.40) with a negative sign, and a negative sign in (10.40) with a positive sign for terms other than the vanilla option expressions:

$$\begin{aligned} DINP = \left(\frac{H}{S}\right)^{2v/\sigma^2} & \left\{C_{bs}\left(\frac{H^2}{S},\ K\right) - C_{bs}\left(\frac{H^2}{S},\ H\right)\right. \\ & \left. - (H-K)e^{-r\tau}N[d_{bs}(H,\ S)]\right\}B_{K>H} \\ & + \Big(P_{bs}[S,\ \min(H,\ K)] - [\min(H,\ K) - K]e^{-r\tau} \\ & \times N\Big\{-d_{bs}[S,\ \min(H,\ K]\Big\}\Big)\,, \end{aligned} \tag{10.43}$$

where $P_{bs}(A,\ B)$ stands for the vanilla put option price with spot price A and strike price B, and $P_{bs}(A,\ B)$ is given in (10.31) with $\omega = -1$, and all parameters and intermediate functions are the same as in (10.40).

Example 10.7. Find the prices of the corresponding down-in barrier put options in Example 10.1.

Substituting $S =$ \$100, $K =$ \$92, $H =$ \$95, $sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.43) yields,

$$\begin{aligned} B_{K>H} &= 0,\ \min(H,\ K) = \min(95,\ 92) = 92\,, \\ DINP &= P_{bs}(100,\ 92) - (92-92)e^{-r\tau}N[-d_{bs}(100,\ 92)] \\ &= P_{bs}[100,\ 92] \\ &= -100e^{-0.03\times 0.5}N[-d_{1bs}(100,\ 92)] \\ &\quad - 92e^{-0.08 times 0.5}N[-d_{bs}(100,\ 92)] \\ &= \$1.68; \end{aligned}$$

and substituting $S = \$100$, $K = \$98$, $H = \$95$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.43) yields,

$$B_{K>H} = 1,\ \min(H,\ K) = \min(95,\ 98) = 95\,,$$

$$\begin{aligned} C_{bs}\left(\frac{95^2}{100},\ 98\right) &= 90.25e^{-0.03\times 0.5}N[-d_{1bs}(90.25,\ 98) \\ &\quad - 98e^{-0.08\times 0.5}N[-d_{bs}(100,\ 98)] \\ &= \$2.953\,, \end{aligned}$$

$$\begin{aligned} C_{bs}\left(\frac{95^2}{100},\ 95\right) &= 90.25e^{-0.03\times 0.5}N[-d_{1bs}(90.25,\ 95)] \\ &\quad - 95 \times e^{-0.08\times 0.5}N[-d_{bs}(100,\ 95)] \\ &= \$3.982\,, \end{aligned}$$

$$\begin{aligned} P_{bs}(100,\ 95) &= -100e^{-0.03\times 0.5}N[-d_{1bs}(100,\ 95)] \\ &\quad - 95e^{-0.08x0.5}N[-d_{bs}(100,\ 95)] \\ &= \$2.490, \end{aligned}$$

$$\begin{aligned} DINP &= \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2}\left\{C_{bs}]\left(\frac{95^2}{100},\ 98\right) - C_{bs}\left(\frac{95^2}{100},\ 95\right)\right. \\ &\quad \left. - (95-98)e^{-0.08\times 0.5}N[d_{bs}(95,\ 100)]\right\} \\ &\quad + \left\{P_{bs}(100,\ 95) - (95-98)e^{-0.08\times 0.5}N[-d_{bs}(100,\ 95)]\right\} \\ &= \$3.522\,. \end{aligned}$$

Example 10.8. Find the prices of the down-in barrier put options when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the options in Example 10.7.

Since the present value of the rebate at the option maturity if the barrier is not touched is the same for both down-in call and put options, the prices of the down-in put options are the sums of the values of the put options without rebates given in Example 10.7 and the present value of the rebate given in Example 10.2:

The price of the down-in put option with strike price $K = \$92$

$$= DINP(K = 92) + RBIN = 1.680 + 0.449 = \$2.129\,,$$

and the price of the down-in put option with strike price $K = \$98$

$$= DINP(K = 98) + RBIN = 3.522 + 0.449 = \$3.971\,.$$

Up-In Barrier Put Options

Our last task in this section is to price up-in barrier put options. As in obtaining the pricing formula of a down-in put option using that of an up-in call option, we can readily obtain the pricing formula of an up-in barrier put option without rebate (UINP) using the pricing formula given in (10.36) for a down-in call option by making the same five substitutions:

$$UINP = \left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{P_{bs}\left[\frac{H^2}{S}, \min(H, K)\right] - [\min(H, K) - K]e^{-r\tau} \times N\left[-d_{bs}\left(\frac{H^2}{S}, \min(H, K)\right)\right]\right\} + \left\{C_{bs}(S, K) - C_{bs}(S, H) - (H - K)e^{-r\tau}N[d_{bs}(S, H)]\right\}B_{K>H}\,, \quad (10.44)$$

where all parameters and intermediate functions are the same as in (10.43).

Example 10.9. Find the prices of the corresponding up-in barrier put options in Exercise 10.4.

Substituting $S = \$100$, $K = \$102$, $H = \$95$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.40) yields

$$v = r - g - \sigma^2/2 = 0.03\,,\ B_{H>K} = 0,$$

$$\min(H, K) = K = 102\,,\ H^2/S = 110.25\,,$$

$$UINP = \left(\frac{105}{100}\right)^{2\times 0.03/0.20^2}\left\{P_{bs}(110.25,\ 102) - (102 - 102)e^{-r\tau} \times N[-d_{bs}(110.25,\ 102)]\right\}$$

$$= 1.05^{1.5}[-110.25e^{-0.03\times 0.5}N[-d_{1bs}(110.25,\ 102)] + 102e^{-0.08\times 0.5}N[-d_{bs}(110.25,\ 102)]$$

$$= \$2.140\,,$$

and substituting $S =$ \$100, $K =$ \$108, $H =$ \$105, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.44) yields

$$B_{H>K} = 1,\ \min(H,\ K) = \min(105,\ 108) = 105\,,$$

$$d_{bs}\left(\frac{H^2}{S},\ K\right) = \frac{\ln[(H^2/S)/K] + v\tau}{\sigma\sqrt{\tau}} = 0.4511\,,$$

$$d_{1bs}\left(\frac{H^2}{S},\ K\right) = d_{bs}\left(\frac{H^2}{S},\ K\right) + \sigma\sqrt{\tau} = 0.5925\,,$$

$$\begin{aligned}
C_{bs}(100,\ 105) &= 100\mathrm{e}^{-0.03\times 0.5}N[d_{1bs}(100,\ 105)\\
&\quad - 105\mathrm{e}^{-0.08\times 0.5}N[d_{bs}(100,\ 105)]\\
&= \$4.513\,,\\
C_{bs}(100,\ 108) &= 100\mathrm{e}^{-0.03\times 0.5}N[d_{1bs}(100,\ 108)\\
&\quad - 108\mathrm{e}^{-0.08\times 0.5}N[d_{bs}(100,\ 108)]\\
&= \$3.454\,,\\
P_{bs}(110.25,\ 105) &= -110.25\mathrm{e}^{-0.03\times 0.5}N(-0.5925)\\
&\quad + 105\mathrm{e}^{-0.08\times 0.5}N(-0.4511)\\
&= \$2.826\,,\\
UINP &= \left(\frac{105}{100}^{2\times 0.03/0.20^2}\left\{P_{bs}(110.25,\ 105)\right.\right.\\
&\quad \left. - (105 - 108)\mathrm{e}^{-0.08\times 0.5}N[-d_{bs}(110.25,\ 105)]\right\}\\
&\quad C_{bs}(100,\ 108) - C_{bs}(100,\ 105) - (105 - 108)\mathrm{e}^{-0.08\times 0.5}\\
&\quad \left.\times N[d_{bs}(100,\ 105)]\right)\\
&= \$4.162\,.
\end{aligned}$$

Example 10.10. Find the prices of the up-in barrier put options when the rebate is paid \$1.5 at maturity if the barrier is not touched within the lives of the options in Example 10.9.

Using the prices of the up-in barrier put options without rebates in Example 10.9 and the present value of the rebate in Example 10.5, we can obtain the prices of the down-in barrier put options with a \$1.5 rebate paid at maturity:

the price of the up-in put option with strike price $K = \$102$

$$= UINP(K = 102) + RBIN = 2.140 + 0.351 = \$2.491\,,$$

and the price of the up-in put option with strike price $K = \$108$

$$= UINP(K = 108) + RBIN = 4.162 + 0.351 = \$4.513\,.$$

10.5.3. Knockout Options

Knockout options are the complements of knock-in options as we described earlier in this chapter. The most important difference between an in option and its corresponding out option is that whereas the payoff of an in option is the corresponding vanilla option as soon as the barrier is touched or a rebate if the barrier is never touched, the payoff of an out option is a rebate as soon as the barrier is touched or its corresponding vanilla option if the barrier is never touched. The uncertain first hitting time creates some difficulties for us to price out-barrier options because we have to use the first passage time to discount the rebate. We will follow similar steps as in the earlier parts of this section in pricing in-barrier options to price out-barrier options. We will first find the values of vanilla options if the barriers are never touched and then find the present values of both nondeferrable and deferrable rebates if the barriers are touched.

Values of Out Options Without Rebates

There are two parts of the payoff of a "down-outer" in (7.3), one including the nondeferrable rebate if the barrier is reached some time within the life of the option, and the other the payoff of the corresponding vanilla option if the barrier is never reached within the life of the option. Using the restricted density function in (10.24) for a down barrier, we can readily find the value of a down-out barrier call option without a rebate (DOTC):

$$\begin{aligned} DOTC = {} & C_{bs}[S,\ \max(H,\ K)] - \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right] \\ & + [\max(H,\ K) - K]e^{-r\tau}\Bigg(N\Big\{d_{bs}[S, \max(H,\ K)]\Big\} \\ & - \left(\frac{H}{S}\right)^{2v/\sigma^2} N\left\{d_{bs}\left[\frac{H^2}{S},\ \max(H,\ K)\right]\right\}\Bigg)\,, \end{aligned} \tag{10.45}$$

where C_{bs} is the extended Black-Scholes formula given in (10.31) and other parameters are the same as in (10.33)–(10.44).

Example 10.11. Find the values of the down-out barrier call options without rebates with strike prices $K = \$92$ and $98 to expire in half a year, given the spot price $S = \$100$, the down barrier $L = H = \$95$, interest rate $r = 8\%$, the yield of the underlying asset $g = 3\%$, the volatility of the underlying asset 20%.

When $K = \$98, \max(H,\ K) = \max(95, 98) = \$98 = K$, the second term in (10.44) is zero. Substituting $\max(H,\ K) = 98$, $S = \$100$, $K = \$98$, $H = L = \$95$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau = 0.5$, $v = r - g - \sigma^2/2$ $= 0.03$ into (10.44) yields

$$
\begin{aligned}
DOTC &= C_{bs}(100,\ 98) - \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2} C_{bs}\left(\frac{95^2}{100},\ 98\right) \\
&= 100\mathrm{e}^{-0.03\times 0.5}N[d_{1bs}(100,\ 98)] - 98\mathrm{e}^{-0.08\times 0.5}N[d_{bs}(100,\ 98)] \\
&\quad - 0.9259\Big\{90.25N[d_{1bs}(90.25,\ 98)] \\
&\quad - 98\mathrm{e}^{-0.08\times 0.5}N[d_{bs}(90.25,\ 98)]\Big\} \\
&= \$5.148\,.
\end{aligned}
$$

When $K = \$92$, $\max(H,\ K) = \max(95,\ 92) = \$95$. Substituting $K = \$92$, max(H, K) = \$95, and other parameters into (10.44) yields

$$
\begin{aligned}
DOTC &= C_{bs}(100,\ 95) - \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2} C_{bs}\left(\frac{95^2}{100},\ 95\right) \\
&\quad + (95 - 92)\mathrm{e}^{-0.08\times 0.5}\left\{N[d_{bs}(100, 95)] - \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2}\right. \\
&\quad \left.\times N\left[d_{bs}\left(\frac{95^2}{100},\ 95\right)\right]\right\} \\
&= \$6.936\,.
\end{aligned}
$$

Using the restricted density function given in (10.20) for an up-barrier, we can readily find the value of an up-out call option without a rebate ($UOTC$):

$$UOTC = B_{H>K}\left\{C_{bs}(S,\ K) - C_{bs}(S,\ H) - (H-K)\mathrm{e}^{-r\tau}N[d_{bs}(S,\ H)]\right.$$
$$-\left(\frac{H}{S}\right)^{2v^2/\sigma^2}\left[C_{bs}\left(\frac{H^2}{S},\ K\right) - C_{bs}\left(\frac{H^2}{S},\ H\right)\right.$$
$$\left.\left.-(H-K)\mathrm{e}^{-r\tau}N[d_{bs}(H,\ S)]\right]\right\}, \tag{10.46}$$

where all parameters are the same as in (10.31)–(10.44).

It is obvious that the value of the up-out call option without any rebate is zero when the strike price K is greater than or equal to the up barrier, which is consistent with the intuition that the call option has no payoff because the density function above K is zero above the up-barrier. Only when the strike price is lower than the barrier does the up-out call option without a rebate has a value because its expected payoff equals the integration from K to H.

Example 10.12. Find the prices of the up-out barrier call options without rebates with strike prices $K =$ \$108 and \$102 to expire in half a year, given the up-barrier $U = H =$ \$105, and other parameters are the same as in Example 10.11.

When $K=$ \$108, $B_{H>K} = 0$, therefore $UOTC = 0$. Substituting $K =$ \$102, $H =$ \$105, $H^2/S = 110.25$ into (10.45) yields

$$\begin{aligned}
C_{bs}(S,\ K) &= 100\mathrm{e}^{-0.03\times0.5}N[d_{1bs}(100,\ 102)] \\
&\quad - 102\mathrm{e}^{-0.08\times0.5}N[d_{bs}(100,\ 102)] \\
&= \$5.798, \\
C_{bs}(S,\ H) &= 100\mathrm{e}^{-0.03\times0.5}N[d_{1bs}(100,\ 105)] \\
&\quad - 105\mathrm{e}^{-0.08\times0.5}N[d_{bs}(100,\ 105)] \\
&= \$4.513, \\
C_{bs}\left(\frac{H^2}{S},\ K\right) &= 110.25\mathrm{e}^{-0.03\times0.5}N[d_{1bs}(110.25,\ 102)] \\
&\quad - 102\mathrm{e}^{-0.08\times0.5}N[d_{bs}(100,\ 102)] \\
&= \$12.597,
\end{aligned}$$

$$
\begin{aligned}
C_{bs}\left(\frac{H^2}{S},\ H\right) &= 110.25\mathrm{e}^{-0.03\times 0.5}N[d_{1bs}(110.25,\ 105)] \\
&\qquad - 105\mathrm{e}^{-0.08\times 0.5}N[d_{bs}(110.25,\ 105)] \\
&= \$10.552\,, \\
UOTC(K=102) &= [C_{bs}(100,\ 102) - C_{bs}(100,\ 105) \\
&\qquad - (105-102)\mathrm{e}^{-0.08\times 0.5}N(-0.2389)] \\
&\qquad - 1.05^{2\times 0.03/0.20^2}[C_{bs}(110.25,\ 102) \\
&\qquad - C_{bs}(110.25,\ 105) - 3\mathrm{e}^{-0.08\times 0.5}N(0.4511)] \\
&= \$0.006\,.
\end{aligned}
$$

Using the symmetry between an up-out put option and a down-out call option, and the pricing formula of a down-out call option given in (10.45), we can readily find the price of an up-out barrier put option without a rebate ($UOTP$):

$$
\begin{aligned}
UOTP = P_{bs}[S, \min(H,\ K)] &- \left(\frac{H}{S}\right)^{2v/\sigma^2} P_{bs}\left[\frac{H^2}{S}, \min(H,\ K)\right] \\
&- [\min(H,\ K) - K]\mathrm{e}^{-r\tau}\left(N\left\{-d_{bs}[S, \min(H,\ K)]\right\} - \left(\frac{H}{S}\right)^{2v/\sigma^2}\right. \\
&\left.\times N\left\{-d_{bs}\left[\frac{H^2}{S}, \min(H,\ K)\right]\right\}\right), \qquad (10.47)
\end{aligned}
$$

where C_{bs} is the extended Black-Scholes formula given in (10.31) and other parameters are the same as in (10.33)–(10.44).

It is obvious that the pricing formula (10.46) can be readily obtained from (10.45) using the five substitutions in obtaining (10.43) from (10.40). This is because there is a symmetry between the open integration domain from $-\infty$ to the $\min(H,\ K)$ for an up-out put option and that from $\max(H,\ K)$ to ∞ for a down-out call option.

Example 10.13. Find the values of the up-out barrier put options without rebates with strike prices $K = \$102$ and $\$108$ to expire in half a year, given the spot price $S = \$100$, the up-barrier $U = H = \$105$, interest rate $r = 8\%$, the yield of the underlying asset $g = 3\%$, the volatility of the underlying asset 20%.

When $K = \$102, \min(H,\ K) = \min(105, 102) = \$102$, the second term in (10.47) disappears. Substituting $K = \min(H,\ K) = \$102$, $S = \$100$, $H = U = \$105$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau = 0.5$, $v = r - g - \sigma^2/2 = 0.03$ into (10.46) yields

$$\begin{aligned} UOTP &= P_{bs}(100,\ 102) - \left(\frac{105}{100}\right)^{2\times 0.03/0.20^2} P_{bs}\left(\frac{105^2}{100},\ 102\right) \\ &= \$3.147\,. \end{aligned}$$

When $K = \$108, \min(H,\ K) = \min(105,\ 108) = \$105$. Substituting $K = \$108, \min(H,\ K) = \$105$, and other parameters into (10.47) yields

$$\begin{aligned} UOTP &= P_{bs}(100,\ 105) - \left(\frac{105}{100}\right)^{2\times 0.03/0.20^2} P_{bs}\left(\frac{105^2}{100},\ 105\right) \\ &\quad - (105 - 108)\mathrm{e}^{-0.08\times 0.5}\Bigg\{ N[-d_{bs}(100,\ 105)] \\ &\quad - \left(\frac{105}{100}\right)^{2\times 0.03/0.20^2} N\left[-d_{bs}\left(\frac{105^2}{100},\ 105\right)\right]\Bigg\} \\ &= \$4.547. \end{aligned}$$

Similar to (10.46), the pricing formula of a down-out put option without a rebate can be obtained using the symmetry between a down-out put option and an up-out call option and using the pricing formula given in (10.46)

$$\begin{aligned} DOTP = B_{K>H}\Bigg\{ & P_{bs}(S,\ K) - P_{bs}(S,\ H) \\ & + (H - K)\mathrm{e}^{-r\tau} N[-d_{bs}(S,\ H)] \\ & - \left(\frac{H}{S}\right)^{2v/\sigma^2} \left[P_{bs}\left(\frac{H^2}{S},\ K\right) - P_{bs}\left(\frac{H^2}{S},\ H\right)\right. \\ & \left. + (H - K)\mathrm{e}^{-r\tau} N[-d_{bs}(H,\ S)]\right]\Bigg\}, \end{aligned} \tag{10.48}$$

where P_{bs} stands for the Black-Scholes pricing formula for a put option and all parameters are the same as in (10.45).

Eample 10.14. Find the values of the corresponding down-out barrier put options without rebates in Example 10.11.

When $K =$ \$92 $<$ \$95 $= H$, $B_{K>H} = 0$, the down-out put option price is zero from (10.47). When $K =$ \$98 $>$ \$95 $= H$, $B_{K>H} = 1$. Substituting $S =$ \$100, $K =$ \$98, $H = L =$ \$95, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau = 0.5$, $v = r - g - \sigma^2/2 = 0.03$, and $H^2/S = 90.25$ into (10.47) yields

$$\begin{aligned}P_{bs}(S,\ K) &= -100e^{-0.03\times0.5}N[-d_{1bs}(100,\ 98)]\\ &\quad + 98e^{-0.08\times0.5}N[-d_{bs}(100,\ 98)]\\ &= \$3.528\,,\\ P_{bs}(S,\ H) &= -100e^{-0.03\times0.5}N[d_{1bs}(100,\ 95)]\\ &\quad + 95e^{-0.08\times0.5}N[d_{bs}(100,\ 95)]\\ &= \$2.490\,,\\ P_{bs}\left(\frac{H^2}{S},\ K\right) &= -90.25e^{-0.03\times0.5}N[-d_{1bs}(90.25,\ 98)]\\ &\quad + 98e^{-0.08\times0.5}N[-d_{bs}(90.25,\ 98)]\\ &= \$8.204\,,\\ P_{bs}\left(\frac{H^2}{S},\ H\right) &= -90.25e^{-0.03\times0.5}N[-d_{1bs}(90.25,\ 95)]\\ &\quad + 95e^{-0.08\times0.5}N[-d_{bs}(90.25,\ 95)]\\ &= \$6.350\,.\end{aligned}$$

Thus,

$$\begin{aligned}DOTP(K = 98) &= P_{bs}(100,\ 98) - P_{bs}(100,\ 95)\\ &\quad + (95 - 98)e^{-0.08\times0.5}N[-d_{bs}(100,\ 95)]\\ &\quad - 0.95^{2x0.03/0.20^2}\{P_{bs}(90.25,\ 98) - P_{bs}(90.25,\ 95)\\ &\quad - 3e^{-0.08\times0.5}N[-d_{bs}(95,\ 100)]\}\\ &= \$0.005\,.\end{aligned}$$

Present Values of Rebates of Out Options

Rebater of barrier options are popular in the market because they can mitigate the all-or-nothing components of barrier risk. Large rebates would de-leverage the transactions, making it more expensive than a variable barrier.

The expected nondeferrable rebate can be obtained by integrating $R(T)$ in (10.5) using the density function of the first passage time given in (10.30). The derivation of the expected rebate is rather long because the time at which the rebate is paid is uncertain. Many steps are involved and we will skip these steps here. Interested readers may check them in Appendix of this Chapter. The present value of the time-dependent part of the rebate for an out option ($RBTOT$) given in (10.5) is

$$\begin{aligned} RBTOT(\eta,\ \theta,\ \tau) &= R\{e^{\eta T} | S(t) < H \text{ and } S(T) \geq H \text{ for some } t < T \leq t^*\} \\ &= R\left\{\left(\frac{H}{S}\right)^{q_1(r-\eta)} N[\theta Q_1(r-\eta)] \right. \\ &\quad \left. + \left(\frac{H}{S}\right)^{q_{-1}(r-\eta)} N[\theta Q_{-1}(r-\eta)]\right\}, \qquad (10.49a) \end{aligned}$$

if the rebate growth rate $\gamma \leq r + \upsilon^2/(2\sigma^2)$, where

$$\begin{aligned} \psi(s) &= \sqrt{\upsilon^2 + 2s\sigma^2}\,, \\ Q_\nu(s) &= \frac{\ln(H/S) + \nu\tau\psi(s)}{\sigma\sqrt{\tau}},\ \nu = 1 \text{ or } -1\,, \\ q_\nu(s) &= \frac{\upsilon + \nu\psi(s)}{\sigma^2}\,; \end{aligned}$$

and

$$\begin{aligned} RBTOT(\eta,\ \theta,\ \tau) = RRe&\left\{\left(\frac{H}{S}\right)^{q'_1(r-\eta)} N[\theta Q'_1(r-\eta)] \right. \\ &\left. + \left(\frac{H}{S}\right)^{q'_1(r-\eta)} N[\theta Q'_{-1}(r-\eta)]\right\}, \qquad (10.49b) \end{aligned}$$

if the rebate growth rate $\eta > r + \upsilon^2/(2\sigma^2)$, where

$$\begin{aligned} \psi'(s) &= i\sqrt{-\upsilon^2 - 2s\sigma^2}\,, \\ Q'_\nu(s) &= \frac{\ln(H/S) + \nu\tau_e\psi'(s)}{\sigma\sqrt{\tau_e}},\ \nu = 1 or - 1\,, \\ q'_\nu(s) &= \frac{\upsilon + \nu\psi'(s)}{\sigma^2}\,, \end{aligned}$$

$i = \sqrt{-1}$ is the standard unit of an imaginary number, $Re(\alpha + \beta i) = \alpha$ is the function to choose the real part of an imaginary number $\alpha + \beta i$ (both α and β are real numbers).

When the rebate growth rate η is smaller than $r+v^2/(2\sigma^2)$, the present value of the time-dependent rebate can be calculated directly using the closed-form solution given in (10.49a). When the rebate growth rate is larger than $r+v^2/(2\sigma^2)$, there is one more step involved because the formula given in (10.49b) involves imaginary numbers. The present value of the time-dependent rebate is actually the real part of the imaginary number given in (10.49b). The function to choose the real part of an imaginary number can be readily found in most computer systems. We can easily find the trivial case of constant rebate $R(T) = R$ as a special case of (10.49a) for zero growth rate $= 0$. Substituting $\eta = 0$ into (10.48a) yields the present value of the time-independent rebate or constant rebate (CRBOT):

$$CRBOT(\theta) = R\left\{\left(\frac{H}{S}\right)^{q_1(r)} N[\theta Q_1(r)] + \left(\frac{H}{S}\right)^{q_{-1}(r)} N[\theta Q_{-1}(r)]\right\}, \quad (10.50)$$

where all parameters are the same as in (10.49a).

With the present value of the time-dependent rebate in (10.49) and the constant rebate in (10.50) for an out barrier option, we can express the present value of the rebate of an out option ($RBOT$) given in (10.5) for $\xi = 1$ as follows

$$RBOT(\eta,\ \theta) = RBTOT(\eta,\ \theta) - RBTOT(0,\ \theta)\,, \quad (10.51)$$

where RBTOT stands for the present value of the time-dependent rebate with the growth rate η given in (10.49).

Example 10.15. Find the present value of the rebate if a constant rebate \$1 is paid any time the options are knocked-out within the lives of the down-out barrier options in Example 10.1.

Substituting $S =$ \$100, $H =$ \$95, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, $\theta = 1$, and $v = r - g - \sigma^2/2 = 0.03$ into (10.50) yields

$$\psi(r) = \sqrt{v^2 + 2r\sigma^2} = 0.08544\,.$$

The present value of the constant rebate of the down-out options is

$$\begin{aligned} CRBOT(1) = 1\Bigg[&\left(\frac{95}{100}\right)^{(0.03+0.0854])/0.20^2} N\left(\frac{(\ln(0.95) + 0.50 \times 0.08544}{0.20\sqrt{0.50}}\right) \\ &+ \left(\frac{95}{100}\right)^{(0.03-0.0854)/0.20^2} N\left(\frac{(\ln(0.95) - 0.50 \times 0.08544}{0.20\sqrt{0.50}}\right)\Bigg] \\ = {}&\$0.682\,. \end{aligned}$$

Example 10.16. Find the present value of the rebate if a constant rebate \$1 is paid any time the option is knocked-out within the life of an up-out barrier option with the up barrier = \$105 to mature in half a year, and other parameters are the same as in Example 10.15.

Substituting $S =$ \$100, $H =$ \$105, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, $\theta = -1$ and $v = r - g - \sigma^2/2 = 0.03$ into (10.50) yields the present value of the constant rebate for the up-out option

$$CRBOT(1) = 1\left[\left(\frac{105}{100}\right)^{(0.03+0.0854)/0.20^2} N\left(-\frac{\ln(1.05) + 0.50 \times 0.08544}{0.20\sqrt{0.50}}\right)\right.$$
$$\left. + \left(\frac{105}{100}\right)^{(0.03-0.0854)/0.20^2} N\left(-\frac{(\ln 1.05 - 0.50 \times 0.08544}{0.20\sqrt{0.50}}\right)\right]$$
$$= \$0.749\,.$$

Example 10.17. Find the present values of the rebates of the down- and up-out barrier options in Examples 10.15 and 10.16 if the rebates increase exponentially at a constant rate of 5%, and other parameters remain the same as in Examples 10.15 and 10.16.

As $\eta = 0.05 < r + v^2/(2\sigma^2)$, we use (10.49a) for the present values of the time-dependent rebates. Substituting $\eta = 0.05$, $R = 1$, $S =$ \$100, $H =$ \$95, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (10.49a) yields

$$\psi(r - \eta) = \sqrt{v^2 + 2(r - \eta)\sigma^2}$$
$$= 0.0574\,.$$

The present value of the rebate of the down-out barrier option is

$$RBTOT(0.05,\ 1) = 1\left[\left(\frac{95}{100}\right)^{(0.03+0.0574)/0.20^2} N\left(\frac{\ln 0.95 + 0.50 \times 0.0574}{0.20\sqrt{0.50}}\right)\right.$$
$$\left. + \left(\frac{95}{100}\right)^{(0.03-0.0755)/0.20^2} N\left(\frac{(\ln 0.95 - 0.50 \times 0.0574}{0.20\sqrt{0.50}}\right)\right]$$
$$= \$0.690\,,$$

and the present value of the rebate of the up-out barrier option is

$$RBTOT(0.05,-1)=1\left[\left(\frac{105}{100}\right)^{(0.03+0.0574)/0.20^2}N\left(-\frac{\ln 1.05+0.50\times 0.0574}{0.20\sqrt{0.50}}\right)\right.$$
$$\left.+\left(\frac{105}{100}\right)^{(0.03-0.0755)/0.20^2}N\left(-\frac{\ln 1.05-0.50\times 0.0574}{0.20\sqrt{0.50}}\right)\right]$$
$$=\$0.758\,.$$

Comparing the results in Examples 10.15, 10.16, and 10.17, we find that the present values of the rebates for the down- and up-out options increase from \$0.682 and \$0.749 in Examples 10.15 and 10.16 to \$0.690 and \$0.758 in Example 10.17, respectively, resulting from the 5% growth in the time-dependent rebates.

The rebate can sometimes be deferrable, i.e., it can be paid at the maturity of the option if the barrier is touched within the life of the option. If the rebate for an out option is deferrable to the maturity of the option and it is also dependent on the time to maturity $Rd(\tau)$ given in (10.3a′) and (10.4a′), the present value of the deferrable rebate (DRB) can be obtained:[2]

$$\begin{aligned} DRB &= E(PUKO) \\ &= e^{-r\tau}Rd(\tau)\left\{\left(\frac{H}{S}\right)^{2v/\sigma^2}N\left[\theta\left(\frac{a+v\tau}{\sigma\sqrt{\tau}}\right)\right]\right. \\ &\quad \left.+N\left[\theta\left(\frac{a-v\tau}{\sigma\sqrt{\tau}}\right)\right]\right\}, \end{aligned} \tag{10.52}$$

where $Rd(\tau)$ is the deferred rebate to be paid at the maturity of the option.

Example 10.18. Find the present value of the deferrable rebates of both the down- and up-out barrier options in Examples 10.15 and 10.16 if the rebates are deferred to be paid \$1 at maturity.

Substituting $S=\$100$, $H=\$95$, $\sigma=0.20$, $r=0.08$, $g=0.03$, $\tau=0.50$, $\theta=1$ and $v=r-g-\sigma^2/2=0.03$ into (10.52) yields the present

[2]The rebate paid at maturity $Rd(\tau)$ must first be discounted to the time T when the option is knocked out with the discounting factor $\exp[-r(\tau-T)]=e^{-r\tau}\exp(rT)$. Consider $e^{-r\tau}\exp(rT)$ as a rebate with a constant growth rate $\eta=r$. Substituting $\eta=r$ into (10.49a), we can find $\psi(r-r)=v$ and the result given in (10.52) is immediately from (10.49a).

value of the deferrable rebate of the down-out option

$$DRB = e^{-0.08\times0.5}\left[\left(\frac{95}{100}\right)^{2\times0.03/0.20^2} N\left(\frac{\ln 0.95 + 0.03\times0.50}{0.20\sqrt{0.50}}\right) + N\left(\frac{\ln 0.95 - 0.03\times0.50}{0.20\sqrt{0.50}}\right)\right]$$

$$= \$0.662\,,$$

and substituting $S =$ \$100, $H =$ \$105, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, $\theta = -1$ and $v = r - g - \sigma^2/2 = 0.03$ into (10.52) yields the present value of the deferrable rebate of the up-out option

$$DRB = e^{-0.08\times0.5}\left[\left(\frac{105}{100}\right)^{2\times0.03/0.20^2} N\left(-\frac{\ln 1.05 + 0.03\times0.50}{0.20\sqrt{0.50}}\right) + N\left(-\frac{\ln 1.05 - 0.03\times0.50}{0.20\sqrt{0.50}}\right)\right]$$

$$= \$0.727\,.$$

Comparing the results in Examples 10.15, 10.16, and 10.18, we find that the present values of the deferred rebates of the down- and up-out options, \$0.662 and \$0.727, are slightly lower than the corresponding values \$0.682 and \$0.749 in Examples 10.15 and 10.16. This is because the one-dollar rebates can be obtained before maturity for nondeferrable rebates in Examples 10.15 and 10.16, and rebates received earlier have smaller discounting factors and therefore higher present values.

We can now express the price of a down-out barrier call option ($PDOTC$) and the price of an up-out barrier call ($PUOTC$) by summing up the values of the out-barrier options without rebates and the present values of the rebates

$$PDOTC = DOTC + RBTOT(\theta = 1)\,, \tag{10.53}$$

and

$$PUOTC = UOTC + RBTOT(\theta = -1)\,, \tag{10.54}$$

where $DOTC$ stands for the value of the down-out call option without rebates given in (10.45), $UOTC$ stands for the value of the up-out call option without barrier given in (10.46), and $RBTOT$ is value of the rebate given in (10.49).

Example 10.19. Find the prices of the down-out barrier options in Example 10.11 with \$1 nondeferrable rebates.

Using the pricing formula in (10.53), we can obtain the prices of the down-out calls by adding up the values of the down-out barrier options without rebates in Example 10.11 and the present value of the \$1 nondeferrable rebate in Example 10.15. Thus, the prices of the down-out barrier call options are 6.936 + 0.682 = \$7.618 with strike price $K =$ \$92 and 5.148 + 0.682 = \$5.830 with strike price $K =$ \$98.

The price of an up-out barrier put option ($PUOTP$) and that of a down-out barrier put option ($PDOTP$) can be similarly expressed using (10.47), (10.48), and the present value of the rebate given in (10.49):

$$PUOTP = UOTP + RBTOT(\theta = -1)\,, \tag{10.55}$$

and

$$PDOTP = DOTP + RBTOT(\theta = 1)\,, \tag{10.56}$$

where $UOTP$ stands for the value of the up-out put option without rebates given in (10.47), and $DOUP$ stands for the value of the down-out put option without rebates given in (10.48), and $RBTOT$ is the value of the rebate given in (10.49).

Example 10.20. Find the prices of the up-out barrier put options in Example 10.13 with \$1 nondeferrable rebates.

Using the pricing formula in (10.54), we can obtain the prices of the up-out puts by adding up the values of the up-out barrier options without rebates in Example 10.13 and the present value of the \$1 nondeferrable rebate in Example 10.16. Thus, the prices of the up-out barrier put options are 3.147 + 0.749 = \$3.896 with strike price $K =$ \$102 and 4.547 + 0.749 = \$5.296 with strike price $K =$ \$108.

10.5.4. Relationship Between the Prices of An In Option and Its Corresponding Out Option

Consider a portfolio consisting of an in option and its corresponding out option without any rebate. If the barrier is never reached, the in option will end up worthless, yet the out option will have its corresponding vanilla option as its payoff. On the other hand if the barrier is reached any time during the life of the option, the in option will have the corresponding vanilla option as its payoff whereas the out option will end up worthless. Thus, the portfolio including an out and its corresponding in options will have the same payoff as that of their corresponding vanilla option regardless of whether the barrier is reached. Since the portfolio and the corresponding vanilla option have exactly the same payoff at the option maturity, arbitrage argument guarantees that the price of the portfolio and the vanilla option must be the same. Put it algebraically, our above argument states

$$PI + POT = C_{bs}(S,\ K)\,, \tag{10.57}$$

where PI and POT stand for the prices of an in option and its corresponding out option, respectively, and $C_{bs}(S,\ K)$ is the price of their corresponding vanilla option.

Since the identity in (10.57) holds for any pair of an in and its corresponding out options, we can also check the validity of the pricing formulas of the four pairs of vanilla barrier options obtained earlier in this section. We leave some of the checking as exercises of this chapter.

The identity in (10.57) can also be used to find the price of either an in option or an out option conveniently given the price of one of them and their corresponding vanilla option.

Example 10.21. Find the prices of the corresponding in call options using the results of the prices of the out call options in Example 10.11.

Substituting $S = \$100$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, and $K = \$92$ and \$98 into (10.31) yields the prices of the two vanilla call options \$11.799 and \$7.882, respectively. From Example 10.11, we know that the prices of the down-out call options with the two strike prices are \$5.418 and \$6.936, respectively. Using the identity given in (10.57), we can readily find the prices of the two corresponding in call options as

$$11.799 - 5.418 = \$6.381 \text{ and } 7.882 - 6.936 = \$0.946\,,$$

respectively.

Figure 10.6 depicts the prices of the knock-in and knockout at-the-money call options for various barriers from \$60 to \$200, given the time to maturity 1 year, volatility of the underlying asset 20%, interest rate 10%, spot price \$100, yield on the underlying asset zero, and all the rebates zero. The dotted and undotted curves represent the prices of the knockout and knock-in call options, respectively, and the line above stands for the sums of the prices of the complementary knock-ins and knockouts. The line above is obviously a straight line parallel to the horizontal axis. The sum of the prices is indeed the price of the corresponding vanilla option \$13.27. Figure 10.6 clearly demonstrates that the knock-in call option prices are about the same as the vanilla option price for barriers not significantly larger than the spot price, and the corresponding knockout call options are almost worthless. It

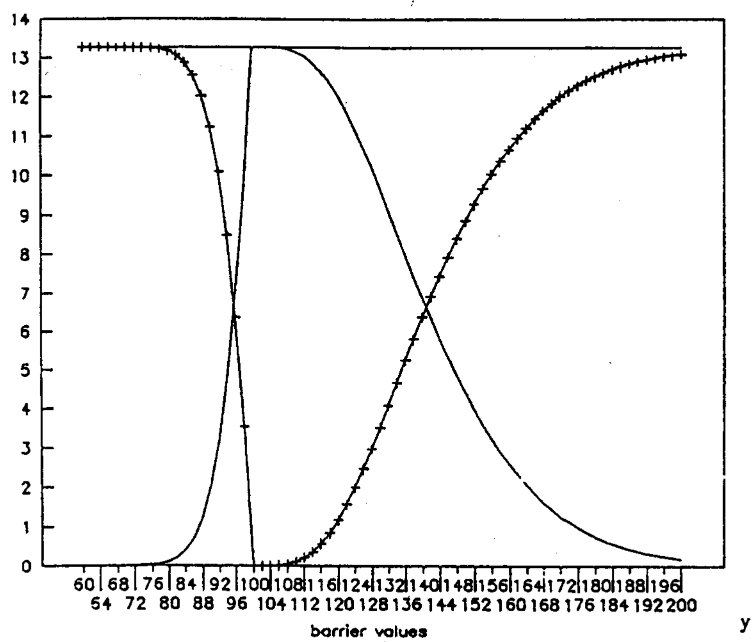

Fig. 10.6. Prices of knock-ins and outs for various barriers.

is worth noticing that the knock-in (resp. -out) call options appreciate (resp. depreciate) in value dramatically for barriers moderately smaller than the spot price. We will analyze these changes in more details in the following section.

Although this is by far the longest section of this book, there are some related materials we have not included in this section. We have provided formulas for all eight types of barrier options, and these formulas are clearly expressed in Black-Scholes type for illustrative purposes. In order to have a convenient source of the pricing formulas for all eight types of vanilla options, we will provide a table in Appendix of this chapter and interested readers may find it very useful.

10.6. GREEKS OF VANILLA BARRIER OPTIONS

The Greeks of barrier options are very different from those of vanilla options in many aspects because of their unique characteristics. We will analyze them in this section. We will emphasize the deltas, gammas, and vegas of standard barrier options and other sensitivities can be obtained accordingly. In order to express the Greeks of vanilla barrier options concisely, we need to obtain the delta, gamma, and vega of the call option pricing formula $C_{sb}(H^2/S,\ K)$ given in (10.33). Let $\delta_{h^2/s}$, $\gamma_{h^2/s}$, and $v_{h^2/s}$ stand for the delta, gamma, and vega of $C_{sb}(H^2/S,\ K)$ in (10.33) respectively. After simplifying, we can obtain[3]

$$\delta_{h^2/s} = -\omega\frac{H^2}{S^2}e^{-g\tau}N[\omega d_{bs1}(H^2/S,\ K)]\,, \tag{10.58}$$

$$\gamma_{h^2/s} = \frac{H^2}{S^3}e^{-g\tau}\left\{2\omega N\left[\omega d_{bs1}\left(\frac{H^2}{S},\ K\right)\right] + \frac{f[d_{bs1}(H^2/S,\ K)]}{\sigma\sqrt{\tau}}\right\}, \tag{10.59}$$

$$v_{h^2/s} = \sqrt{\tau}Ke^{-r\tau}f\left[d_{bs}\left(\frac{H^2}{S},\ K\right)\right] > 0\,, \tag{10.60}$$

where

$$d_{bs}\left(\frac{H^2}{S},\ K\right) = \frac{\ln[H^2/(SK)] + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\,,$$

and

$$d_{1bs}\left(\frac{H^2}{S},\ K\right) = d_{bs}\left(\frac{H^2}{S},\ K\right) + \sigma\sqrt{\tau}\,.$$

The delta in (10.58) has an opposite sign to that of a vanilla option in (2.43). The opposite sign results from the fact that the spot is in the denominator of the expression. We can obtain the delta, gamma, and vega of a down-in barrier option using (10.33) and the results in (10.58)–(10.60):

$$\delta_{DI} = \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[\delta_{h^2/s} - \frac{2v}{S\sigma^2}C_{bs}\left(\frac{H^2}{S},\ K\right)\right], \tag{10.61}$$

$$\gamma_{DI} = \left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{\frac{2v(2v+\sigma^2)}{S^2\sigma^4}C_{bs}\left(\frac{H^2}{S},\ K\right) + \gamma_{h^2/s} - \frac{4v}{S\sigma^2}\delta_{h^2/s}\right\}, \tag{10.62}$$

$$\mathrm{v}_{DI} = \left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{v_{h^2/s} - \frac{4(r-g)}{\sigma^3}C_{bs}\left(\frac{H^2}{S},\ K\right)\ln\left(\frac{H}{S}\right)\right\}, \tag{10.63}$$

where $\delta_{h^2/s}$, $\gamma_{h^2/s}$ and $v_{h^2/s}$ are given in (10.58), (10.59e), and (10.60), respectively.

Example 10.22. Find the delta, gamma, and vega of the down-in call option without rebate with strike price $K = \$98$ in Example 10.1.

[3]Using the function $d_{bs}(H^2/S\ K)$ in (10.31), we can readily show the identity

$$f[d_{bs}(H^2/S,\ K)]/f[d_{bs}(H^2/S,\ K) + \sigma\sqrt{\tau}]$$
$$= (H^2/SK)e^{(r-g)^\tau} = (H^2/S)e^{-g\tau}/Ke^{-r\tau}\,,$$

with which the Greeks can be simplified significantly.

Substituting $S = \$100$, $K = \$98$, $H = \$95$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$, $v = r - g - \sigma^2/2 = 0.08 - 0.03 - 0.20^2/2 = 0.03$, $H^2/S = 95^2/100 = 90.25$ into (10.58)–(10.60) yields

$$d_{bs}\left(\frac{H^2}{S}, K\right) = \frac{\ln[(H^2/S)/K] + v\tau}{\sigma\sqrt{\tau}} = -0.4765\,,$$

$$\begin{aligned} d_{1bs}\left(\frac{H^2}{S}, K\right) &= d_{bs}\left(\frac{H^2}{S}, K\right) + \sigma\sqrt{\tau} \\ &= -0.4765 + 0.20 \times \sqrt{0.50} = -0.3351\,, \end{aligned}$$

$$\begin{aligned} \delta_{h^2/s} &= -\frac{95^2}{100^2}e^{-0.03\times 0.50}N(-0.3351) \\ &= -0.9025 \times 0.9851 \times 0.3688 = -0.3279\,, \end{aligned}$$

$$\gamma_{h^2/s} = \frac{95^2}{100^3}e^{-0.03\times 0.50}\left\{2N(-0.3351) + \frac{f(-0.3351)}{0.20\sqrt{0.50}}\right\} = -0.017\,,$$

$$v_{h^2/s} = \sqrt{0.50} \times 98 \times e^{-0.08\times 0.50}f(-0.4765) = 23.711\,.$$

Substituting the above values into (10.61)–(10.63) yields

$$\delta_{DI} = \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2}\left\{-0.3279 - \frac{2 \times 0.03}{100 \times 0.20^2} \times 2.953\right\} = -0.3446\,,$$

$$\begin{aligned} \gamma_{DI} = \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2} &\left\{\frac{2 \times 0.03 \times (2 \times 0.03 + 0.20^2)}{100^2 \times 0.20^4} \times 2.953\right. \\ &\left. -0.017 - \frac{4 \times 0.03}{100 \times 0.20^2}(-0.3279)\right\} = -0.0056\,, \end{aligned}$$

$$\begin{aligned} v_{DI} &= \left(\frac{95}{100}\right)^{2\times 0.03/0.20^2}\left\{23.711 - \frac{4 \times (0.08 - 0.03)}{0.20^3} \times 2.953 \times \ln(0.95)\right\} \\ &= 25.4614\,. \end{aligned}$$

Assuming that all rebates are zero, the delta, gamma, and vega of an out barrier option can be readily obtained using the identity given in (10.57) and the results given in (10.61)–(10.63):

$$\delta_{UOUT} = \delta_{vanilla} - \delta_{DI}\,, \tag{10.64}$$

$$\gamma_{UOUT} = \gamma_{vanilla} - \gamma_{DI}\,, \tag{10.65}$$

$$v_{UOUT} = v_{vanilla} - v_{DI}\,, \tag{10.66}$$

where $\delta_{vanilla}$, $\gamma_{vanilla}$, and $v_{vanilla}$ are the delta, gamma, and vega of a vanilla option given in (3.32), (3.37), and (3.33), respectively; and δ_{DI}, γ_{DI}, and v_{DI} are the delta, gamma, and vega of an in barrier option given in (10.61), (10.62), and (10.63), respectively.

Example 10.23. Find the delta, gamma, and vega of the corresponding down-out call option in Example 10.1.

Substituting $S = \$100$, $K = \$98$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, and $\tau = 0.50$ into (3.32), (3.37), and (3.33) yields the delta, gamma, and vega of the corresponding vanilla option:

$$\delta_{vanilla} = \mathrm{e}^{-0.03\times 0.50} N[d_{1sb}(100,\ 98)] = 0.9851 \times 0.6519 = 0.6422\,,$$

$$\delta_{vanilla} = \mathrm{e}^{-0.03\times 0.50} f[d_{1sb}(100,\ 98)]/(100 \times 0.20\sqrt{0.50}) = 0.0258, \text{ and}$$

$$\delta_{vanilla} = 100 \times \sqrt{0.50} \times \mathrm{e}^{-0.03\times 0.50} f[d_{1sb}(100,\ 98)] = 25.7510\,.$$

Substituting the above Greeks of the vanilla option and the Greeks of the down-in call option in Example 10.22 into (10.64)–(10.66) yields the Greeks of the corresponding down-out call option:

$$\delta_{UOUT} = \delta_{vanilla} - \gamma_{DI} = 0.6422 - (-0.3446) = 0.9868\,,$$

$$\gamma_{UOUT} = \gamma_{vanilla} - \gamma_{DI} = 0.0258 - (-0.0056) = 0.0314\,,$$

and

$$v_{UOUT} = v_{vanilla} - v_{DI} = 25.7510 - 22.4614 = 3.2896\,.$$

The results in Examples 10.22 and 10.23 indicate that (i) the delta of call option can be negative and that it can be greater than 1, (ii) the gamma of an up-out barrier can be negative and it can be many times greater than the gamma of its corresponding vanilla option (the gammas of the down-in and down-out call options are 110 and 109 times greater than that of their corresponding vanilla option, respectively); and (iii) the vegas of both the down-in and down-out call options are smaller than that of their corresponding vanilla option. The delta of a vanilla call option is always between 0 and 100% and its gamma is always much smaller than its delta value. The extraordinarily different values of both the deltas and gammas of barrier options result from the fact that the barriers change the characteristics of the options significantly, especially the underlying asset prices around the barriers. Simulation results show that the gammas of barrier options can be extremely large for barriers close to the underlying spot prices.

Examples 10.22 and 10.23 show that whereas the deltas and gammas of barrier options exhibit properties significantly different from those of their corresponding vanilla options, the vegas of the two down barrier options are smaller than that of their corresponding vanilla option. We have some general results for the vegas of down-in and -out barrier options.

Corollary 10.1. The vegas of a down in barrier option and its corresponding down out option are given as follows if $K > H$

$$V_{din} = (vegafactor)Vega\,, \tag{10.67a}$$

and

$$V_{dout} = (1 - vegafactor)Vega\,, \tag{10.67b}$$

where

$$0 < vegafactor = \left(\frac{S}{H}\right)^{\frac{2\ln(S/K)}{\sigma^2\tau}} \psi^{\xi 2a^2/(\sigma^2\tau\kappa)} < 1\,,$$

V_{din}, V_{dout}, and $Vega$ stand for the vegas of the down-in barrier, down-out barrier, and the corresponding vanilla call options, respectively.

Proof. Substituting $d_{bs}(H^2/S,\ K) = d_{bs}(S,\ K) + 2\ln(H/S)$ into (10.60) and using the vega of a vanilla call option given in (3.37) yields the results given in (10.67) after simplifications. The proof of this corollary is given as an exercise of this chapter. □

The results in Corollary 10.1 indicate that the vegas of down barrier options are always smaller than, and sum up to, the vega of their corresponding vanilla option. The reason behind this is that vegas of any options cannot be negative, thus they cannot surpass the vega value of their corresponding vanilla option.

The rho of an up-in option can be obtained as follows using the identity given in footnote 3 in this chapter:

$$Rho_{DI} = \tau\omega e^{-r\tau} K \left(\frac{H}{S}\right)^{2v/\sigma^2} N[\omega d_{bs}(H^2/S,\ K)]\,, \tag{10.68}$$

which indicates that an up-in call (put) option becomes more expensive (resp. cheaper) as the interest rate goes higher (resp. lower).

We have analyzed the Greeks of down barrier options in this section. Those of the other types of vanilla barrier options can be analyzed accordingly. It would take a lot of space to list the formulas of the major Greeks of all eight types of vanilla barrier options. We will not list them here for the sake of simplicity.

10.7. SUMMARY AND CONCLUSIONS

We introduced all eight types of standard barrier options and discussed how to obtain conditional density functions to price them. These eight types of standard barrier options are also called vanilla barrier options. They are very different from vanilla options because an additional barrier condition is imposed on the distribution of the underlying asset price. These differences can be clearly seen from the Greeks of these options.

Our pricing formulas for the standard barrier options have a few additional generalizations and more flexibility over existing results. We provided closed-form solutions for the time-dependent rebate of out options — a more realistic and reasonable rebate payment; secondly, we provided closed-form solutions for both nondeferrable and deferrable rebates for out options; and lastly, our analysis allows the three kinds of rebates to be different: the rebate at maturity $Rm(\tau)$ if the out option is not knocked out during the life of the option, the nondeferrable rebate if the out option is knocked-out $R(T)$, and the deferrable rebate paid at maturity $R_d(\tau)$.

Barrier options are among the few most popular exotic options in the OTC marketplace because they are cheaper than vanilla options in general. Besides the vanilla barrier options, other barrier options have been designed to increase the flexibility of vanilla exotic options or to capture some more general features. We call the other barrier options exotic barrier options. The analysis of vanilla barrier options provides the foundation for the analysis of exotic barrier options, as the study of vanilla options provides the basis for exotic options. We will study these exotic options in the following chapter.

QUESTIONS AND EXERCISES

Questions

10.1. What are knock-in options?

10.2. What are knockout options?

10.3. Why are barrier options path-dependent?

10.4. What are trig options?

10.5. What are vanilla barrier options?

10.6. How many types of basic vanilla barrier options are there? What are they?

10.7. Why do investors use barrier options?

10.8. Are barrier options generally more expensive or cheaper than their corresponding vanilla options?

10.9. Is there any relationship between a knock-in and its corresponding knockout?

10.10. Is there any relationship between the deltas of a knock-in barrier option, its corresponding knockout option, and their corresponding vanilla option?

10.11. Can the deltas of barrier call options be negative? Why?

10.12. Are the deltas of barrier options generally similar to or different from those of their corresponding vanilla options?

10.13. Can the deltas of barrier options be greater than 100%? Why?

10.14. Are the gammas of barrier options generally similar to or different from those of their corresponding vanilla options?

10.15. Is there any relationship between the gammas of a knock-in barrier option, its corresponding knockout option, and their corresponding vanilla option?

10.16. What is the "image source" of the origin for a barrier option?

10.17. What are deferrable and nondeferrable rebates?

10.18. Why should the nondeferrable rebates for knockout options be time-dependent?

10.19. How many types of rebates are there for vanilla barrier options?

10.20. What is an absorbing barrier and what is its use in pricing barrier options?

10.21. What is a reflecting barrier? What is the reflection principle?

10.22. What is the unrestricted distribution of the underlying asset log-return in a Black–Scholes environment? And what is its use in option pricing?

10.23. What is the first passage time? Why is it necessary in pricing vanilla barrier options?

10.24. What are the general characteristics of the curvature of the distribution of the first passage time for down and up barriers?

10.25. Why are the functional forms of the pricing formulas of a down-in call and an up-in put options the same?

10.26. Why do we have to distinguish the relative magnitudes of the strike price and the barrier in pricing all kinds of vanilla barrier options?

10.27. Is there any symmetry between a down-in call and an up-in call options? Why?

10.28. What is the relationship between the vegas of an in option and its corresponding out option?

10.29. Can the vegas of barrier options be negative? Why?

10.30. Can the vegas of barrier options be greater than those of their corresponding vanilla options?

Exercises

10.1. Find the prices of the down-in barrier call options without rebates with strike prices $K = \$100$ and \$90 to mature in four months, given the spot price $S = \$100$, the down barrier $L = \$95$, interest rate $r = 10\%$, the yield of the underlying asset $g = 3\%$, the volatility of the underlying asset 15%.

10.2. Find the prices of the down-in barrier call options to mature in one year in Exercise 10.1.

10.3. Find the present value of the rebate if the rebate is paid \$2 at maturity when the barrier is not touched within the lives of the call options in Example 10.1.

10.4. Find the prices of the down-in call options with the rebate \$2 and other parameters are the same as in Examples 10.1 and 10.3.

10.5. Find the deltas, vegas, gammas, and lamdas of the down-in call options in Exercise 10.1.

10.6. Find the deltas, vegas, gammas, and lamdas of the down-in call options in Exercise 10.2.

10.6.* Find the prices of the up-in barrier call options without rebates with strike prices $K = \$100$ and \$110 to mature in half a year, given the spot price $S = \$100$, the up barrier $U = \$105$, interest rate $r = 7\%$, the yield of the underlying asset $g = 3\%$, the volatility of the underlying asset 15%.

10.7. Find the present value of the rebate if the rebate is paid \$2 at maturity when the barrier is not touched within the lives of the call options in Example 10.6.

10.8. Find the prices of the up-in call options with the rebate \$2 and other parameters are the same as in Examples 10.6 and 10.7.

10.9. Find the prices of the corresponding down-in barrier put options in Exercise 10.1.

10.10. Find the prices of the corresponding down-in barrier put options in Exercise 10.3.

10.11. Find the prices of the corresponding up-in barrier put options in Exercise 10.6.

10.12. Find the present value of the down-out barrier option if the rebate is paid \$3 as soon as the barrier is touched with the rebate growth rate $\eta = 10\%$, given the same information as in Exercise 10.1.

10.13. Find the present value of the up-out barrier option if the rebate is paid \$3 as soon as the barrier is touched with the rebate growth rate 6%, given the same information as in Exercise 10.6.

10.14.* Show the identity

$$Ke^{-r\tau}f\left[d_{bs}\left(\frac{H^2}{S},\ K\right)\right] = (H^2/S)e^{-g\tau}f\left[d_{bs}\left(\frac{H^2}{S},\ K\right)\right]$$

10.15.* Show that the sum of the price of the down-in call option without rebate given in (10.36) and the price of the down-out call option without rebate given in (10.45) is the same as the corresponding vanilla option price for both $K > H$ and $K < H$.

10.16. Find the prices of the corresponding down-out call options in Exercise 10.1.

10.17. Find the deltas, vegas, gammas, and lamdas of the down-in call options in Exercise 10.2.

10.17.* Show that the identity (10.57) holds for an up-in call option and its corresponding out option.

10.18. Find the prices of the corresponding up-out call options in Exercise 10.6.

10.19.* Show that the vega factor of Corollary 10.1 in (10.67) is between 0 and 1.

10.20. Find the vega factor in Corollary 10.1.

10.21. Find the vegas of the down-in and down-out call options using the vega factor obtained in Exercise 10.20 and compare them with those obtained in Examples 10.22 and 10.23.

10.22.* Show that the identity (10.57) holds for an up-in put option and its corresponding out-put option.

10.23. Find the prices of the corresponding down-out barrier put options in Exercise 10.10.

10.24. Find the corresponding up-out barrier put options in Exercise 10.11.

10.25. Find the present value of the rebate \$2 if it is paid at maturity, given other information the same as in Exercise 10.1.

10.26. Find the prices of the up-out barrier put options without rebates with strike prices $K = \$103$ and $\$107$ to expire in half a year, given the spot price $S = \$100$, the up-barrier $U = \$105$, interest rate $r = 10\%$, the yield of the underlying asset $g = 2\%$, the volatility of the underlying asset 10%.

10.27. Find the prices of the corresponding up-in barrier put options in Exercise 10.26.

10.28. Find the prices of the up-out put options in Exercise 10.26 if the rebate is paid \$3 as soon as the barrier is touched and the growth of the rebate is 10%.

10.29. Find the prices of the up-out put options if the rebate is time-dependent with a growth rate 10% starting from zero.

10.30. Find the prices of the corresponding down-out put options in Exercise 10.26 if the down barrier is \$98 and other parameters are the same as in Exercise 10.26.

APPENDIX

A10.1. The Derivation of the Density Function If the Down-Barrier is Never Hit

If $y \leq 0$, the cumulative function of the log-return of the minimum value y_τ is given [see Harrison (1985), p. 13]

$$P(y_\tau \geq y) = N\left(\frac{-y + v\tau}{\sigma\sqrt{\tau}}\right) - e^{2yv/\sigma^2} N\left(\frac{y + v\tau}{\sigma\sqrt{\tau}}\right). \qquad \text{(A10.1)}$$

The probability that y_τ is no-smaller than $a = \ln(H/S)$ can be obtained directly from (A10.1)

$$P(y_\tau \geq a) = N\left(\frac{-a + v\tau}{\sigma\sqrt{\tau}}\right) - e^{2ya/\sigma^2} N\left(\frac{a + v\tau}{\sigma\sqrt{\tau}}\right). \qquad \text{(A10.2)}$$

The following identity is always true for any x and a:

$$P(X_\tau \leq x,\ y_\tau \geq a) + P(X_\tau > x,\ y_\tau \geq a) = P(y_\tau \geq a). \qquad \text{(A10.3)}$$

Since the right-hand side of (A10.3) is given in (A10.2) which is independent of x, the conditional density function of the log-return of underlying asset conditioned on that the down-barrier is not hit within the life of the option is the first-order derivative of $P(X_\tau \leq x, y_\tau \geq a)$ with respect to x, or the negative of the first-order derivative of $P(X_\tau > x, y_\tau \geq a)$ given in (10.25) with respect to x.

A10.2. The Expected Rebate For An Out-Barrier Option (EROB)

We may simply concentrate on the following integration

$$EROB = \int_0^{\tau} e^{-rT} e^{\eta T} h(T|a) dT = \int_0^{\tau} e^{-(r-\eta)T} h(T|a) dT \,. \tag{A10.4}$$

Making the substitution $z = 1/T$, the integration becomes

$$EROB = \theta a \int_{1/\tau}^{\infty} \frac{1}{\sigma\sqrt{2\pi z}} \exp\left[-\frac{(az-v)^2 + 2\sigma^2(n-\eta)}{2\sigma^2 z} \right] dz \,. \tag{A10.5}$$

Making the substitution $y = \sqrt{z}$, we can obtain the following from (A10.5) for $\eta \le r + v^2/(2\sigma^2)$

$$\begin{aligned} EROB =& 2\theta a e^{av/\sigma^2} \int_{1/\sqrt{\tau}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \times \exp\left\{ \frac{a^2y^2 + [\psi(r-\eta)]^2/y^2}{2\sigma^2 z} \right\} dy \\ =& 2\theta a e^{av/\sigma^2} \int_{1/\sqrt{\tau}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \\ & \times \exp\left\{ -\frac{[ay - \psi(r-\eta)/y]^2 + 2a\psi(r-\eta)}{2\sigma^2} \right\} dy \\ =& 2\theta a \left(\frac{H}{S} \right)^{[v-\psi(r\eta)]/\sigma^2} \int_{1/\sqrt{\tau}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \\ & \times \exp\left\{ -\frac{[ay - \psi(r-\eta)]/y^2}{2\sigma^2} \right\} dy \,. \end{aligned} \tag{A10.6}$$

Making the substitution $x = ay - \psi(r-\eta)/y$, we can solve for y in terms x (the negative root is not reasonable because y is always positive)

$$y = \frac{1}{2a}\left[x + \sqrt{x^2 + 4a\psi(r-\eta)} \right] \text{ and } dy = \frac{1}{2}\left[1 + \frac{x}{\sqrt{x^2 + 4a\psi(r-\eta)}} \right] . \tag{A10.7}$$

Substituting (A10.7) into (A10.6) yields the following, for $a > 0$

$$\begin{aligned} EROB =& \left(\frac{H}{S} \right)^{[v-\psi(r-\eta)]/\sigma^2} \left\{ 1 - N\left[\frac{a - \tau\psi(r-\eta)}{\sigma\sqrt{\tau}} \right] \right. \\ & \left. + \frac{1}{\sigma\sqrt{2\pi}} \int_{low}^{\infty} \frac{x}{\sqrt{x^2 + 4a\psi(r-\eta)}} \exp\left(-\frac{x^2}{2\sigma^2} \right) dx \right\} , \end{aligned} \tag{A10.8}$$

where $low = [-a\tau\psi(r-\eta)]/\sqrt{\tau}$.

Making the last substitution $s = \sqrt{x^2 + 4a\psi(r-\eta)}$, the second term of (A10.8) can be simplified to

$$(H/S)^{2\psi(r-\tau)/\sigma^2} N\{-[a + \tau\psi(r-\eta)]/(\sigma\sqrt{\tau})\},$$

thus completes the integration and the result given in (A10.8) is precisely the same as that given in the first brace in (10.49a). If $a < 0, x$ is always negative, the integration domain in (A10.8) is from $-\infty$ to $low = [a - \tau\psi(r-\eta)]/\sqrt{\tau}$ instead of being from low to $+\infty$. Thus, the directional binary operator θ appears in the formula because the integration domains are different.

When $\eta \geq r + v^2/(2\sigma^2)$, the rebate growth rate is significantly high, we cannot find any perfect square in real numbers to represent

$$a^2y^2 - [\psi_1(r-\eta)]^2/y^2, \ \psi_1(r-\eta) = \sqrt{-v^2 - 2(r-\tau)\sigma^2},$$

because the two terms are always opposite in signs. Yet, we can use the imaginary number $i^2 = -1$ to find a perfect square for

$$a^2y^2 - [\psi_1(r-\eta)]^2/y^2 = a^2y^2 + \psi'(r-\eta)]^2/y^2 .$$

With $i^2 = -1$, we can follow exactly the same procedure as above to obtain an expression for the present value of the rebate. Since the present value is an imaginary number, we have to choose the real part to represent the actual value.

A10.3. Pricing Formulas for Vanilla Barrier Options In Compact Forms

Making use of the direction binary operator θ (1 stands for a down barrier and -1 for an up barrier) in (10.30), the option binary operator ω (1 stands for a call option and -1 for a put option) in (10.31), and the symmetry between the two functions max(H, K) and min(H, K) [min(H, K) $=$ $-$max($-H, -K$)], we can simplify the pricing formulas of a down-in call option and an up in put option (DCUP), those of an up-in call option and a down-in put option (UICDNP), and those of a down-out call option and an up-out put option (DCUPOT) into one formula

$$DCUP(\omega,\ \theta) = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left(C_{bs}\left[\frac{H^2}{S},\ \max(\omega),\ \omega\right] \right.$$
$$\left. + \ \theta[\max(\omega) - K]e^{-r\tau} N\left\{\omega d\left[\frac{H^2}{S},\ \max(\omega)\right]\right\}\right)$$

$$+ \{C_{bs}(S,\ K,\ -\theta) - \{C_{bs}(S,\ H,\ -\theta)$$

$$+ \theta(H - K)e^{-r\tau}N[\theta d_{bs}(s,\ H)]\}B_{\theta H > \theta K} \qquad \text{(A10.9)}$$

$$UCDP(\omega,\ \theta) = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}}\left\{C_{bs}\left(\frac{H^2}{S},\ K,\ \theta\right) - C_{bs}\left(\frac{H^2}{S},\ H,\ \theta\right)\right.$$

$$\left. + \omega(H - K)e^{-r\tau}N[\theta d_{bs}(H,\ S)]\right\}B_{\omega K > \omega}$$

$$+ (C_{bs}[S,\ \max(\omega),\ -\theta] + \omega\max(\omega)$$

$$- Ke^{-r\tau}N\{\omega d_{bs}[S,\ \max(\omega)]\}), \qquad \text{(A10.10)}$$

$$DCUPOT[\omega,\ \theta,\ \max(\omega)] = C_{bs}[S,\ \max(\omega),\ \theta]$$

$$- \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} C_{bs}\left[\frac{H^2}{S},\ \max(\omega),\ \theta\right]$$

$$+ \theta[\max(\omega) - K]e^{-r\tau}\left(N\left\{\theta d_{bs}[s,\ \max(\omega)]\right\}\right.$$

$$\left. - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{\theta d_{bs}\left[\frac{H^2}{S},\ \max(\omega)\right]\right\}\right), \qquad \text{(A10.11)}$$

$$UCDPOT[\omega,\ \theta] = B\left(\left\{C_{bs}(S,\ K,\ \omega) - C_{bs}(S,\ H,\ \omega)\right.\right.$$

$$\left. + \theta(H - K)e^{-r\tau}N[\omega d_{bs}(S,\ H)]\right\}$$

$$- \frac{H}{S}^{\frac{2v}{\sigma^2}}\left\{\left(\frac{H^2}{S},\ K,\ \omega\right) - \left(\frac{H^2}{S},\ H,\ \omega\right)\right.$$

$$\left.\left. + \theta(H - K)e^{-r\tau}N[\omega d_{bs}(H,\ S)\right\}\right), \qquad \text{(A10.12)}$$

where $\max(\omega) = \omega(\max)(\omega H, \omega K), C_{bs}(X, Y, \omega)$ is the same as the formula in (10.31), standing for the extended Black-Scholes pricing formula with ω as the binary operator (1 for a call and -1 for a put option).

The present value of the rebate of a down-in and an up-in barrier options ($RBIN$), and that of an out option ($RBOT$) can be similarly expressed in one expression using (10.38), (10.41), and (10.49)

$$RBIN(\theta) = \mathrm{e}^{-r\tau} Rm(\tau) \left\{ N[\theta d_{bs}(S,\ H)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N[\theta d_{bs}(H,\ S)] \right\}, \tag{A10.13}$$

and

$$RBTOT(\eta,\theta) = R\left\{ \left(\frac{H}{S}\right)^{q_1(r-\eta)} N[\theta Q_1(r-\eta)] + \left(\frac{H}{S}\right)^{q_1(r-\eta)} N[\theta Q_1(r-\eta)] \right\}, \tag{A10.14}$$

when $\eta \le r + v^2/(2\sigma^2)$.

With the six pricing formulas in (A10.9)–(A10.14), we can obtain the pricing formulas for all eight types of vanilla barrier options as follows:

$$\begin{aligned}
\text{Down-In Call} &= DCUP(1,1) + RBIN(1)\,,\\
\text{Up-In Call} &= UCDP(1,-1) + RBIN(-1)\,,\\
\text{Down-In Put} &= UCDP(-1,1) + RBIN(1)\,,\\
\text{Up-In Put} &= DCUP(-1,-1) + RBIN(-1)\,,\\
\text{Down-Out Call} &= DCUPOT[1,1,\max(1)] + RBTOT(\eta,1)\,,\\
\text{Up-Out Call} &= UCDPOT[1,-1] + RBTOT(\eta,1)\,,\\
\text{Down-Out Put} &= UCDPOT[-1,1] + RBTOT(\eta,-1)\,,\\
\text{Up-Out Put} &= DCUPOT[-1,-1,\max(-1)] + RBTOT(\eta,-1)\,.
\end{aligned}$$

The above pricing formulas essentially state that there are only four independent formulas for the prices of the eight types of vanilla barrier options without rebates because there are four pairs of them, each pair sharing one pricing formula resulting from symmetry. With the independent formulas of the rebates for both in and out options, there are altogether six pricing formulas. If the nondeferrable rebate of an out option starts from zero, then we simply subtract the present value of the constant nondeferrable rebate from the present value of the time dependent rebate and other parameters remain unchanged.

Although these six formulas are convenient to use, there are still quite a few number of formulas. In the following chapter for earlier-ending barrier and outside barrier options, we will find unified formulas which include the four formulas for all eight types of vanilla barrier options as special cases.

Chapter 11

EXOTIC BARRIER OPTIONS

11.1. INTRODUCTION

Besides the vanilla barrier options studied in Chapter 10, there are many other kinds of barrier options: time-dependent barrier options, Asian barrier options or barrier options on the average of underlying asset prices, forward-start barrier options, window or limited-time barrier options, dual-barrier or corridor options, and so on. We may call them exotic barrier options compared to the vanilla barrier options studied before. These exotic barrier options expand the functions of the vanilla barrier options significantly and increase their flexibility. Although they are different from one another, they share one thing in common: low premiums. The premiums of exotic barrier options are, in general, even lower than those of vanilla barrier options. The low premiums of barrier options make them particularly attractive to hedgers and speculators. We will illustrate and price all these exotic barrier options in this chapter.

11.2. FLOATING BARRIER OPTIONS

In pricing a vanilla barrier option in Chapter 10, the barrier is assumed to be constant throughout the life of the option. However, it may change with time in many applications. In general, the barrier may either increase or decrease with time, or follow some other deterministic paths. For simplicity, we assume that the barrier changes exponentially:

$$H(T) = H\mathrm{e}^{\gamma T}, H > 0, \tag{11.1}$$

where H a constant coefficient of the floating barrier or the constant barrier, γ is the rate of change or the floating rate of the barrier, and $0 < T < \tau$ is any time within the life of the option.

Figure 11.1 shows the change of the floating barrier for $\gamma >, =,$ and < 0. We can observe from the figure that it is more difficult (resp. easier) for the

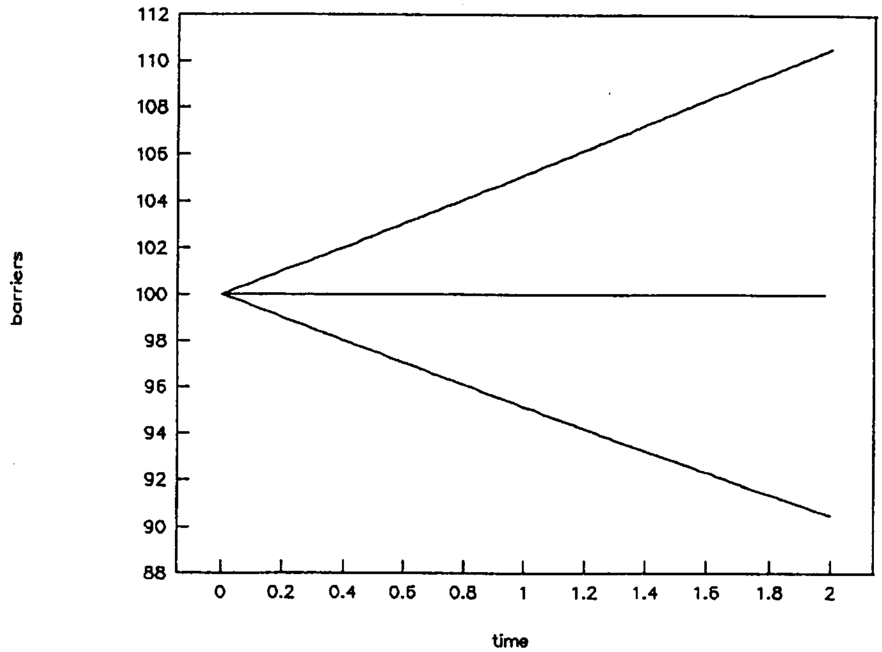

Fig. 11.1. Floating barriers with $f = 0.05$ and -0.05.

underlying asset price to reach an up-floating barrier from below if it increases (resp. decreases) with time, and that it is more difficult (resp. easier) for the underlying asset price to reach a down-floating barrier from above if it decreases (resp. increases) with the barrier. Thus, up-in barrier options without rebates should be cheaper (resp. more expensive) if their barriers increase (resp. decrease) with time, and down-in options should be cheaper (resp. more expensive) if their barriers decrease (resp. increase) with time.

The floating rate of the barrier given in (11.1) can be any real number. If it is positive (resp. negative), the barrier increases (resp. decreases) with time, and if it is zero, (11.1) degenerates to the constant barrier H as covered in Chapter 10. To price floating barrier options, we may first consider the following equivalence for any $0 < T < \tau$:

$$\text{the probability of } \left\{ S(T) = S\exp\left[\left(r-g-\frac{1}{2}\sigma^2\right)T+\sigma z(T)\right] \text{ hits } H(T) = He^{\gamma T} \right\}$$

$$= \text{the probability of } [S(T)e^{-\gamma T} \text{ hits } H] .$$

From above we may consider the underlying asset price $S(T)$ hitting the floating barrier $H(T)$ the same as $S(T)e^{-\gamma T}$ hitting the constant barrier H. Thus, we can simply use the formulas developed in Chapter 10 to price vanilla barrier options because the barrier is the same constant H for the new process $S(T)e^{-\gamma T}$. Let $S_\gamma(T) = S(T)e^{-\gamma T}$ stands for the new process with the constant barrier H. The new process $S_\gamma(T)$ can be expressed conveniently as

$$S_\gamma(T) = S(T)e^{-\gamma T} = S\exp\left\{\left[r - (g+f) - \frac{1}{2}\sigma^2\right]T + \sigma z(T)\right\}. \tag{11.2}$$

Comparing the expression of the new process in (11.2) with that of $S(T)$ in (5.3), we can find that the new process $S_\gamma(T)$ cna be obtained from the expression of $S(T)$ easily be changing the payout rate from g to $g_\gamma = g+\gamma$. Therefore, barrier options with floating barriers specified in (11.1) can be priced using the same formulas for vanilla barrier options developed in Chapter 10 by substituting r and υ with

$$g_\gamma = g + f \tag{11.3}$$

and

$$\upsilon_\gamma = r - g_\gamma - \sigma^2/2. \tag{11.4}$$

Example 11.1. Find the down-in call option price with strike price $K = \$98$ in Example 10.1 if the barrier increases exponentially by 4%

Substituting $\gamma = 0.04$ into (11.3) and (11.4) yields

$$g_\gamma = g + \gamma = 0.03 + 0.04 = 0.07\,,$$
$$\upsilon_\gamma = r - g_\gamma - \sigma/2 = 0.08 - 0.07 - 0.20^2/2 = -0.01.$$

Substituting $S = \$100, K = \$98$, $H = \$95$, $\sigma = 0.20, r = 0.08, g_\gamma = 0.07$, and $\tau = 0.50$ into (10.36) and replacing $g_\gamma = 0.07$ and $\upsilon_\gamma = -0.01$ with g and υ in (10.36) yields

$$H^2/S = 90.25, \max(H,K) = \max(95,98) = 98,$$

$$d_{bs}\left(\frac{H^2}{S}, K\right) = -0.4765,\ d_{1bs}\left(\frac{H^2}{S}, K\right) = d_{bs}\left(\frac{H^2}{S}, K\right) + \sigma\sqrt{\tau} = -0.331.$$

$$\text{The down-in call price } (\gamma = 0.04) = \left(\frac{95}{100}\right)^{-2.03/0.20^2} C_{bs}(90.25, 98)$$
$$= \$3.108$$

Example 11.2. Find the down-in call option price with strike price $K = \$98$ in Example 10.1 if the barrier decreases exponentially by 4%.

Substituting $\gamma = -0.04$ into (11.3) and (11.4) yields

$$r_\gamma = r - \gamma = 0.08 - (-0.04) = 0.12\,,$$
$$v_\gamma = r_\gamma - g - \sigma^2/2 = 0.12 - (0.03) - 0.20^2/2 = 0.07.$$

$$\text{The down-in call price } (\gamma = -0.04) = \left(\frac{95}{100}\right)^{2\times 0.07/0.20^2} C_{bs}(90.25, 98)$$
$$= \$2.4673.$$

Comparing the results in Examples 11.1 and 11.2 with Example 10.1, we can find that the down-in call option price with an increasing barrier is greater than that of the corresponding vanilla down-in call option with a constant barrier, and that the down-in call option price with a decreasing barrier is smaller than that of the vanilla down-in call option with a constant barrier. This is consistent with the intition that an increasing (resp. decreasing) down barrier is easier (resp. more difficult) to touch as discussed earlier in this section.

11.3. ASIAN BARRIER OPTIONS

Most barrier options are written on one underlying asset directly. The underlying asset price can be manipulated and fluctuates dramatically at any time before maturity. The payoffs of barrier options can in turn be manipulated. To overcome this, many institutions have traded barrier options based on averages of the underlying asset prices rather than on a single underlying asset prices. The reason of doing this is to avoid underlying manipulation, as in the case of Asian options. We may call these options Asian barrier options. We will price them in this section.

The averages in Asian barrier options can be either arithmetic or geometric as in Asian options in Chapters 5 and 6. Since the geometric average is lognormally distributed and the corresponding arithmetic average is not, we will start with geometric Asian barrier options and using their results, approximate arithmetic Asian barrier options using the approximation method developed in Chapter 6.

11.3.1. Flexible Geometric Asian Barrier Options

Flexible geometric Asian barrier options are barrier options written on flexible geometric averages of the underlying asset prices, rather than on the underlying asset prices directly as vanilla barrier options. As shown in Chapter 7, standard geometric averages with equal weights to all observations are special cases of flexible geometric averages. Thus, the flexible geometric Asian barrier options also include standard geometric Asian barrier options as special cases.

Suppose that the flexible geometric average is defined as in (7.4) and the underlying asset price is specified as in (5.3). Theorem 7.1 indicates that the natural logarithm of $FGA(n)/S$ or $\ln[FGA(n)/S]$ is normally distributed with mean $(r - g - \sigma^2/2)T^f_{\mu,n-j} + \ln B^f(j)$ and variance $\sigma^2 T^f_{n-j}$, where

$$B^f(0) = 1,\ B^f(j) = \prod_{i=1}^{j} \{S[\tau(n-i)h]/S\}^{w(i)}\,, \text{ for } 1 \geq j \geq n\,,$$

$$T^f_{\mu,n-j} = \sum_{i=j+1}^{n} w(i)[\tau - (n-i)h]\,,$$

$$T^f_{n-j} = \sum_{i=j+1}^{n} w^2(i)[\tau - (n-i)h] + 2\sum_{i=j+1}^{n-1}\sum_{k=i+1}^{n} w(i)w(k)[\tau - (n-k)h]\,,$$

and $B^f(j)$ is the weighted geometric average of the gross returns of those observations that have already passed; τ is the time to maturity of the option; n is the number of observations specified in the contract; h is the observation frequency or the time interval between two consecutive observations; j is the number of observations already passed; $w(i)$ is the weight assigned to the ith observation, and other parameters are the same as in Chapter 7.

We can rearrange the results indicated in Theorem 5.1 for later use.

Proposition 11.1. The flexible geometric average $FGA(\tau)$ can be expressed with the current underlying asset price $S(t) = S$, the effective payout rate of the flexible geometric average g_{fga}, and the effective volatility of the flexible geometric average σ_{fga}:

$$FGA(\tau) = S\exp\left[\left(r - g_{fga} - \frac{1}{2}\sigma^2_{fga}\right)\tau + \sigma_{fga}w(\tau)\right]\,, \tag{11.5}$$

where

$$\sigma_{fga} = \sigma\sqrt{\frac{T^f_{n-j}}{\tau}}\,,$$

$$g_{fga} = r - \frac{1}{2\tau}\sigma^2 T^f_{n-j} - \frac{1}{\tau}\left[vT^f_{\mu,n-j} + \ln B^f(j)\right]\,,$$

and all other parameters are the same as in Chapters 5 and 10.

Proof. Taking logarithm to (11.5) and comparing it with Theorem 7.1, we can solve for g_{ga} and σ_{ga} immediately. □

If we compare (11.5) with (5.3), we can find that (11.5) can be readily obtained from (5.3) by substituting g and σ with g_{fga} and σ_{fga}, respectively. Because of this similarity, we can price flexible geometric Asian barrier options using the same pricing formula for vanilla barrier options obtained in Chapter 10 by substituting g, σ, and v with g_{fga}, σ_{fga}, and $v_{fga} = r - g_{ga} - \sigma^2_{fga}/2$, respectively.

Example 11.3. Find the effective yield and volatility of the flexible geometric average given the volatility of the underlying asset $\sigma = 0.20$, the interest rate $r = 0.08$, the yield $g = 0.03$, and the time to maturity $\tau = 1$, if there are 12 observations in the geometric average, observation frequency is monthly, the averaging period has not started, the time to maturity of the option is one year, and the weight parameter $\alpha = 0.50$.

We can use the effective mean and variance time obtained in Example 7.3, $T^f_{\mu,n-j} = 0.629$ and $T^f_{n-j} = 0.476$. Substituting these two effective time values, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau = 1$, and $v = r - g - \sigma^2/2 = 0.03$ into (11.5) yields

$$\sigma_{fga} = \sigma\sqrt{T^f_{n-j}/\tau} = 0.20\sqrt{0.476} = 0.138$$

and

$$g_{fga} = r - \frac{1}{2\tau}\sigma^2 T^f_{n-j} - \frac{1}{\tau}[vT^f_{\mu,n-j} - \ln B^f(j)] = 0.05161.$$

Example 11.4. Find the price of the flexible geometric Asian down-in barrier call option given the spot price \$100, the strike price \$98, and other information the same as in Examples 11.3 and 10.1.

Substituting $S = \$100$, $K = \$98$, $H = \$95$, $\sigma_{fga} = 0.138, r = 0.08$, $g_{fga} = 0.05161$, and $\tau = 1$ into (10.36) yields

$$v = r - g_{fga} - \sigma^2_{fga}/2 = 0.0189,$$

$$H^2/S = 90.25, \max(H, K) = \max(95, 98) = 98,$$

$$d_{bs}\left(\frac{H^2}{S}, K\right) = \frac{\ln[(H^2/S)/K] + v\tau}{\sigma\sqrt{\tau}} = 0.4603,$$

$$d_{1bs}\left(\frac{H^2}{S}, K\right) = d_{bs}\left(\frac{H^2}{S}, K\right) + \sigma\sqrt{\tau} = -0.3223.$$

Since $K = 98 > 95 = H$, the call option price is $B_{H>K} = 0$, we can find the down-in call price from (10.36) as follows:

$$\text{The down-in call price } (K = 98) = \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S}, K\right)$$
$$= 0.95^{2\times 0.0189/0.20^2} \times 2.8338 = \$2.5595\,.$$

In general, it is not easy to obtain any simple comparative statics results as to how the geometric Asian barrier option price changes with various parameters of the averaging process, such as the number of observations and observation frequency. We can, however, analyze the limiting case with continuous observation for standard geometric Asian barrier options. Assume that averaging is just to start and the averaging period is the same as the time to maturity of the option. As Chapter 7 showed that the two effective time functions $T^{sa}_{\mu,n-j}$ and T^{sa}_{n-j} (the effective time functions for standard geometric averages with equal weights to all observations) approach $\tau/2$ and $\tau/3$, respectively, the standard deviation and the payout rate of the geometric average given in (11.5) approach:

$$\sigma_{gac} = \frac{\sigma}{\sqrt{3}}\,,$$
$$g_{gac} = r - \frac{1}{6}\sigma^2 - \frac{1}{2}v = \frac{1}{2}\left(r + g + \frac{1}{6}\sigma^2\right), \tag{11.6}$$

respectively, where σ_{gac} and g_{gac} represent the standard deviation and the payout rate of the standard geometric average when observation is continous.

It can be shown that the continuous geometric payout rate g_{gac} is greater than or equal to the underlying payout rate g if $r \geq g - \sigma^2/6$. If we assume $v = r - g - \sigma^2/2 \geq 0$ as in pricing knockout options, $r - g + \sigma^2/6 - 2\sigma^2/3 \geq 0$ implies $r - g + \sigma^2/6 \geq 2\sigma^2/3 > 0$ or $r \geq g - \sigma^2/6$. Thus, the difference between the interest rate and the continuous geometric payout rate is smaller than or equal to the same difference for the underlying asset $r - g$. Since the rho of a down-in barrier call option is always positive as indicated in (10.67), the smaller difference between the interest rate and the continuous geometric payout rate implies that the continuous geometric Asian down-in barrier call options are cheaper than their corresponding up-in barrier call options.

Example 11.5. Find the effective yield and volatility of the continuous standard geometric average given the volatility of the underlying asset $\sigma = 0.20$, the interest rate $r = 0.08$, the yield $g = 0.03$, and the time to maturity $\tau = 1$.

Substituting $r = 0.08$, $g = 0.03$, $\tau = 1$, and $\sigma = 0.20$ into (11.6) yields

$$\sigma_{gac} = \sigma/\sqrt{3} = 0.1155 = 11.55\%,$$

and

$$g_{gac} = (r + g + \sigma^2/6)/2 = 0.0583 = 5.83\%.$$

Example 11.6. Find the price of the continuous standard geometric Asian down-in call option with other information the same as in (11.4).

Substituting $S = \$100$, $K = \$98$, $H = \$95$, $\sigma_{gac} = 0.1155, r = 0.08$, $g_{gac} = 0.0583$, and $\tau = 1$ into (10.36) yields

$$\begin{aligned} v = r - g_{gac} - \sigma_{gac}^2/2 &= 0.08 - 0.0583 - 0.1155^2/2 = 0.0084, \\ H^2/S &= 95^2/100 = 90.25, \\ \max(H, K) &= \max(95, 98) = 98. \end{aligned}$$

Since $K = \$98 > \$95 = H$, the call option price is $B_{H>K} = 0$, we can find the down-in call price from (10.36) as follows

$$\begin{aligned} \text{The down-in call price } (K = 98) &= \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S}, K\right) \\ &= 0.95^{2\times0.0084/0.20^2} \times 1.925 = \$1.552. \end{aligned}$$

The call option price in Example 11.6 is much lower than that given in Example 11.4. The lower price of the down-in call option results from the lower volatility in the continuous average in Example 11.5 than that with monthly observation given in Example 11.4. In general, Asian barrier options have lower premiums because the effective volatilities of averages are lower than those of the underlying assets. The effective continuous geometric volatility is only about $57.7\% = 1/\sqrt{3}$ of the underlying volatility shown in (11.7), the volatility effect is significant here. Since the volatility of the continuous geometric average σ_{gac} is always much smaller than the underlying asset volatility σ, and the vega for both down-in and down-out barrier call options are positive as indicated in (10.66), the continuous geometric Asian up-in barrier call options are cheaper than their corresponding vanilla down-in barrier call options. These two effects together indicate that Asian up-in barrier call options are cheaper than their corresponding vanilla up-in barrier call options. These analyses also hold for up-in arithmetic Asian barrier call options, as we can see in the following section.

11.3.2. Flexible Arithmetic Asian Barrier Options

Standard arithmetic averages with equal weights to all observations are special cases of flexible arithmetic averages. Thus, flexible arithmetic Asian barrier options also include standard arithmetic Asian barrier options as special cases. Chapters 6 and 7 show that a flexible arithmetic average is not lognormally distributed in a Black-Scholes environment even when all the observations are lognormally distributed. Thus, exact closed-form solutions for flexible arithmetic Asian options are not possible in a Black-Scholes environment. The same is also true for arithmetic Asian barrier options. However, we can approximate their prices from their corresponding geometric options using the method developed in Chapter 7.

Suppose that the flexible arithmetic average is defined as in (7.1) and the underlying asset price is specified as in (5.3). Theorem 7.3 of Chapter 7 shows that a flexible arithmetic average (FAA) of the underlying asset prices defined in (7.5) can be approximated with its corresponding geometric average as follows:

$$FAA(\tau) \cong \kappa^f FGA(\tau)\,, \tag{11.7}$$

where

$$\kappa^f = 1 + \frac{1}{2}E(v^f) + \frac{1}{4}\left\{[E(v^f)]^2 + Var(v^f)\right\}\,, \tag{11.8}$$

$E(v^f)$ and $Var(v^f)$ are given in (7.10) and (7.11), respectively.

Using (11.7) and following the same procedure as in Proposition 11.1, we can obtain

$$FAA(\tau) = S\exp\left[\left(r - g_{faa} - \frac{1}{2}\sigma_{faa}^2\right)\tau + \sigma_{faa}s(\tau)\right]\,, \tag{11.9}$$

where

$$\sigma_{faa} = \sigma_{fga}, \tag{11.9a}$$

$$g_{faa} = g_{fga} - (\ln\kappa^f)/\tau, \tag{11.9b}$$

and other parameters are the same as in (11.6).

If we compare (11.9) with (5.3), we can find that (11.9) can be obtained from (5.3) by substituting g and σ with g_{faa} and σ_{fga} respectively. Because of this similarity, we can price arithmetic Asian barrier options using the same pricing formulas for vanilla barrier options obtained in Chapter 10 by substituting g, σ, abd v with g_{faa}, σ_{fga}, and $v_{faa} = r - g_{faa} - \sigma_{faa}^2/2$, respectively.

Example 11.7. Find the effective yield and volatility of the arithmetic average, given the time to maturity $\tau = 1$, the weight parameter $\alpha = 0.5$, interest rate 7%, yield on the gold is zero, volatility of gold return is 20%, the number of observation is 12, and observation frequency is 1/12.

Following the same steps as in Example 7.7, we can find the log-normalization factor $\kappa^f = 1.0376$. Substituting $\kappa^f = 1.0376$ into (11.9) using the results given in Example 11.3 yields

$$\sigma_{faa} = \sigma_{fga} = \sigma/\sqrt{3} = 0.1380 = 13.80\%,$$

and

$$g_{faa} = g_{fga} - (\ln k^f)/\tau = 0.0147 = 1.47\%.$$

Example 11.8. Find the price of the flexible arithmetic Asian down-in call option with information the same as in Examples 11.6 and 11.7.

Substituting $S = \$100$, $K = \$98$, $H = \$95$, $\sigma_{faa} = 0.138, r = 0.08$, $g_{faa} = 0.0147$, and $\tau = 1$ into (10.36) yields

$$v = r - g_{faa} - \sigma_{faa}^2/2 = 0.08 - 0.0147 - 0.138^2/2 = 0.04626,$$
$$H^2/S = 95^2/100 = 90.25,$$
$$\max(H, K) = \max(95, 98) = 98.$$

Since $K = \$98 > \$95 = H$, the call option price is $B_{H>K} = 0$, we can find the down-in call price from (10.36) as follows

$$\text{The down-in call price } (K = 98) = \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S}, K\right),$$
$$= 0.95^{2\times 0.04626/0.20^2} \times 1.925 = \$4.156.$$

We have considered Asian barrier options written on flexible geometric and arithmetic averages. In practice, there are many barrier options written on moving averages of some underlying asset prices. Yet, these topics are beyond the scope of this book.

11.4. FORWARD-START BARRIER OPTIONS

Barriers are immediately effective after the contracts are signed for most barrier options. Some users of barrier options however, may not want the barriers to be effective immediately but some time in the future within the life of the option. Forward-start barrier options can meet their needs. Since the barriers are not effective immediately as in vanilla barrier options studied in Chapter 10, the pricing formulas for vanilla barrier options in Chapter 10 are not appropriate for forward-start barrier options. In this section, we try to price forward-start barrier options within a Black-Scholes environment.

Assume that the barrier becomes effective at $t_1, t < t_1 < t^*$, where t and t^* represent current and maturity time, respectively. Figure 11.2 depicts the effective time of the barrier for a forward-start barrier option.

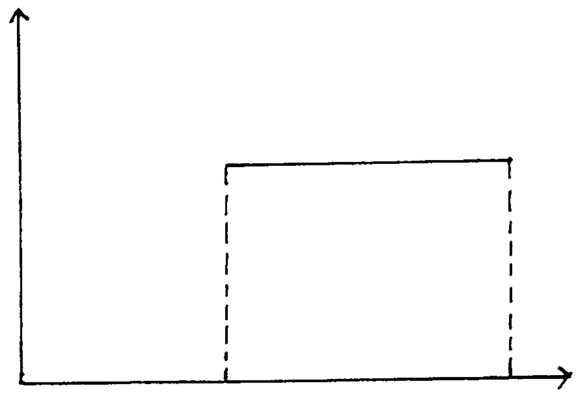

Fig. 11.2. Effective time for a forward-start barrier.

11.4.1. Pricing Forward-Start Barrier Options

For convenience, we repeat the price solution of the standard geometric Brownian motion with yield g:

$$S(t_1) = S\exp[\upsilon\tau_1 + \sigma w(\tau_1)], \tag{11.10}$$

where $\upsilon = r - g - \sigma^2/2, \tau_1 = t_1 - t, S = S(t)$ is the spot price, and $w(\tau_1)$ is a Gauss-Wiener process with time τ_1.

As the barrier starts to be effective at time t_1, the forward-start option can be either an "up" barrier option or a "down" option depending whether $S(t_1)$ is greater or smaller than the barrier H. If the underlying asset price at $S(t_1)$ is greater (resp. smaller) than the barrier H, the barrier option will be a "down" (resp. an "up") barrier option. Although we are not certain that $S(t_1)$ will be greater or smaller than H, we do know the distribution of $S(t_1)$. Let $z = \ln[S(t_1)/S], S(t_1) = Se^z$, we know from (10.9) that z is normally distributed with mean $\upsilon\tau_1$ and variance $\sigma^2\tau_1$. Using the distribution of $z = \ln[S(t_1)/S]$, we can find the expected value of the "up" and "down" portion of the forward-start barrier option.

Using the condensed pricing formula for vanilla barrier options given in Appendix of Chapter 10, we can express the expected value of an forward-start-in (FWIN) barrier option:

$$\begin{aligned} E(FWIN) &= \int_{-\infty}^{\ln(H/S)} UCDP[S(t_1), \tau-\tau_1, 1, -1] f(z)dz \\ &\quad + \int_{\ln(H/S)}^{\infty} DCUP[S(t_1), \tau-\tau_1, 1, 1] f(z)dz\,, \\ &= \int_{-\infty}^{\ln(H/S)} UCDP(Se^z, \tau-\tau_1, 1, -1) f(z)dz \\ &\quad + \int_{\ln(H/S)}^{\infty} DCUP(Se^z, \tau-\tau_1, 1, 1) f(z)dz\,, \end{aligned} \tag{11.11}$$

where $UCDP(S, \tau_2, \omega, \theta)$ is the price of an up-in call ($\theta = -1$ and $\omega = 1$) or down-in put ($\theta = 1$ and $\omega = -1$) and $DCUP(S, \tau_2, \omega, \theta)$ is the price of a down-in call ($\theta = 1$ and $\omega = 1$) or an up-in put ($\theta = -1$ and $\omega = -1$) with the spot price S and the time to maturity τ_2 given in Appendix of Chapter 10.

Making the substitution $u = (z - \upsilon\tau_1)/(\sigma\sqrt{\tau_1})$, we can express (11.11) alternatively

$$\begin{aligned} E(FWIN) &= \int_{-\infty}^{-d_{bs}} UCDP(Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, -1) f(u)du \\ &\quad \int_{-d_{bs}}^{\infty} DCUP(Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, 1) f(u)du\,, \end{aligned} \tag{11.12}$$

where $\qquad d_{bs} = d_{bs}(S, H, \tau_1)\dfrac{\ln(S/H) + \upsilon\tau_1}{\sigma\sqrt{\tau_1}}, \upsilon = r - g - \sigma^2/2,$

and $d_{bs}(S, H, \tau_1)$ is the same argument as in the Black-Scholes formula with the spot price S, the strike price H, and time τ_1, and $f(u)$ is the density function of a standard nowmal distribution.

Discounting the expected value of the forward-start-in option given in (11.12) yields the price of the forward-start-in option (FWIN) without rebates:

$$\begin{aligned} FWIN = e^{-\tau_1 r}\Bigg[&\int_{-\infty}^{-d_{bs}} UCDP\left(Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, -1\right) f(u)du \\ &+ \int_{-d_{bs}}^{\infty} DCUP\left(Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, 1\right) f(u)du\Bigg], \end{aligned} \tag{11.13}$$

where all parameters are the same as in (11.12).

Similarly, the price of the forward-start-out option (PEWOT) can be given

$$PEWOT = \mathrm{e}^{-r_1\tau}\left[\int_{-\infty}^{-d_{bs}} UCDPOT\left(S\mathrm{e}^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, -1\right) f(u)du \right.$$
$$\left. + \int_{-d_{bs}}^{\infty} DCUPOT\left(S\mathrm{e}^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}}, \tau-\tau_1, 1, 1\right) f(u)du\right], \tag{11.14}$$

where $UCDPOT(S, \tau_2, \omega, \theta)$ stands for the price of an up-out call ($\theta = -1$ and $\omega = 1$) or down-out put ($\theta = 1$ and $\omega = -1$) and $DCUP(S, \tau_2, \omega, \theta)$ stands for the price of a down-out call ($\theta = 1$ and $\omega = 1$) or an up-out put ($\theta = -1$ and $\omega = -1$) with the spot price S and the time to maturity τ_2 given in Appendix of Chapter 10, and all parameters are the same as in (11.12). The pricing formulas given in (11.13) and (11.14) are somewhat complicated because integrations are involved. The analysis of some special cases of these two formulas can help us understand them better. We can look at some limiting cases of the two formulas. If $\tau_1 \to 0$, the underlying asset price at t_1 will approach S, and both $UCDP$ and $DCUP$ are independent with u, and the forward-start-in option price given in (11.13) can be shown to be

$$PFWIN(\tau_1 \to 0) = N(-d_{bs})UCDP(S) + N(d_{bs})DCUP(S)\,. \tag{11.15}$$

If $S > H, d_{bs} \to +\infty$, thus $N(-d_{bs}) \to 0$ and $N(d_{bs}) \to 1$, the forward-start barrier option price becomes $PFWIN(\tau_1 \to 0) = DCUP(S)$ which is consistent with our intuition; if $S < H, d_{bs} \to -\infty$, thus $N(-d_{bs}) \to 1$ and $N(d_{bs}) \to 0$, the forward-start barrier option price becomes $FWDIN(\tau_1 \to 0) = UCDP(S)$; if $S = H, d_{bs} = 0, N(-d_{bs}) = N(d_{bs}) = 1/2$, and the forward price becomes $PFWIN(\tau_1 \to 0) = [UCDP(S) + DCUP(S)]/2 = [C_{sb} + +C_{sb}]/2 = C_{sb}$ because both "up-in" and "down-in" options become vanilla options when $S = H$. These three cases show that the forward-start-in option pricing formula given in (11.13) includes the standard "in" barrier option pricing formula as a special case. The same can be shown for "out" options when $S \neq H$. If $S = H, d_{bs} = 0, N(-d_{bs}) = N(d_{bs}) = 1/2$, and the forward-start-out barrier option price becomes $PFWOT(\tau_1 \to 0) = [UCDPOT(S) + DCUPOT(S)]/2 = [0+0]/2 = 0 =$ the price of a standard "out" barrier option because all "up-out", "down-out", the vanilla "out" barrier options become worthless when $S = H$.

We can also show that as $t_1 \to t^*$ or $\tau_1 \to \tau$, the pricing formula of forward-start barrier options will become that of vanilla options. As $t_1 \to t^*$ or $\tau_1 \to \tau$,

$$\begin{aligned} UIN\left(Se^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}}, \tau-\tau_1\right) &\to UIN\left(Se^{\upsilon\tau+u\sigma\sqrt{\tau}}, 0\right) \\ &= \max[Se^{\upsilon\tau+u\sigma\sqrt{\tau}} - K, 0], \end{aligned} \tag{11.16}$$

and

$$\begin{aligned} DIN\left(Se^{\upsilon\tau+u\sigma\sqrt{\tau}}, \tau-\tau_1\right) &\to DIN\left(Se^{\upsilon\tau+u\sigma\sqrt{\tau}}, 0\right) \\ &= \max[Se^{\upsilon\tau+u\sigma\sqrt{\tau}} - K, 0]. \end{aligned} \tag{11.17}$$

Substituting (11.16) and (11.17) into (11.13) yields

$$\begin{aligned} PFWIN &= e^{-\tau r}\left[\int_{-\infty}^{-\infty} \max\left(Se^{\upsilon\tau+u\sigma\sqrt{\tau}} - K\right) f(u)du\right] \\ &= C_{bs}(S, K), \end{aligned} \tag{11.18}$$

where $C_{bs}(S, K)$ is the extended Black-Scholes pricing formula in (10.31).

11.4.2. Present Values of Rebates

So far in this section, we have covered the prices of forward-start barrier options without rebates. The present value of the rebate of a forward-start barrier option (FWINRBT) can be similarly obtained using the present-value formula for the rebate of a vanilla in-barrier option given in (A10.12) of Chapter 10:

$$\begin{aligned} FWINRBT = e^{-\tau_1 r}\Bigg[&\int_{-\infty}^{-d_{bs}} RBIN\left(-1, Se^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}}, \tau-\tau_1\right) f(u)du \\ &+ \int_{-d_{bs}}^{\infty} RBIN\left(1, Se^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}}, \tau-\tau_1\right) f(u)du\Bigg], \end{aligned} \tag{11.19}$$

where $RBIN(\theta, S, \tau_2)$ stands for the present value of the rebate of an "in" option given in (A10.12) with the binary operator θ (1 for down and -1 for up), the spot price S, the time to maturity τ_2, and other parameters are the same as in (11.12) and (11.13).

11.4.3. Pricing Forward-Start Barrier Options in Closed-Form

The pricing formulas for forward-start barrier options without rebates given in (11.13) and (11.14) and the present value of the rebate for a forward-start barrier option in (11.19) are all in integrals. They can actually be expressed in closed-form in terms of cumulative functions of standard bivariate normal distributions. In order to avoid the long mathematical derivations, we will summarize the outline of the derivations in Appendix of this chapter. We use the forward-start-in barrier call option pricing formula to illustrate how the closed-form solution of a forward start barrier option pricing formula looks like. The pricing formulas for the other three kinds of forward-start barrier options (out-call, in-put, and out-put) can be similarly obtained. We can obtain the pricing formulas for a down-in and an up-in call options using (10.36) and (10.40):

$$DCUP(1,1) = \left[\frac{H}{S(\tau_1)}\right]^{2v/\sigma^2} C_{bs}\left[\frac{H^2}{S(\tau_1)}, K, \tau - \tau_1\right], \tag{11.20a}$$

and

$$UCDP(1,-1) = C_{bs}[S(\tau_1), K, \tau - \tau_1], \tag{11.20b}$$

for $K > H$.

$$\begin{aligned} DCUP(1,1) = \left[\frac{H}{S(\tau_1)}\right]^{2v/\sigma^2} & \left(C_{bs}\left[\frac{H^2}{S(\tau_1)}, H\right]\right. \\ & \left. + \ (H-K)e^{-r\tau}N\left\{d_{bs}\left[\frac{H^2}{S(\tau_1)}, H\right]\right\}\right) \\ & + \ P_{bs}[S(\tau_1), K] - P_{bs}[S(\tau_1), H] \\ & + \ H(H-K)e^{-r\tau}N[-d_{bs}[S(\tau_1), H], \end{aligned} \tag{11.21a}$$

and

$$\begin{aligned} UCDP(1,-1) = \left[\frac{H}{S(\tau_1)}\right]^{2v/\sigma^2} & \left(P_{bs}\left[\frac{H^2}{S(\tau_1}, K\right] - P_{bs}\left[\frac{H^2}{S(\tau_1)}, H\right]\right. \\ & \left. + (H-K)e^{-r\tau}N\left\{-d_{bs}[H, S(\tau_1)]\right\}\right) \\ & + \ C_{bs}[S(\tau_1), H] + (H-K)e^{-r\tau}N\{d_{bs}[S(\tau_1), H]\}, \end{aligned} \tag{11.21b}$$

for $K \leq H$.

Substituting (11.20) and (11.21) into (11.13) and using the results obtained in Appendix yields the pricing formula of a forward-start-in call option

(PFWIN):

$$
\begin{aligned}
&PFWIN(K > H) \\
&= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ \frac{H^2}{S} \mathrm{e}^{-g\tau} N_2 \left[d_{1bs}(S, H, \tau_1) \right.\right. \\
&\quad + \left(\frac{2v + \sigma^2}{\sigma}\right) \sqrt{\tau_1}, -d_{1bs}\left(\frac{H^2}{S}, K, \tau\right), -\left.\sqrt{\frac{\tau_1}{\tau}}\right] \\
&\quad - K\mathrm{e}^{-r\tau} N_2 \left.\left[d_{bs}(S, H, \tau_1) + \frac{2v}{\sigma}\sqrt{\tau 1}, -d_{bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}} \right] \right\} \\
&\quad + S\mathrm{e}^{-g\tau} N_2 \left[-d_{1bs}(S, H, \tau_1), d_{1bs}(S, K, \tau), \sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad - K\mathrm{e}^{-r\tau} N_2 \left[-d_{bs}(S, H, \tau_1), d_{bs}(S, K, \tau), \sqrt{\frac{\tau_1}{\tau}} \right],
\end{aligned} \tag{11.22a}
$$

and

$$
\begin{aligned}
&PFWIN(K \leq H) \\
&= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ \frac{H^2}{S} \mathrm{e}^{-g\tau} N[-d_{1bs}(H, S, \tau)] \right. \\
&\quad - K\mathrm{e}^{-r\tau} N_2 \left[d_{bs}(S, H, \tau_1) + \frac{2v}{\sigma}\sqrt{\tau_1}, -d_{bs}(H, S, \tau), \sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad - \frac{H^2}{S} \mathrm{e}^{-g\tau} N_2 \left[-d_{1bs}(S, H, \tau_1) + \left(\frac{2v + \sigma^2}{\sigma}\right) \sqrt{\tau_1}, -d_{1bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad + K\mathrm{e}^{-r\tau} N_2 \left[-d_{bs}(S, H, \tau_1) + \frac{2v}{\sigma}\sqrt{\tau_1}, -d_{bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad - H\mathrm{e}^{-r\tau} N_2 \left.\left[-d_{bs}(S, H, \tau_1) + \frac{2v}{\sigma}\sqrt{\tau_1}, -d_{bs}(H, S, \tau), -\sqrt{\frac{\tau_1}{\tau}} \right] \right\} \\
&\quad + S\mathrm{e}^{g\tau} N[d_{1bs}(S, H, \tau)] - S\mathrm{e}^{-g\tau} N_2 \left[d_{1bs}(S, H, \tau_1), d_{1bs}(S, K, \tau), \sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad + K\mathrm{e}^{-r\tau} N_2 \left[d_{bs}(S, H, \tau_1), d_{bs}(S, K, \tau), \sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad - K\mathrm{e}^{-r\tau} N_2 \left[-d_{bs}(S, H, \tau_1), d_{bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}} \right] \\
&\quad - H\mathrm{e}^{-r\tau} N_2 \left[d_{bs}(S, H, \tau_1), d_{bs}(S, H, \tau), \sqrt{\frac{\tau_1}{\tau}} \right].
\end{aligned} \tag{11.22b}
$$

Example 11.9. Find the price of the forward-start barrier call option to start in three months, given the spot price \$100, the strike price \$102, the barrier \$98, the time to maturity of the option half a year, the interest rate 10%, the pay out rate of the underlying asset 5%, and the volatility of the underlying asset 20%.

Substituting $S = \$100, K = \$102, H = \$98, \tau = 0.5, \tau_1 = 0.25, r = 0.10, g = 0.05, \sigma = 0.20, v = 0.10 - 0.05 - 0.20^2/2 = 0.03, H^2/S = 96.04$, $\tau_1/\tau = 0.50$, and $\sqrt{\tau_1}(2v + \sigma^2)/\sigma = 025$ into (11.22a) yields the price of the forward start option

$$
\begin{aligned}
&0.98^{2\times 0.03/0.20^2}\Big\{96.04e^{-0.03\times 0.5}N_2[d_{1bs}(100, 98, 0.25)\\
&\quad + 0.25, -d_{1bs}(96.04, 102, 0.5), \sqrt{0.5}\,] - 102e^{-0.10\times 0.5}\\
&\quad + N_2\left[d_{bs}(100, 98, 0.25) + \frac{2\times 0.03}{0.20}\sqrt{0.25}, -d_{bs}(100, 102, 0.50), -\sqrt{0.50}\right]\Big\}\\
&\quad + 100e^{0.03\times 0.5}N_2[-d_{1bs}(100, 98, 0.25), d_{1bs}(1001020.50), \sqrt{0.50}\,]\\
&\quad - 102e^{-0.10\times 0.5}N_2[-d_{bs}(100, 98, 0.25), d_{bs}(100, 102, 0.50), \sqrt{0.50}\,]\\
&= 0.9702\{96.04 \times 0.9851 \times N_2[0.627, 0.1785, -0.7071]\\
&\quad - 102 \times 0.9512 \times N_2[0.427, 0.034, -0.7071]\}\\
&\quad + 100 \times 0.9851 \times N_2[-0.377, 0.1075, 0.7071]\\
&\quad - 102 \times 0.9512 \times N_2[-0.277, -0.034, 0.7071]\\
&= \$5.579\,.
\end{aligned}
$$

We can readily find the prices of the corresponding vanilla barrier option and vanilla option to be \$3.747 and the \$5.74, respectively. It is obvious that the price of the forward start barrier option \$5.579 is greater than that of its corresponding vanilla barrier option \$3.747 and smaller than that of its corresponding vanilla option \$5.740. The fact that the price of a forward-start barrier option is between the prices of its corresponding vanilla barrier option and its corresponding vanilla option is very intuitive because the barrier is effective not as much as in the vanilla barrier option and more than in the vanilla option. Thus, we can expect the prices of forward-start barrier options more expensive with the forward-start time further in the future or closer to the time to maturity.

11.4.4. Rebates of Forward-Start Barrier Options

Using the same results given in Appendix of this chapter, we can obtain the present value of the rebate of a forward-start-in barrier option in (11.19) in closed-form:

$$
\begin{aligned}
&FWINRBT \\
&= e^{-r\tau} R_m(\tau)\Bigg(N_2\left[-d_{bs}(S,H,\tau_1), -d_{bs}(H,S,\tau), \sqrt{\frac{\tau_1}{\tau}}\right] \\
&\quad \times N_2\left[d_{bs}(S,H,\tau_1), d_{bs}(H,S,\tau), \sqrt{\frac{\tau_1}{\tau}}\right] \\
&\quad - \left(\frac{H}{S}\right)^{\frac{2\upsilon}{\sigma^2}}\Bigg\{N_2\left[-d_{bs}(S,H,\tau_1) + \frac{2\upsilon}{\sigma}\frac{\tau_1}{\sqrt{\tau}}, -d_{bs}(H,S,\tau) - \frac{2\upsilon}{\sigma}\frac{\tau_1}{\sqrt{\tau}}, -\sqrt{\frac{\tau_1}{\tau}}\right] \\
&\quad \times N_2\left[d_{bs}(S,H,\tau_1) - \frac{2\upsilon}{\sigma}\frac{\tau_1}{\sqrt{\tau}}, d_{bs}(H,S,\tau) + \frac{2\upsilon}{\sigma}\frac{\tau_1}{\sqrt{\tau}}, -\sqrt{\frac{\tau_1}{\tau}}\right]\Bigg\}\Bigg).
\end{aligned} \tag{11.23}
$$

Example 11.10. Find the present value of the forward-start-in barrier option if the rebate is paid \$1 and other information is the same as in Example 11.9.

Substituting $S = \$100, K = \$102, H = \$98, \tau = 0.5, \tau_1 = 0.25, r = 0.10, g = 0.05, \sigma = 0.15, \upsilon = 0.10-0.05-0.20^2/2 = 0.03, H^2/S = 96.04, \tau_1/\tau = 0.50$, and $2\upsilon\tau_1/(\sigma\sqrt{\tau} = 0.1061$into (11.23) yields

$$
\begin{aligned}
&FWINRBT \\
&= e^{0.10\times 0.5} \times 1(N_2[-d_{bs}(100,98,0.25), -d_{bs}(98,100,0.50), 0.7071] \\
&\quad + N_2\,[d_{bs}(100,98,0.25), d_{bs}(98,100,0.50), 0.7071] \\
&\quad - 0.98^{2\times 0.03/0.20^2}\{N_2[d_{bs}(100,98,0.25) + 0.1061, \\
&\quad - d_{bs}(98,100,0.50) - 0.1061, -0.7071] \\
&\quad \times N_2[d_{bs}(100,98,0.25) - 0.1061, d_{bs}(98,100,0.50 + 0.1061, -0.7071]\} \\
&= \$0.457.
\end{aligned}
$$

With the present value of the forward-start-in call option given in (11.22) and the present value of the rebate given in (11.23), we can express the price of a forward-start-in call option (PFWINC) as follows:

$$PFWINC = FWIN + FWINRBT. \tag{11.24}$$

Example 11.11. Find the price of the forward-start-in barrier option if the rebate is paid \$1 and other information is the same as in Example 11.9.

Substituting the price of the in forward-start-in barrier option in Example 11.9 and the present value of the rebate in Example 11.10 into (11.24) yields the price of forward-start-in barrier option with rebate:

$$PFWINC = FWIN + FWINRBT = 5.579 + 0.457 = \$6.036.$$

Using the same method to obtain the present value of a forward-start-in barrier option given in (11.23), we can obtain the present value of the rebate of an out forward-start barrier option (OTFWRBT) in closed-form:

$$\begin{aligned} OTFWRBT = R\Bigg\{ & \left(\frac{H}{S}\right)^{q_1} e^{-(r+\upsilon q_1-\sigma^2 q_1^2/2)\tau_1} \\ & \times \left[N_2\left(D_1, Q_1, \sqrt{\frac{\tau_1}{\tau}}\right) + N_2\left(-D_1, -Q_1, \sqrt{\frac{\tau_1}{\tau}}\right)\right] \\ & + \left(\frac{H}{S}\right)^{q_{-1}} e^{-(r+\upsilon q_{-1}-\sigma^2 q_{-1}^2/2)\tau_1}\left[N_2\left(D_{-1}, Q_{-1}, \sqrt{\frac{\tau_1}{\tau}}\right)\right. \\ & \left.+N_2\Big(-D_{-1}, -Q_{-1}, \sqrt{\frac{\tau_1}{\tau}}\Big)\right]\Bigg\}, \end{aligned} \tag{11.25}$$

where

$$\begin{aligned} Q_\nu &= d_{bs}(S, H, \tau) - \sigma q_\nu \sqrt{\tau}, \\ D_\nu &= d_{bs}(S, H, \tau_1) - \sigma q_\nu \sqrt{\tau_1}, \\ q_\nu &= \frac{\upsilon + \nu\psi(r-\eta)}{\sigma^2}, \nu = 1, \text{ or } -1, \end{aligned}$$

$d_{bs}(S, H, s)$ is the same argument in the extended Black-Scholes formula given in (10.31) with the spot and strike prices S and H, and the time to maturity s, respectively.

We can check that (11.25) degenerates to the present value of a vanilla out-barrier option given in (10.48a) when the forward-start time approaches zero. We leave this as an exercise.

Example 11.12. Find the present value of the rebate for an out forward-start barrier option if the rebate increases 8% from \$1 and other information is the same as in Example 11.9.

Substituting $S = \$100, H = \$98, \eta = 0.08, \tau = 0.50, \tau_1 = 0.25, r = 0.10, g = 0.05, \sigma = 0.20, v = 0.10 - 0.05 - 0.20^2/2 = 0.03, \tau_1/\tau = 0.50$, and $\psi = \sqrt{v^2 + 2\sigma^2(r-\eta)} = 0.05$ into (11.25) yields

$$q_1 = \frac{v + \psi(r-\eta)}{\sigma^2} = 2.00\,,$$

$$q_{-1} = \frac{v - \psi(r-\eta)}{\sigma^2} = -0.50\,,$$

$$D_1 = d_{bs}(S, H, \tau_1) - \sigma q_1\sqrt{\tau_1} = 0.2095 - 0.20 = 0.0095\,,$$

$$D_{-1} = d_{bs}(S, H, \tau_1) - \sigma q_1\sqrt{\tau_1} = 0.2095 + 0.05 = 0.2595\,,$$

$$Q_1 = d_{bs}(S, H, \tau) - \sigma q_1\sqrt{\tau} = 0.1482 - 0.2828 = -0.1347\,,$$

$$Q_{-1} = d_{bs}(S, H, \tau) - \sigma q_{-1}\sqrt{\tau} = 0.1482 + 0.0707 = 0.2189\,,$$

$$(r + vq_1 - \sigma^2 q_1^2/2)\tau_1 = 0.02\,,$$

$$(r + vq_{-1} - \sigma^2 q_{-1}^2/2)\tau_1 = 0.02\,,$$

$$\begin{aligned} FWORBT = {} & 0.98^2 e^{-0.02}[N_2(0.0095, -0.1347, 0.7071) \\ & + N_2(-0.0095, 0.1347, 0.7071)] \\ & + 0.98^{-0.5} e^{-0.02}[N_2(0.2595, 0.2189, 0.7071) \\ & + N_2(-0.2595, 0.2189, 0.7071)] \\ = {} & \$0.449\,. \end{aligned}$$

11.5. FORCED FORWARD-START BARRIER OPTIONS

As we discussed in the previous section, a forward-start barrier option can start either as a down-barrier or up-barrier option when the effective starting time becomes valid, depending on whether the underlying asset price at the starting time is below or above the prespecified barrier. In some applications, however, the buyers of forward-start barrier options would like to have a guaranteed down- or up-barrier option, depending on whether the spot price is above or below the given barrier as in vanilla barrier options.

For instance, if the spot price is below the given barrier and the barrier is to become valid in three months, a forced forward-start down-barrier option is a forward-start barrier option as studied in Section 11.4 if the underlying asset price turns out to be below the barrier, and it is knocked out with some rebates if the underlying asset price turns out to be above the barrier. Thus, a forced forward-start barrier option possesses some properties of both a vanilla barrier option and a forward-start barrier option.

A forced forward-start barrier option is also called a protected forward-start barrier option. Actually, a forced forward-start barrier option is simpler to price than a standard forward-start barrier option because we only need to find the present value of the up (resp. down) portion of a forced down (resp. up) forward-start barrier option for all possible prices of the underlying asset above (resp. below) the barrier. In other words, we only need to find the value of a forced forward-start barrier option using an appropriate integration domain (the up part above the barrier for a forced down forward-start barrier option, and the down part below the barrier for a forced up forward-start barrier option), and the rebate at the forward-start time can be obtained by integrating over the other part of the integration domain. If the underlying asset price at the forward-start time τ_1 is known, the present value of the rebate at τ_1 for a forced forward-start barrier option (FFWRBT) can be found:

$$FFWRBT = e^{-r\tau_1} R(\tau_1) N[-\theta d_{bs}(S, H, \tau_1)]\,, \tag{11.26}$$

where θ stands for the binary operator (1 for a down barrier and -1 for an up barrier), and $R(\tau_1)$ is the rebate at the forward-start time.

We can modify the pricing expression given in (11.14) to incorporate the "forced" characteristic:

$$\begin{aligned} &FPFWOT(\omega, \theta, S, H, \tau_1, \tau) \\ &\quad = e^{-r\tau_1} \left\{ \left(\frac{1+\theta}{2}\right) \int_{-d_{bs}}^{\infty} DCUP\left[Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau - \tau_1, 1, 1\right] f(u)du \right. \\ &\quad\quad + \left. \left(\frac{1-\theta}{2}\right) \int_{-\infty}^{-d_{bs}} \left[Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau - \tau_1, 1, -1\right] f(u)du \right\} \\ &\quad\quad + R(\tau_1) e^{-r\tau_1} N(-\theta d_{bs})\,, \end{aligned} \tag{11.27}$$

where $FPFWOT$ stands for the price of a forced forward-start-out barrier option, $R(\tau_1)$ is the rebate if the forward-start barrier is knocked out at the forward-start time and $N(-\theta d_{bs})$ is the probability that the forward-start barrier is knocked out at the forward-start time, and all other parameters are the same as in (11.14).

Example 11.13. Find the price of the corresponding forced down forward-start barrier option in Example 11.9.

Substituting $\theta = 1, S = \$100, K = \$102, H = \$98, \tau = 0.5, \tau_1 = 0.25, r = 0.10, g = 0.05, \sigma = 0.15, v = 0.10 - 0.05 - 0.20/2 = 0.03$, $H^2/S = 96.04, \tau_1/\tau = 0.25/0.50 = 0.50$, and $\sqrt{\tau_1}(2v + \sigma^2)/\sigma = 0.25$ into (11.27) using the results given in (11.22a) yields

$$\begin{aligned} &FPFWOT(1,1,100,98,0.25,0.50) \\ &= 0.98^{2\times 0.03/0.20^2}\Big\{96.04e^{-0.03\times 0.5}N_2[d_{1bs}(100,98,0.25) \\ &+ 0.25, -d_{1bs}(96.04,102,0.5), -\sqrt{0.5}] \\ &- 102e^{-0.10\times 0.5}N_2\Big[d_{bs}(100,98,0.25) \\ &+ \frac{2\times 0.03}{0.20}\sqrt{0.25}, -d_{bs}(100,102,0.50), -\sqrt{0.50}\Big]\Big\} = \$5.068 \end{aligned}$$

11.6. EARLY-ENDING BARRIER OPTIONS

We priced forward-start barrier options in the previous section. The complements of forward-start barrier options are early-ending barrier options. An early-ending barrier option is a barrier option with the barrier stopping to be effective before the expiration of the option. Early-ending barrier options provide users with more flexibility as they can capture the need that barriers are not expected to be effective all the time during the lives of the options. We will price early-ending barrier options in this section.

Suppose that the barrier stops to be effective at time $t_e, t < t_e < t^*$, where t and t^* represent the current time and the maturity time of the option, respectively. Figure 11.3 depicts the effectiveness of the barrier of an early-ending barrier option.

As in most other situations, we can price all kinds of options as long as we know the distribution of the underlying asset prices at maturity. We will first find a unified density function of the underlying asset price at maturity using the density function of the underlying asset price at the early-ending time. Using this unified density function, we can price all types of early-ending barrier options and the present values of their rebates.

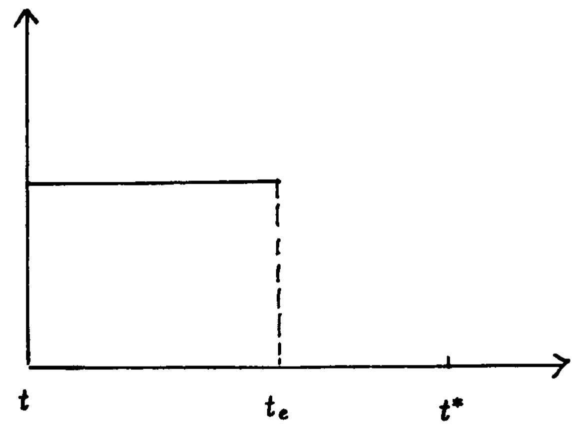

Fig. 11.3. Effective time for an early-ending barrier.

11.6.1. The Density Function at Maturity

We need to define another binary operator to represent whether a barrier option is an in-barrier option or an out-barrier option in order to price early-ending barrier options in concise forms. Let $\zeta = 1$ and -1 stand for an out- and in-barrier options, respectively, and let $ENDN(\zeta, \tau_e)$ stand for the density function of the underlying asset at the early-ending time. From our analysis about the restricted and unrestricted density functions in Section 10.4, we know the density function of the underlying asset price at the early-ending time for an out-barrier option is:

$$ENDN(x, 1, \tau_e) = f(x) - e^{2av/\sigma^2} f(x - 2a), \quad \text{if } \theta x > \theta a, \tag{11.28a}$$

$$= 0, \quad \text{if } \theta x \le \theta a, \tag{11.28b}$$

where $f(x)$ stands for the unrestricted density function of the underlying asset price given in (10.10), θ stands for the binary operator (1 for a down-barrier and -1 for an up-barrier), and $a = \ln(H/S)$ is the barrier.

Similarly, the density function of the underlying asset price at the early-ending time for an in-barrier option is given:

$$ENDN(x, -1, \tau_e) = e^{2av/\sigma^2} f(x - 2a), \quad \text{if } \theta x > \theta a, \tag{11.29a}$$

$$= f(x), \quad \text{if } \theta x \le \theta a, \tag{11.29b}$$

where all parameters are the same as in (11.28).

Given the underlying asset price at the time when the barrier becomes ineffective at $\tau_e, ENDN(x, \zeta, \tau_e)$ in (11.28) and (11.29), we can express the underlying asset price at the maturity of the option τ. If we know the underlying asset price at τ_e, the underlying asset price at maturity $S(\tau)$ can be expressed using the solution of the standard geometric Brownian motion starting from $S(\tau_e)$:

$$S(\tau) = S(\tau_e)\exp[\upsilon(\tau - \tau_e) + \sigma z_{\tau_e}(\tau)]\,, \tag{11.30}$$

where $\upsilon = r - g - \sigma^2/2$ and $\sigma_{\tau_e}(\tau)$ stands for Brownian motion starting from τ_e.

Let x and z stand for the log-returns of the underlying asset at the early-ending time and maturity, respectively, and let $y = \upsilon(\tau - \tau_e) + \sigma z(\tau - \tau_e)$. Using (11.30), we can obtain the relationship between x, y, and z :

$$z = x + y\,. \tag{11.31}$$

The density function of x is known as $ENDN(x, \zeta, \tau_e)$ given in (11.28) and (11.29), and the density function of y is a normal distribution starting from τ_e:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi(\tau - \tau_e)}}\exp\left\{-\frac{[y - \upsilon(\tau - \tau_e)]^2}{2\sigma^2(\tau - \tau_e)}\right\}. \tag{11.32}$$

Since we know the density functions of both x and y, we can obtain the density function of z. It is not simply the sum of the density functions of x and y, but has to be found using the standard method of random-variable transformation. We can obtain it as follows (see Appendix of this chapter for an outline of the proof):

$$\psi(z) = f(z)N\left[\theta\zeta\frac{\tau_e z - \tau a}{\sigma\sqrt{\tau\tau_e(\tau - \tau_e)}}\right] - \zeta e^{2a\upsilon/\sigma^2} f(z - 2a)N\left[\theta\frac{\tau_e z - a(\tau - 2\tau_e)}{\sigma\sqrt{\tau\tau_e(\tau - \tau_e)}}\right], \tag{11.33}$$

where θ and ζ stand for the two binary operators ($\theta = 1$ and -1 for down and up barriers, respectively, and $\zeta = 1$ and -1 for out- and in-barrier options, respectively) and $N(.)$ is the cumulative function of a standard normal distribution.

We can readily check that the density function of the log-return of the underlying asset price at maturity includes all the four density functions given in (11.28) and (11.29) for vanilla barrier options. When $\tau_e \to \tau$, the density function given in (11.33) becomes

$$\psi(z) = f(z)N\left(\theta\zeta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) - \zeta e^{2av/\sigma^2} f(z-2a)N\left(\theta\frac{\sigma\sqrt{\tau-\tau_e}}{z-a}\right), \quad (11.34)$$

where all parameters are the same as in (11.28) and (11.29).

For a down-out barrier option, $\theta = \zeta = 1$, the arguments in both the cumulative functions in (11.34) approach $+\infty$ as $\tau_e \to \tau$ if $z > a$, and they approach $-\infty$ as $z < a$. Thus,

$$N\left(\theta\zeta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 1 \text{ and } N\left(\theta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 1, \text{ if } z > a,$$

and

$$N\left(\theta\zeta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 0 \text{ and } N\left(\theta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 0, \text{ if } z < a.$$

Therefore the density function given in (11.34) becomes exactly the same as that of a down-out barrier option given in (11.28). Similarly, for a down-in barrier option, $\theta = 1$ and $\zeta = -1$, the arguments in the cumulative functions in (11.34) approach the following:

$$N\left(\theta\zeta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 0 \quad \text{and} \quad N\left(\theta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 1, \quad \text{if} \quad z > a,$$

and

$$N\left(\theta\zeta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to 1 \quad \text{and} \quad N\left(\theta\frac{z-a}{\sigma\sqrt{\tau-\tau_e}}\right) \to, 0, \quad \text{if} \quad z < a.$$

Thus the density function given in (11.34) becomes exactly the same as the density function of a down-in barrier option given in (11.29). We leave the confirmation that the density function given in (11.33) includes the density functions for up vanilla options given in (11.28) and (11.29) as exercises of this chapter.

As shown above, the density function given in (11.33) is a unified density function for all four types of barrier options, up-in, up-out, down-in, and down-out. We can find the density function for each of the four types of vanilla barrier options very conveniently by specifying the binary operators θ and ζ. Using this unified density function, we can find a unified pricing formula for all eight types of earlier ending-barrier options.

11.6.2. A Unified Pricing Formula for Early-Ending Barrier Options

With the density function given in (11.34), we can find the price of an option (END)

$$END(\omega,\theta,\zeta) = e^{-r\tau}\int \max[\omega S e^{z} - \omega K, 0]\psi(z)dz, \tag{11.35}$$

where ω a binary operator (1 for a call option and -1 for a put option), the integration is taken from $-\infty$ to $-d_{bs}(S,K)$ for a put option, and from $-d_{bs}(S,K)$ to ∞ for a call option.

Using the method to express forward-start barrier options in closed-form in terms of cumulative functions of bivariate normal distributions illustrated in Appendix of this chapter, we can find the closed-form solution for an early-ending barrier option as follows:

$$\begin{aligned}
END(\omega,\theta,\zeta) &= \omega S e^{-g\tau} N_2\left[\omega d_{1bs}(S,K,\tau), \theta\zeta d_{1bs}(S,H,\tau_e), \omega\theta\zeta\sqrt{\frac{\tau_e}{\tau}}\right] \\
&- \omega K e^{-r\tau} N_2\left[\omega d_{bs}(S,K,\tau), \theta\zeta d_{bs}(S,H,\tau_e), \omega\theta\zeta\sqrt{\frac{\tau_e}{\tau}}\right] \\
&- \left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{\omega\frac{H^2}{S}e^{-g\tau}N_2\left[\omega d_{1bs}\left(\frac{H^2}{S},K,\tau\right), \theta d_{1bs}\left(\frac{H^2}{S},H,\tau_e\right),\right.\right. \\
&\left.\omega\theta\sqrt{\frac{\tau_e}{\tau}}\right] - \omega K e^{-r\tau} N_2\left[\omega d_{bs}\left(\frac{H^2}{S},K,\tau\right),\right. \\
&\left.\left.\theta d_{bs}\left(\frac{H^2}{S},H,\tau_e\right), \omega\theta\sqrt{\frac{\tau_e}{\tau}}\right]\right\},
\end{aligned} \tag{11.36}$$

where ω, θ, ζ are the option, direction, and in/out binary operators, respectively.

The pricing formula given in (11.36) can be applied to all eight types of early-ending barrier options because we can simply choose the appropriate combination of the three binary operators. It should include vanilla options as a special case when the earlier ending time is zero and all eight types of vanilla barrier options as a special case when $\tau_e \to \tau$. To check its generality, let's examine a few special cases.

11.6.2A. Vanilla Options

Substituting $\tau_e = 0$ into (11.36), we can show that (11.36) will be simplified as follows[1]

$$\omega S e^{-g\tau} N[\omega d_{1bs}(S,K)] - \omega K e^{-r\tau} N[\omega d_{bs}(S,K)]\,,$$

which is exactly the same as the pricing formula for a vanilla option given in (10.31). Notice that the direction and the in/out binary operators are absent in the above formula. This is because the barrier does not exist, therefore the direction and the in/out binary operators are simply irrelevant.

11.6.2B. Down-Out Vanilla Barrier Call Options

For a down-out early-ending barrier call option, $\omega = \theta = \zeta = 1$. Substituting $(\omega,\theta,\zeta) = (1,1,1)$ and $\tau_e = \tau$ into (11.36) yields

$$\begin{aligned} END(1,1,1) &= S e^{-g\tau} N_2\left[d_{1bs}(S,K,\tau), d_{1bs}(S,H,\tau), 1\right] \\ &\quad - K e^{-r\tau} N_2\left[d_{bs}(S,K,\tau), d_{bs}(S,K,\tau), 1\right] - \left(\frac{H}{S}\right)^{2v/\sigma^2} \\ &\quad \times \left\{ \frac{H^2}{S} e^{-g\tau} N_2\left[d_{1bs}\left(\frac{H^2}{S},K,\tau\right), d_{1bs}\left(\frac{H^2}{S},H,\tau\right), 1\right] \right. \\ &\quad \left. - K e^{-r\tau} N_2\left[d_{bs}\left(\frac{H^2}{s},K,\tau\right), d_{bs}\left(\frac{H^2}{S},H,\tau\right), 1\right] \right\}. \end{aligned} \tag{11.37}$$

The correlation coefficients are all one in (11.37). We can simplify (11.37) using the mathematical identity (see Appendix for a proof)

$$N_2(A,B,1) = N[\min(A,B)], \tag{11.38}$$

for any real numbers A and B.

[1]Making use of the following identity

$$N_2(A,B,0) = N(A)N(B)\,,$$

for any real numbers A and B, where $N_2(.,.,.)$ and $N(.)$ are cumulative funtions of standard bivariate and univariate normal distributions, respectively. The proof is left as an exercise.

It can be shown that if $K > H$, then the following identities always hold

$$\begin{aligned} d_{1bs}(S,K,\tau) &< d_{1bs}(S,H,\tau)\,, \\ d_{bs}(S,K,\tau) &< d_{bs}(S,H,\tau)\,, \\ d_{1bs}\left(\frac{H^2}{S},K,\tau\right) &< d_{1bs}\left(\frac{H^2}{S},H,\tau\right)\,, \end{aligned}$$

and

$$d_{bs}\left(\frac{H^2}{S},K,\tau\right) < d_{bs}\left(\frac{H^2}{S},H,\tau\right)\,.$$

Thus, (11.37) can be simplified using (11.38) and the above inequalities

$$\begin{aligned} END(1,1,1) &= Se^{-g\tau}N\left[d_{1bs}(S,K,\tau)\right] - Ke^{-r\tau}N\left[d_{bs}(S,K,\tau)\right] \\ &\quad - \left(\frac{H}{S}\right)^{2\upsilon/\sigma^2}\left\{\frac{H^2}{S}e^{-g\tau}N\left[d_{1bs}\left(\frac{H^2}{S},K,\tau\right)\right]\right. \\ &\quad \left. - Ke^{-r\tau}N\left[d_{bs}\left(\frac{H^2}{S},K,\tau\right)\right]\right\} \\ &= C_{bs}(S,K) - \left(\frac{H}{S}\right)^{2\upsilon/\sigma^2} C_{bs}\left(\frac{H^2}{S},K\right)\,, \quad \text{for } K > H, \end{aligned} \tag{11.39}$$

which is exactly the same as the pricing formula for a down-out call option given in (10.44) for $K > H$.

It can also be shown that if $K \leq H$, then the following identities always hold

$$\begin{aligned} d_{1bs}(S < K,\tau) &\leq d_{1bs}(S,H,\tau)\,, \\ d_{bs}(S,K,\tau) &\leq d_{bs}(S,H,\tau)\,, \\ d_{1bs}\left(\frac{H^2}{S},K,\tau\right) &\leq d_{1bs}\left(\frac{H^2}{S},H,\tau\right)\,, \end{aligned}$$

and

$$d_{bs}\left(\frac{H^2}{S},K,\tau\right) \geq d_{bs}\left(\frac{H^2}{S},H,\tau\right)\,.$$

Thus, (11.37) can be simplified using (11.38) and the above inequalities

$$\begin{aligned}END(1,1,1) &= S\mathrm{e}^{-g\tau}N[d_{1bs}(S,H,\tau)] - K\mathrm{e}^{-r\tau}N[d_{bs}(S,H,\tau)]\\ &\quad - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{\frac{H^2}{S}\mathrm{e}^{-g\tau}N\left[d_{1bs}\left(\frac{H^2}{S},H,\tau\right)\right]\right.\\ &\quad \left. - K\mathrm{e}^{-r\tau}N\left[d_{bs}\left(\frac{H^2}{S},H,\tau\right)\right]\right\}\\ &= C_{bs}(S,H) - \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S},H\right)\\ &\quad + (H-K)\mathrm{e}^{-r\tau}\left\{N[d_{bs}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\right.\\ &\quad \left. N\left[d_{bs}\left(\frac{H^2}{S},H\right)\right]\right\}, \text{ for } K \le H, \end{aligned} \tag{11.40}$$

which is exactly the same as the pricing formula for a down-out call option given in (10.44) for $K \le H$.

11.6.2C. Down-In Vanilla Barrier Call Options

For a down-in early-ending barrier call option, $\omega = \theta = 1$ and $\zeta = -1$. Substituting $(\omega, \theta, \zeta) = (1, 1, -1)$ and $\tau_e = \tau$ into (11.36) yields

$$\begin{aligned}END(1,1,-1) &= S\mathrm{e}^{-g\tau}N_2[d_{1bs}(S,K,\tau), -d_{1bs}(S,H,\tau), -1]\\ &\quad - K\mathrm{e}^{-r\tau}N_2[d_{bs}(S,K,\tau), -d_{bs}(S,K,\tau), -1] + \left(\frac{H}{S}\right)^{2v/\sigma^2}\\ &\quad \times\left\{\frac{H^2}{S}\mathrm{e}^{-g\tau}N_2\left[d_{1bs}\left(\frac{H^2}{S}H,\tau\right) d_{1bs}\left(\frac{H^2}{S}H,\tau\right),1\right]\right.\\ &\quad \left. - K\mathrm{e}^{-r\tau}N_2\left[d_{bs}\left(\frac{H^2}{S},K,\tau\right), d_{bs}\left(\frac{H^2}{S},H,\tau\right),1\right]\right\}. \end{aligned} \tag{11.41}$$

Similar to the identity given in (11.38) with perfect positive correlations, we can use the following two identities for perfect negative correlations to simplify (11.41) (see Appendix for a proof)

$$N_2(A, B, -1) = 0, \text{ if } A + B \le 0 \tag{11.42}$$

and

$$N_2(A,B,-1) = N[\max(A,B)] - N[\min(A,B)], \quad \text{if} \quad A+B>0, \qquad (11.43)$$

for any real numbers A and B.

It can be shown that if $K \geq H$, then the followings always hold

$$d_{1bs}(S,K,\tau) - d_{1bs}(S,H,\tau) = \ln\left(\frac{H}{S}\right)/(\sigma\sqrt{\tau}) \leq 0\,,$$

$$d_{bs}(S,K,\tau) - d_{bs}(S,H,\tau) = \ln\left(\frac{H}{S}\right)/(\sigma\sqrt{\tau}) \leq 0\,,$$

$$d_{1bs}\left(\frac{H^2}{S},K,\tau\right) < d_{1bs}\left(\frac{H^2}{S},H,\tau\right), \text{ and } d_{bs}\left(\frac{H^2}{S},K,\tau\right) < d_{bs}\left(\frac{H^2}{S},H,\tau\right).$$

Thus, (11.41) can be simplified using (11.42) and (11.38) and the above inequalities

$$END(1,1,-1) = \left(\frac{H}{S}\right)^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S},H\right) \qquad (11.44)$$

which is exactly the pricing formula of a down-in call option given in (10.37) when $K \geq H$.

It can be shown that if $K < H$, then the followings always hold

$$d_{1bs}(S,K,\tau) - d_{1bs}(S,H,\tau) = \ln\left(\frac{H}{K}\right)/(\sigma\sqrt{\tau}) > 0\,,$$

$$d_{bs}(S,K,\tau) - d_{bs}(S,H,\tau) = \ln\left(\frac{H}{K}\right)/\sigma\sqrt{\tau}) > 0\,,$$

$$d_{1bs}\left(\frac{H^2}{S},K,\tau\right) > d_{1bs}\left(\frac{H^2}{S},H,\tau\right), \text{ and } d_{bs}\left(\frac{H^2}{S},K,\tau\right) > d_{bs}\left(\frac{H^2}{S},H,\tau\right).$$

Thus, (11.41) can be simplified using (11.38) and (11.43) and the above inequalities

$$\begin{aligned} END(1,1,-1) = {} & \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{C_{bs}\left(\frac{H^2}{S},H\right) + (H-K)e^{-r\tau}N[d_{bs}(H,S)]\right\} \\ & + \{C_{bs}(S,K) - C_{bs}(S,H) - (H-K)e^{-r\tau}N(S,H)]\}\,, \end{aligned} \qquad (11.45)$$

which is exactly the pricing formula for a down-in call option given in (10.37) when $K < H$.

11.6.2D. Up-In Vanilla Barrier Call Options

For an up-in early-ending barrier call option, $\omega = 1$ and $\theta = \zeta = -1$. Substituting $(\omega, \theta, \zeta) = (1, -1, -1)$ and $\tau_e = \tau$ into (11.36) yields

$$\begin{aligned} END(1,-1,-1) &= Se^{-g\tau} N_2[d_{1bs}(S,K,\tau), d_{1bs}(S,H,\tau), 1] \\ &\quad - Ke^{-r\tau} N_2[d_{bs}(S,K,\tau), d_{bs}(S,K,\tau), 1] + \left(\frac{H}{S}\right)^{2v/\sigma^2} \\ &\quad \times \left\{ \frac{H^2}{S} e^{-g\tau} N_2\left[d_{1bs}\left(\frac{H^2}{S}, K, \tau\right), -d_{1bs}\left(\frac{H^2}{S}, H, \tau\right), -1\right] \right. \\ &\quad \left. - Ke^{-r\tau} N_2\left[d_{bs}\left(\frac{H^2}{S}, K, \tau\right), -\ d_{bs}\left(\frac{H^2}{S}, H, \tau\right), -1\right] \right\}. \end{aligned} \tag{11.46}$$

Using the identity given in (11.38) with perfect positive correlations and the identities given in (11.42) and (11.43) with perfect negative correlations, we can readily simplify (11.46) to[2]

$$END(1,-1,-1) = C_{bs}(S,K), \text{for} K > H, \tag{11.47a}$$

and

$$\begin{aligned} END(1,-1,-1) &= C_{bs}(S,K) - C_{bs}(S,H) \\ &\quad + (H-K)e^{-r\tau} N[d_{bs}(S,H)] \\ &\quad + \left(\frac{H}{S}\right)^{2v/\sigma^2} \left\{ C_{bs}\left(\frac{H^2}{S}, K\right) - C_{bs}\left(\frac{H^2}{S}, H\right) \right. \\ &\quad \left. - (H-K)e^{-r\tau} N[d_{bs}(H,S)] \right\}, \text{ for } K \le H. \end{aligned} \tag{11.47b}$$

The pricing formulas given in (11.47a) and (11.47b) are exactly the same formulas for an up-in call option given in (10.41).

The other five special cases of (11.36) for other combinations of (ω, θ, ζ) with $\tau_e \to \tau$ can be checked similarly using the identities given in (11.38), (11.42), and (11.43). We will leave them as exercises.

[2]When $K \le H$, we can easily show that

$$d_{1bs}(H^2/S, K) > -d_{1bs}(H^2/S, H) \text{ and } d_{bs}(H^2/S, K) > -d_{bs}(H^2/S, H),$$

Because $H > S$ for any up-barrier options.

Example 11.14. Find the prices of the early-ending down-in and -out barrier call options with the early-ending time three months before the maturity of the options and other parameters remain the same as in Examples 10.1 and 10.11.

Substituting $\omega = \theta = 1$, $\zeta = -1$, $S = \$100$, $K = \$98$, $H = \$95$, $\sigma = 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, $\tau_e = 0.25$, $v = r - g - \sigma^2/2 = 0.03$, $H^2/S = 90.25$ into (11.36) yields the price of the early-ending down-in call option

$$\begin{aligned} &END(1,1,-1) \\ &= 100\mathrm{e}^{-0.03\times 0.5} N_2\left[d_{1bs}(100,98,0.50), -d_{1bs}(100,95,0.25), -\sqrt{0.5}\right] \\ &\quad -98\mathrm{e}^{-0.08\times 0.5} N_2\left[d_{bs}(100,98,0.50), -d_{bs}(100,95,0.25), -\sqrt{0.50}\right] \\ &\quad + 0.95^{2\times 0.03/0.20^2}\{90.25\mathrm{e}^{-0.03\times 0.5} \\ &\quad \times N_2\left[d_{1bs}(90.25,98,0.50), d_{1bs}(90.25,95,0.25), \sqrt{0.50}\right] \\ &\quad - 98\mathrm{e}^{-0.08\times 0.5} N_2\left[d_{bs}(90.25,98,0.50), d_{bs}(90.25,95,0.25), \sqrt{0.50}\right] \\ &= \$2.547, \end{aligned}$$

and substituting $\omega = \theta = \zeta = 1$, $S = \$100$, $K = \$98$, $H = \$95$, $\sigma 0.20$, $r = 0.08$, $g = 0.03$, $\tau = 0.50$, $\tau_e = 0.25$, $v = r - g - \sigma^2/2 = 0.03$, $H^2/S = 90.25$ into (11.36) yields the price of the down-out barrier call option

$$\begin{aligned} END(1,1,1) &= 100\mathrm{e}^{-0.03\times 0.5} N_2\left[d_{1bs}(100,98,0.50), d_{1bs}(100,95,0.25), \sqrt{0.5}\right] \\ &\quad - 98\mathrm{e}^{-0.08\times 0.5} N_2\left[d_{bs}(100,98,0.50), d_{bs}(100,95,0.25), \sqrt{0.50}\right] \\ &\quad - 0.95^{2\times 0.03/0.20^2}\left\{90.25\mathrm{e}^{-0.03\times 0.5}\right. \\ &\quad \times N_2\left[90.25,98,0.50), d_{1bs}(90.25,95,0.25), \sqrt{0.50}\right] \\ &\quad - 98\mathrm{e}^{-0.08\times 0.5} N_2\,[d_{bs}(90.25,98,0.50), \\ &\quad \left.+d_{bs}(90.25,95,0.25), \sqrt{0.50}]\right\} \\ &= \$5.329\,. \end{aligned}$$

Comparing the price of the down-out barrier option in Example 11.14 and that in Example 10.11, we can find that the price of the down-out call option with early-ending features in Example 11.14 is higher than that without early-ending features in Example 10.11. Similarly, we can find that the price of the down-in barrier call option with early-ending features in Example 11.14 is lower than that without early-ending features in Example 10.1. These relative values of the barrier options result from the fact that the probability that the barrier is touched is less with shorter early-ending time, thus the down-in barrier options become less valuable and the corresponding down-out barrier options become more valuable when the early-ending time gets shorter.

Following similar procedures, we can find the prices of the down-in and down-out barrier options with various earlier ending time from zero to the time to maturity, given all other parameters the same as in Example 11.4. Figure 11.4 depicts the values of the down-in and down-out barrier options with various early-ending time. The straight line above represents the sums of the prices of the down-in barrier call options and their corresponding down-out call options. The curves in the middle and below represent the prices of down-out and down-in barrier options. From Figure 11.4 we can readily observe that whereas the down-in barrier option becomes worthless, the down-out barrier option price becomes the same as that of the

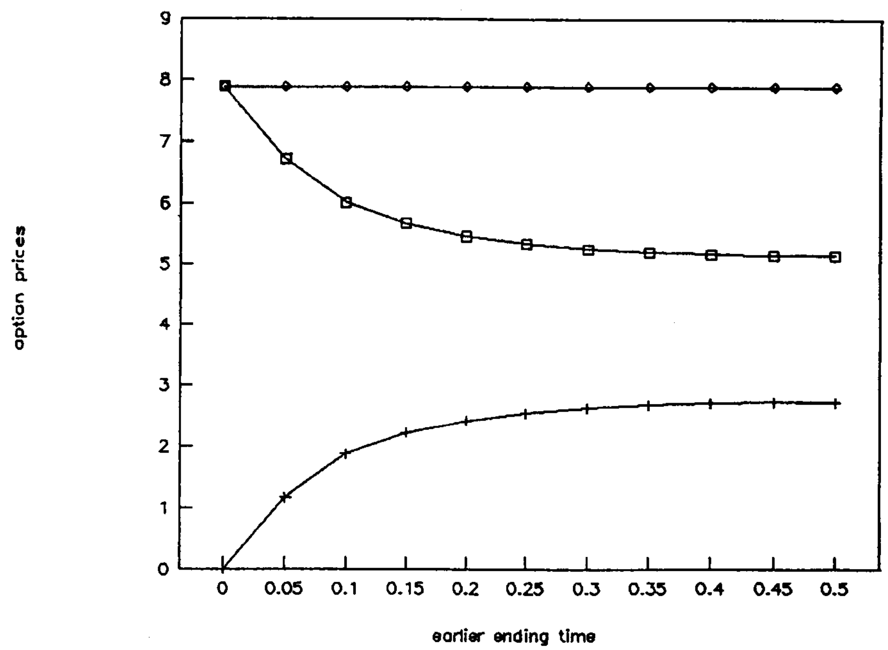

Fig. 11.4. Early-ending barrier options prices for down-in and down-out options.

corresponding vanilla option when the earlier ending time is zero, and both the down-in and down-out barrier options become vanilla barrier options when the early-ending time is the same as the time to maturity of the option.

11.6.3. Present Values of Rebates for Early-Ending Barrier Options

Following the same procedures to derive (10.48) using the density function of the first passage time given in (10.30), we can find the present value of the time-dependent rebate for an early-ending out-barrier option (EDRBTOT)

$$EDRBTOT(\eta,\theta,\tau_e) = R\left\{\left(\frac{H}{S}\right)^{q_1(r-\eta)} N[\theta Q_1(r-\eta)] + \left(\frac{H}{S}\right)^{q_{-1}(r-\eta)} N[\theta Q_{-1}(r-\eta)]\right\}, \qquad (11.48a)$$

if the rebate growth rate $\eta \leq r + \upsilon^2/(2\sigma^2)$, where

$$\psi(s) = \sqrt{\upsilon^2 + 2s\sigma^2},$$

$$Q\nu(s) = \frac{\ln(H/S) + \nu\tau_e\psi(s)}{\sigma\sqrt{\tau_e}}, \nu = 1\,\text{or}\, -1,$$

$$q_\nu(s) = \frac{\upsilon + \nu\psi(s)}{\sigma^2},$$

and

$$RBTOT(\eta,\theta,\tau_e) = \text{RRe}\left\{\left(\frac{H}{S}\right)^{q_1'(r-\eta)} N[\theta Q_1'(r-\eta)] + \left(\frac{H}{S}\right)^{q_1'(r-\eta)} N[\theta Q_{-1}'(r-\eta)]\right\}, \qquad (11.48b)$$

if the rebate growth rate $\gamma > r + \upsilon^2/(2\sigma^2)$, where

$$\psi'(s) = i\sqrt{-\upsilon^2 - 2s\sigma^2},$$

$$Q_\nu'(s) = \frac{\ln(H/S) + \nu\tau_e\psi'(s)}{\sigma\sqrt{\tau_e}}, \nu = 1\,\text{or}\, -1,$$

$$q'\nu(s) = \frac{\upsilon + \nu\psi'(s)}{\sigma^2},$$

$i = \sqrt{-1}$ is the standard unit of an imaginary number, $\text{Re}(\alpha + \beta i) = \alpha$ is the function to choose the real part of an imaginary number $\alpha + \beta i$ (both α and β are real numbers), θ is the same binary operator as in (11.36) (1 to represent a down and -1 to represent an up barrier, respectively).

We can show that the present-value formula given in (11.48) approaches that of vanilla knockout options given in (10.48) when the early-ending time approaches the time to maturity of the option.

Example 11.15. Find the present value of the rebate for the down-out barrier option in Example 10.15 if the early-ending time is two months before the maturity of the option and other parameters remain the same as in Example 10.15.

Substituting $S = K = \$100, r = 0.08, g = 0.03, \sigma = 0.20, \tau = 0.50, \tau_e = (6-2)/12 = 1/3$, and $v = r - g - \sigma^2/2 = 0.03$ into (11.48a) yields

$$\psi(r) = \sqrt{0.03^2 + 2 \times 0.08 \times 0.20^2} = 0.0854,$$

$$q_1(r) = (0.03 + 0.0854)/0.20^2 = 2.886,$$

$$q_{-1}(r) = (0.03 + 0.0854)/0.20^2 = -1.385,$$

$$Q_1 = \ln(95/100) + (1/3)0.0854]/(0.20\sqrt{1/3}] = -0.1974,$$

$$Q_{-1} = \ln(95/100) - (1/3)0.0854]/(0.20\sqrt{1/3}] = -0.6907,$$

$$EDRBTOT(0,1,1/3) = 0.95^{2.886} n(-0.1974) + 0.95^{-1.385} N(-0.6907)$$

$$= \$0.6267.$$

Comparing the present values of the rebates of the down-out barrier options in Examples 10.15 and 11.15, we can find that it is larger in Example 10.15 than in Example 11.15. This is because the barrier is effective throughout the life of the barrier option in Example 10.15 and it is effective only for the first four months in Example 11.15, thus the probability the barrier is touched is larger in Example 10.15 than in Example 11.15.

11.7. WINDOW BARRIER OPTIONS

Window barrier options are also called limited-time barrier options. As the word window implies, a window-barrier option is a barrier option in which the barrier is effective only within one or more than one prespecified periods during the life of the option. Actually, the forward-start barrier options studied in Section 11.4 are one kind of special window barrier options because the barrier is effective from the forward-start time to the expiration time of the option, and the early-ending barrier options studied in Section 11.6 are another kind of special window barrier options because the barrier is effective from the beginning of the option to the ending time of the barrier. Whereas the barrier of a forward-start barrier option is effective in the second part of the life of the barrier option, it is effective in the first part of the life of an early-ending barrier option.

A general window barrier option may include a few windows. A window barrier option with one period within which the barrier is effective may start some time in the future and have the effective ending time before the expiration of the option. It is rather complicated to express the price of a general window barrier option with more than one windows. To illustrate how general window options can be priced, we simply consider the simplest window barrier option with one window. As a matter of fact, a general window barrier option with one window can be readily decomposed into a forward-start barrier option and an early-ending barrier option. Figure 11.5 depicts the effectiveness of the barrier in a window barrier option with one window.

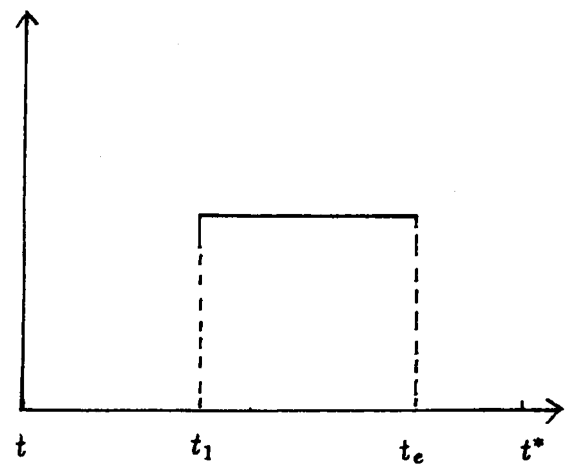

Fig. 11.5. Effective time for a window barrier.

Using the method to price forward-start barrier options in Section 11.4, we can find window barrier option prices readily using the unified pricing formula for early-ending barrier options given in (11.36). Assume for the moment that the underlying asset price at the forward-start time is known as in pricing forward-start barrier options in Section 11.4. The assumed underlying asset price at the window starting time can be considered as the current spot in the unified pricing formula for an early-ending barrier option in (11.36). Let $END(\omega, \theta, \zeta, S, \tau_e, \tau)$ stand for the price of the early-ending barrier option in (11.36) with the spot price S, the ending time τ_e, and the time to maturity τ. Using the pricing formula of early-ending barrier options given in (11.36), we can find the price of a window barrier option (PWD) with one window in terms of integrations

$$\begin{aligned} &FWD(\omega, \zeta, S) \\ &= e^{-\tau_1 r}\left\{ \int_{-\infty}^{-d_{bs}} END\left[\omega, -1, \zeta, Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau_e, \tau - \tau_1\right] f(u)du \right. \\ &\quad \left. + \int_{-d_{bs}}^{\infty} END\left[\omega, 1, \zeta, Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau_e, \tau - \tau_1\right] f(u)du \right\}, \qquad (11.49) \end{aligned}$$

where all parameters are the same as in (11.36).

The pricing formula given in (11.49) is for any window barrier options with one window without any restriction on the ways the options are knocked in or out at the window starting time τ_1. There are forced window barrier options similar to forced or protected forward-start barrier options discussed in Section 11.5. If the barrier is a down (resp. up) barrier compared to the current spot price of the underlying asset for a forced window barrier option and the barrier option is expected to be down (resp. up) after the window starting time, the window barrier option is knocked out and a rebate is paid if the underlying asset price is actually below (resp. above) the barrier at the window starting time. We can modify the pricing expression in (11.49) to incorporate the "forced" characteristic:

$$\begin{aligned} &FFWD(\omega, \zeta, S, H, \tau_1, \tau_e) \\ &= e^{-\tau_1 r}\left\{ \left(\frac{1-\theta}{2}\right) \int_{-\infty}^{-d_{bs}} END\left[\omega, -1, \zeta, Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau_e, \tau - \tau_1\right] f(u)du \right. \\ &\quad \left. + \left(\frac{1+\theta}{2}\right) \int_{-d_{bs}}^{\infty} END\left[\omega, 1, \zeta, Se^{\upsilon\tau_1 + u\sigma\sqrt{\tau_1}}, \tau_e, \tau - \tau_1\right] f(u)du \right\} \\ &\quad + R(\tau_1)e^{-r\tau_1} N(-\theta d_{bs}), \qquad (11.50) \end{aligned}$$

where *FFWD* stands for the price of a forced window barrier option with one window, $R(\tau_1)$ is the rebate if the forced window barrier option is knocked out at the window starting time and $N(-\theta d_{bs})$ is the probability that the forced window barrier is knocked out at the window starting time, and all other parameters are the same as in (11.49).

The term outside the brace in (11.50) represents the present value of the rebate if the window barrier option is knocked out at the window starting time given in (11.26). We can readily check that (11.50) includes both a forced down ($\theta = 1$) and up ($\theta = -1$) window barrier options as special cases. For a forced down (resp. up) window barrier option, $\theta = 1$ (resp. $\theta = -1$), the first (resp. second) term in the brace in (11.50) disappears and only the second (resp. first) term remains with the coefficient one.

11.7.1. Rebates for Window Barrier Options

Using the same method to obtain the present value of a window barrier option in the previous section, we can obtain the present value of the rebate of a window barrier option (WDRBT) in closed-form:

$$
\begin{aligned}
WDRBT = R\Bigg\{ & \left(\frac{H}{S}\right)^{q_1} e^{-\left(r+vq_1-\sigma^2\frac{q_1^2}{2}\right)\tau_1}\left[N_2\left(D_1, Q_1(\tau_e), -\sqrt{\frac{\tau_1}{\tau_e}}\right)\right. \\
& \left. + N_2\left(-D_1, -Q_1(\tau_e), -\sqrt{\frac{\tau_1}{\tau_e}}\right)\right] \\
& + \left(\frac{H}{S}\right)^{q_{-1}} e^{-(r+vq_{-1}-q_{-1}^2\sigma^2/2)\tau_1}\left[N_2\left(D_{-1}, +Q_{-1}(\tau_e), -\sqrt{\frac{\tau_1}{\tau_e}}\right)\right. \\
& \left. + N_2\left(-D_{-1}, -Q_{-1}(\tau_e), -\sqrt{\frac{\tau_1}{\tau_e}}\right)\right]\Bigg\}, \qquad (11.51)
\end{aligned}
$$

where

$$
D_v = d_{bs}(S, H, \tau_1) - \sigma q_v\sqrt{\tau_1}, \quad Q_v(\tau_e) = \frac{\ln\frac{H}{S} + v\tau_e\psi}{\sigma\sqrt{\tau_e}},
$$

q_v is the same as in (11.48), $d_{bs}(S, H, s)$ is the same as in (10.31) with the spot and strike prices S and H, and time to maturity s, respectively.

We can check that (11.51) degenerates to the present value of an early-ending out-barrier option when the starting time is zero, and it degenerates to that of a vanilla out-barrier option when the forward-start time is zero and the early-ending time is the same as the time to maturity of the option.

Example 11.16. Find the present value of the rebate of the down-out barrier option in Examples 10.15 and 11.15 if the forward-start time is three months, and the ending time is two months before the maturity of the option.

Substituting $S = K = \$100$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau_1 = 0.25$, $\tau = 0.50$, $\tau_e = 0.50 - 2/12 = 0.3333$, and $v = r - g - \sigma^2/2 = 0.03$ into

(15.5) and using the values of $\psi(r) = 0.0854,\ q_1(r) = 2.886,\ q_{-1}(r) = -1.385$ in Example 11.15 yields

$$
\begin{aligned}
(r + vq_{-1} - \sigma^2 q_{-1}^2/2)\tau_1 &= 0.0208 \\
d_{bs}(S, H, \tau_1) &= 0.5879\,, \\
d_{bs}(S, H, \tau_1 + \tau_e) &= 0.4504\,, \\
D_1 &= 0.5879 - 0.20 \times 2.886\sqrt{0.25} = 0.2993\,, \\
D_{-1} &= 0.5879 - 0.20 \times (-1.385)\sqrt{0.25} = 0.7264\,, \\
Q_1 &= 0.4504 - 0.20 \times 2.886\sqrt{0.5833} = 0.0096\,, \\
Q_{-1} &= 0.4504 - 0.20 \times (-1.385)\sqrt{0.5833} = 0.6620\,, \\
\rho &= -\sqrt{0.25/0.5833} \\
&= -0.6547\,,
\end{aligned}
$$

and therefore the present value of the rebate is:

$$
\begin{aligned}
P1AD = 0.95^{2.886} e^{-0.0433} &\Big[N_2(0.2993, -0.0096, -0.6547) \\
&+ N_2(-0.2993, 0.0096, -0.6547) \Big] \\
&+ 0.95^{-1.385} e^{-0.0208} \Big[N_2(0.7264, -0.6620, -0.6547) \\
&+ N_2(-0.7264, 0.6620, -0.6547) \Big] \\
= \$0.4567\,.
\end{aligned}
$$

Comparing the results in Examples 10.15, 11.15, and 11.16, we can find that the present value of the down-out barrier option is the lowest with both forward-start and early-ending features in Example 11.16, and the highest with neither forward-start nor early-ending features in Example 10.15. This is because the length of the effective time of the barrier is different in these examples, a longer effective time makes it more likely for the barrier to be touched and in turn the option will have a higher present value of the rebate.

With the pricing formula given in (11.49) for a window out-barrier option and the present value of its rebate given in (11.51), we can express the price of a window barrier option with one window and the time-dependent rebate

(P1WD) as follows:

$$P1WD = FWD(\omega, \zeta, S, H, \tau_1, \tau_e, \tau) + WDRBT. \qquad (11.52)$$

Similarly, we can express the price of a forced window barrier option with one window and the time-dependent rebate (FP1WD) using (11.50) and (11.51):

$$FP1WD = FFWD(\omega, \zeta, S, H, \tau_1, \tau_e, \tau) + WDRBT. \qquad (11.53)$$

11.8. OUTSIDE BARRIER OPTIONS

All the barrier options covered in this chapter so far are barrier options with only one underlying instrument. For them, the asset involved in the payoffs of the options, the payoff asset, is always the same as the measurement asset or the measurement instrument. However, this is not necessarily the case in many applications. If the payoff asset is different from the measurement asset, there are two assets involved in such barrier options; and these barrier options existed in the market. In late 1993, Bankers Trust structured a call option on a basket of Belgian stocks which would be knocked out if the Belgian franc appreciated by more than 3.5%.

Heynen and Kat (1944a) extended vanilla barrier options which involve only one single underlying asset to barrier options with two assets. Heynen and Kat called barrier options with only one underlying asset inside barrier options because whether the barrier are touched only concerns the single asset. They called barrier options with two assets outside barrier options. The idea of separating the measurement asset from the payoff asset was also illustrated in the analysis of correlation digital options by Zhang (1995d). As shown by Heynen and Kat, the correlation coefficient between the returns of the two assets involved plays an important role in determining the prices of outside barrier options. Because of this, we may call outside barrier options correlation barrier options compared to correlation digital options in Zhang (1995d) because in both kinds of options, one asset serves merely as a measurement asset and the other is the payoff asset.

Whether an outside-barrier option is knocked in or out depends on whether the price of the measurement asset touches a prespecified barrier within the life of the option. As we need a conditional density function for the log-return of the underlying asset in pricing vanilla barrier options, we also need a conditional density function for the log-return of the payoff asset conditioned on whether the price of the measurement asset touches the barrier within the life of the option. Although Heynen and Kat (1944a) illustrated the idea of outside barrier options clearly, they used two density functions for the underlying asset of the options, one for down and the other for up. Following the method in expressing the prices of all eight types of early-ending barrier options in one unified pricing formula in Section 11.6, we will find an unified formula for all eight types of outside barrier options in this section.

11.8.1. The Unified Marginal Density Function

Suppose that the measurement asset price follows a similar standard geometric Brownian motion as given in (3.1) in a risk-neutral world:

$$dM = (r - g_2)M dt + \sigma_2 M dz_m(t)\,, \tag{11.54}$$

where g_2 and σ_2 are the payout rate and volatility of the measurement asset, respectively, and $z_m(t)$ is a standard Gauss-Wiener process.

Using the standard method, we can solve the stochastic equation in (11.54) given the spot price of the measurement asset M:

$$M(\tau) = M \exp[\upsilon_2 \tau + \sigma_2 z_m(\tau)]\,, \tag{11.55}$$

where $\upsilon_2 = r - g_2 - \sigma_2^2/2, \tau = t^* - t$, and t and t^* are the current time and the time to maturity of the option, respectively.

Let y stand for the log-return of the measurement asset. $ENDN(y, \zeta, \tau)$ stands for the density function of the measurement asset at the maturity time. The functional form of the density function $ENDN(y, \zeta, \tau)$ for y at τ is exactly the same as the density functions given in (11.28) and (11.29) for out- and in-barrier options, respectively:

$$ENDN(y, 1, \tau) = f(y) - e^{2a\upsilon/\sigma^2} f(y - 2a), \quad \text{if} \quad \theta y > \theta a, \tag{11.56a}$$

$$= 0, \quad \text{if} \quad \theta y \leq \theta a, \tag{11.56b}$$

where $f(y)$ stands for the unrestricted density function of the measurement asset price given in (10.10) (we need to use the volatility and drift parameters σ_2 and υ_2 for y), θ stands for the binary operator (1 for a down-barrier and -1 for an up-barrier), $a = ln(H/M)$ is the barrier; and

$$ENDN(y, -1, \tau) = e^{2a\upsilon/\sigma^2} f(y - 2a), \quad \text{if} \quad \theta y > \theta a, \tag{11.57a}$$

$$= f(y), \quad \text{if} \quad \theta y \leq \theta a, \tag{11.57b}$$

where all parameters are the same as in (11.56).

Let x stand for the log-return of the payment asset or the underlying asset as in previous sections of this chapter. The log-returns of the payment and measurement assets are assumed to be correlated with a constant correlation coefficient ρ. Since both x and y are normally distributed and are correlated with the correlation coefficient ρ, x and y are joint normally distributed with the correlation coefficient ρ. As we know the marginal density function of y given in (11.56) and (11.57), we can find the density function of x using the barrier condition given in (11.56) and (11.57) following a similar procedure in obtaining (11.33) in Section 11.6 for early-ending barrier options (see Appendix for an outline of the proof):

$$\xi(x) = f(u)N\left\{\theta\zeta\frac{[d_{bs}(M,H,\sigma_2)+\rho u]}{\sqrt{1-\rho^2}}\right\} - \zeta e^{2av_2/\sigma_2^2} f\left(u - \frac{2\rho a}{\sigma_2\sqrt{\tau}}\right)$$
$$\times N\left(\theta\left\{\frac{[d_{bs}(M,H,\sigma_2)+\rho u]}{\sqrt{1-\rho^2}} + \frac{2a}{\sigma_2\sqrt{\tau}}\sqrt{1-\rho^2}\right\}\right), \qquad (11.58)$$

where $u = (x - v_\tau)/(\sigma\sqrt{\tau})$ is the standardized normal variable for x, and

$$d_{bs}(A,B,\sigma_2) = [\ln(A/B) + v_2\tau]/(\sigma_2\sqrt{\tau})$$

is the same argument as in the extended Black-Scholes formula with the spot and strike prices A and B and volatility σ_2, respectively, and all other parameters are the same as in (11.33).

We can check that the density function of the log-return of the payment asset price at maturity given in (11.58) includes all the four density functions given in (11.28) and (11.29) for vanilla barrier options. When $\rho \to 1$, $\sigma_2 \to \sigma$, and $g_2 \to g$, the density function given in (11.58) becomes

$$\xi(x) = f(x)N\left\{\theta\zeta\frac{[d_{bs}(M,H,\sigma_2)+u]}{\sqrt{1-\rho^2}}\right\}$$
$$- \zeta e^{2av_2/\sigma_2^2}(x-2a)N\left\{\theta\frac{[d_{bs}(M,H,\sigma_2)+u]}{\sqrt{1-\rho^2}}\right\}, \qquad (11.59)$$

where all parameters are the same as in (11.58).

For a down-out barrier option, $\theta = \zeta = 1$, the arguments in both the cumulative functions in (11.59) approach $+\infty$ as $\rho \to 1, \sigma_2 \to \sigma$, and $g_2 \to g$, because $u > -d_{bs}(M,H,\sigma_2)$ or $d_{bs}(M,H,\sigma_2) + u > 0$, and

$$N\left\{\frac{[d_{bs}(M,H,\sigma_2)+u]}{\sqrt{1-\rho^2}}\right\} \to N(\infty) = 1\,.$$

Therefore, the density function given in (11.59) is simplified to

$$\xi(x) = f(x) - \zeta e^{2av/\sigma^2} f(x - 2a) \ ,$$

which is precisely the same for vanilla out-barrier options given in (11.28).

For a down-in barrier option, $\theta = 1, \zeta = -1$. Substituting $\theta = 1$ and $\zeta = -1$ into (11.59), we can find

$$N\left\{-\frac{[d_{bs}(M, H, \sigma_2) + u}{\sqrt{1-\rho^2}}\right\} \to 0 \text{ and } N\left\{\frac{[d_{bs}(M, H, \sigma_2) + u]}{\sqrt{1-\rho^2}}\right\} \to 1, \text{ if } u > -d(M, H, \sigma_2)$$

and

$$N\left\{-\frac{[d_{bs}(M, H, \sigma_2) + u]}{\sqrt{1-\rho^2}}\right\} \to N\left\{\frac{[d_{bs}(M, H, \sigma_2) + u]}{\sqrt{1-\rho^2}}\right\} \to 0, \text{ if } u < -d(M, H, \sigma_2) \, .$$

Thus the density function in (11.59) becomes exactly the same as that for the down-in barrier option in (11.57). We leave the confirmation that the density function given in (11.59) includes the density functions of the other six types of vanilla barrier options as special cases as exercises at the end of this chapter.

As shown above, the density function given in (11.58) is an unified density function including all four types (up-in, up-out, down-in, and down-out) of outside barrier options. We can simply find the density function for each of the four types very conveniently by specifying the binary operators θ and ζ. Using this unified density function, we can find an unified pricing formula for all eight types of outside barrier options.

11.8.2. The Unified Pricing Formula for Outside Barrier Options

With the unified density function given in (11.58), we can find the price of an outside-barrier option (OTSD)

$$OTSD(\omega, \theta, \zeta, \rho) = e^{-r\tau} \int \max[\omega S e^x - \omega K, 0] \xi(x) dx, \qquad (11.60)$$

where ω is a binary operator (1 for a call option and -1 for a put option), the integration is taken from $-\infty$ to $-d_{bs}(S, K, \sigma)$ for a put option and from $-d_{bs}(S, K, \sigma)$ to ∞ for a call option. Using the method to express the prices of forward start barrier options and early-ending barrier options in closed-form in terms of cumulative functions of bivariate normal distributions, we can find the closed-form solution for an outside barrier option as follows:

$$\begin{aligned} OTSD(\omega,\theta,\zeta,\rho) &= \omega S e^{-g\tau} N_2\left[\omega d_{1bs}(S,K,\sigma),\theta\zeta d_{12},\omega\theta\zeta\rho\right] \\ &\quad - \omega K e^{-r\tau} N_2\left[\omega d_{bs}(S,K,\sigma),\theta\zeta d_{bs}(M,H,\sigma_2),\omega\theta\zeta\rho\right] \\ &\quad - \zeta\left(\frac{H}{M}\right)^{2v_2/\sigma_2^2}\left\{\omega S\left(\frac{H}{M}\right)^{2\rho\sigma/\sigma_2} e^{-g\tau} N_2\left[\omega d_{21},\theta d_{22},\omega\theta\rho\right]\right. \\ &\quad \left. - \omega K e^{-r\tau} N_2\left[\omega(d_{21}-\sigma\sqrt{\tau},)\theta(d_{22}-\rho\sigma\sqrt{\tau}),\omega\theta\rho\right]\right\}, \end{aligned} \tag{11.61}$$

where

$$d_{12} = d_{bs}(M,H,\sigma_2) + \rho\sigma\sqrt{\tau},$$

$$d_{21} = d_{1bs}(S,K,\tau) + \frac{2a\rho}{\sigma_2\sqrt{\tau}},$$

$$d_{22} = d_{12} + \frac{2a}{\sigma_2\sqrt{\tau}},$$

and ω,θ,ζ are the option, direction, and in/out binary operators, respectively, as in (11.39).

The pricing formula in (11.61) can be applied to all eight types of barrier options because we can simply choose the appropriate combination of the three binary operators. It should include the pricing formula of vanilla options and all eight types of vanilla barrier options as special cases. We will illustrate a few special cases of the unified pricing formula.

11.8.3. Vanilla Options As Special Cases

Like the unified pricing formula for early-ending barrier options in (11.36), the one given in (11.61) also includes the pricing formula of vanilla options as a special case. However, the conditions under which the two unified formulas degenerate to the pricing formula of vanilla options are different. Substituting $\rho = 0, \sigma_2 = 0, r = g_2$ into (11.61) using the identity given in Footnote 1 of this chapter yields precisely the same pricing formula of vanilla options in (10.31) after simplifications.

11.8.4. Vanilla Barrier Options As Special Cases

For a down-out vanilla call barrier option, we can simply set $\omega = \theta = \zeta = 1$. Substituting $(\omega,\theta,\zeta) = (1,1,1)$, $\rho \to 1$, $\sigma_2 \to \sigma$, $g_2 \to g$, $M \to S$

into (11.61) yields exactly the same result as in (11.37) which is precisely the pricing formula of a down-out vanilla barrier call option given in (10.44) as shown in Section 11.6. For a down-in vanilla barrier call option, we can set $\omega = \theta = 1$ and $\zeta = -1$. Substituting $(\omega, \theta, \zeta) = (1, 1, -1), \rho \to 1, \sigma_2 \to \sigma, g_2 \to g$, $M \to S$ into (11.61) yields exactly the same result as in (11.41) which is precisely the pricing formula of a down-in vanilla call option given in (10.36) as shown in Section 11.6.

For an up-in vanilla barrier call option, we can set $\omega = 1$ and $\theta = \zeta = -1$. Substituting $(\omega, \theta, \zeta) = (1, -1, -1), \rho \to 1, \sigma_2 \to \sigma, g_2 \to g$ and $M \to S$ into (11.61) yields exactly the same result as in (11.46), which is precisely the pricing formula of an up-in vanilla call option given in (10.40) as shown Section 11.6. We can check that the other five types of vanilla barrier options are also special cases of the unified pricing formula of outside barrier options given in (11.61) using the identities given in (11.38), (11.42), and (11.43), $\rho \to 1, \sigma_2 \to \sigma, g_2 \to g$, and $M \to S$.

11.8.5. The Trivial Case of Zero Correlation

Vanilla options are shown as a special case of outside barrier options when the correlation coefficient approaches perfect positive correlation. To have a better understanding of the unified pricing formula in (11.61), let us consider the special case when the payment and measurement assets are perfectly independent. Substituting $\rho = 0$ into (11.61) yields

$$OTSD(\omega, \theta, \zeta, 0) = \left\{ N[\zeta\theta d_{bs}(M, H, \sigma_2)] - \zeta \left(\frac{H}{M}\right)^{2v_2/\sigma_2^2} \times N[\theta d_{bs}(H, M, \sigma_2)] \right\} C_{bs}(S, K, \omega), \tag{11.62}$$

where $C_{bs}(S, K, \omega)$ stands for the vanilla option price given in (10.31) with the spot and strike prices S and K and the option operator ω, respectively.

The first term in the brace in (11.62) can be interpreted as the probability that the barrier is not touched for an out vanilla option and as that the barrier is touched for an in vanilla option with the measurement asset. Thus, the pricing formula in (11.62) can be understood as the product of the probability that the barrier is touched (resp. not touched) for an in-barrier (resp. out-barrier) option with the measurement asset and the vanilla option price with the payment asset. This interpretation is very consistent with our intuition that the price of an in- (resp. out-) barrier option without any rebates should be the price of the corresponding vanilla option multiplied by the probability that the option is knocked in (not knocked out).

Example 11.17. Find the prices of the down-in and down-out outside barrier call options to expire in half a year, given the spot prices of the payment and the measurement assets \$100, the strike price \$98, the barrier \$95, the volatilities of the two assets 20% and 15%, respectively, the payout rates of the two assets 3% and 5%, respectively, the interest rate 8%, and the correlation coefficient between the two assets 75%.

Substituting $\omega = \theta = \zeta = 1, S = M = \$100,\ K = \$98,\ H = \$95,\ \sigma = 0.20,\ \sigma_2 = 0.15,\ r = 0.08,\ g = 0.03,\ g_2 = 0.05,\ \rho = 0.75$ into (11.61) yields

$$d_{bs}(S, K, \sigma) = 0.2489, d_{1bs}(S, K, \sigma) = 0.3903, d_{bs}(M, H, \sigma_2) = 0.5720,$$
$$d_{12} = 0.6781, d_{21} = 0.3351, d_{22} = 0.2891\,.$$

The price of the down-out outside call option is then

$$\begin{aligned} OTSD(1,1,1) = {} & 100\mathrm{e}^{-0.03\times 0.5}N_2[0.3903, 0.6781, 0.75] \\ & - 98\mathrm{e}^{-0.08\times 0.5}N_2[0.2489, 0.5720, 0.75] \\ & - 0.95^{2\times 0.01875/0.15^2}\{100 \times 0.95^{2\times 0.75\times 0.2/0.15}\mathrm{e}^{-0.03\times 0.5} \\ & \times N_2[0.3351, 0.2891, 0.75] - 98\mathrm{e}^{-0.08\times 0.5^2} \\ & \times N_2[0.3351 - 0.20\sqrt{0.50}, 0.2891 \\ & - 0.75 \times 0.20\sqrt{0.50}, 0.75]\} \\ = {} & \$5.299 \end{aligned}$$

and the price of the down-in outside call option is

$$\begin{aligned} OTSD(1,1,-1) = {} & 100\mathrm{e}^{-0.03\times 0.5}N_2[0.3903, -0.6781, -0.75] \\ & - 98\mathrm{e}^{-0.08\times 0.5}N_2[0.2489, -0.5720, -0.75] \\ & + 0.95^{2\times 0.01875/0.15^2}\{100 \times 0.95^{2\times 0.75\times 0.2/0.5} \\ & \times N_2[0.3351, 0.2891, 0.95] - 98\mathrm{e}^{-0.08\times 0.5}N_2[0.3351 \\ & - 0.20\sqrt{0.50}, 0.2891 - 0.75 \times 0.20\sqrt{0.50}, 0.75]\} \\ = {} & \$2.583\,. \end{aligned}$$

11.9. OUTSIDE ASIAN BARRIER OPTIONS

We discussed and priced Asian barrier options in Section 11.3. They are barrier options with the underlying spot price substituted by an average of the underlying asset prices. As there are eight types of vanilla barrier options and two types of averages (flexible geometric averages including standard equal-weighting geometric averages as special cases and flexible arithmetic averages including standard equal-weighting arithmetic averages as special cases), there are eight types of Asian barrier options with the strike prices replaced by the spot prices for each type of average. As we explained in Chapters 6 and 7, there are Asian options substituting an average for the strike price. There are eight types of Asian average-strike barrier options for each type of average. Thus, there are a total of thirty-two types of Asian barrier options resulting directly from the combination of standard Asian options with vanilla barrier options.

After studying outside barrier options in the previous section, we can have many other types of Asian barrier options. Any outside barrier option always involves two assets, one payment asset and one measurement asset. Let (PM) stands for the combination of a payment asset and its corresponding measurement asset for an outside barrier option. Table 11.1 lists all the four possible combinations of the average and its corresponding underlying asset for an outside barrier option.

Table 11.1. Possible combinations of the underlying asset and its average.

Spot-Spot	Spot-Average
Average-Spot	Average-Average

An outside barrier option with the combination spot-spot is obviously a vanilla barrier option because the underlying asset price is both the payment asset and the measurement asset. An outside barrier option with the combination average-average is clearly an Asian barrier option we studied in Section 11.3 because the average price is both the payment asset and the measurement asset. An outside barrier option with the combination spot-average is an outside barrier option with the average as the measurement instrument and the underlying asset as the payment asset. Such outside barrier options possess more desirable properties than vanilla barrier options, because the average can potentially reduce spot manipulation as standard Asian options. An outside barrier option with an average of the underlying asset prices as the measurement asset price combine Asian options with outside barrier options. The outside barrier option with the combination average-spot has the spot as the measurement instrument and an average of the underlying asset prices as the payment asset.

To illustrate how Asian outside barrier options can be priced or approximated in closed-form, we simply consider an Asian outside barrier option with the underlying asset as the payment asset and a flexible geometric average of the underlying asset prices as the measurement asset. In order to price such an Asian outside barrier option, we need to know the correlation coefficient between the log-return of the underlying asset and that of the flexible geometric average defined in (7.4), the payout rate, and the volatility of the flexible geometric average. Fortunately, the correlation relationship (4.7) is given in Theorem 7.5 of Chapter 7:

$$\rho = \frac{(\sigma^2 + v^2)\left(\tau - \frac{n-1}{2}h\right) - v^2\tau T^f_{\mu,n-j}}{\sigma^2\sqrt{\tau T^f_{n-j}}},$$

where $T^f_{\mu,n-j}$ and T^f_{n-j} are the effective mean time and variance time functions given in (7.7) and (7.8), respectively, and the payout rate and the volatility of the flexible geometric average are given in (11.5). With the above information, we can price an Asian outside barrier option with a flexible geometric average as the measurement instrument by substituting these parameters into the pricing formula in (11.61).

Example 11.18. Find the prices of the down-out and down-in Asian outside barrier options with the measurement instrument as a flexible geometric average with 12 monthly observations as in Example 7.3, the interest rate is 7%, the yield on the underlying asset is zero, the volatility of the underlying asset is 20%, the time to maturity is one year, the spot price, strike price, and barrier are \$100, \$96, and \$95, respectively.

We can use the effective time values in Example 7.3, $T^f_{\mu,n-j} = 0.629$, $T^f_{nj} = 0.476$ because the conditions of Example 7.3 are the same as in this example. Substituting $\tau = 1$, $T^f_{\mu,n-j}0.629$, $T^f_{n=j} = 0.476$, $r = 0.07$, $g = 0.03)$, $\sigma = 0.20$ into (11.5) yields the effective volatility and payout rate of the flexible average as follows

$$\sigma_{fga} = 0.20\sqrt{0.476/1} = 0.138,$$

$$g_{fga} = 0.07 - (0.20^2.476/2)0.692] = 0.02588\,.$$

We can also use the correlation coefficient between the underlying asset price and the flexible geometric average in Example 7.9, $\rho = 0.7772$.

Substituting $\omega = \theta = \zeta = 1, S = M = \$100, K = \$96, H = \$95, \sigma = 0.20, \sigma_2 = \sigma_{fga} = 0.138, r = 0.07, g = 0, g_2 = g_{fga} = 0.02588$, and $\rho = 0.7772$ into (11.61) and following the same procedure as in Example 11.17 yields the price of the down-out Asian outside call option $OTSD(1,1,1) = \$8.336$, the price of the down-in Asian outside call option $OTSD(1, 1, -1) = \$5.582$, the price of the down-out Asian outside put option $OTSD(-1, 1, 1) = \$0.212$, the price for the down-in Asian outside call option $OTSD(-1, 1, -1) = \$3.216$.

Since flexible geometric averages include standard geometric averages with equal weights as special cases, the above procedure can also price Asian outside barrier options with standard geometric averages as measurement instruments. To price Asian outside barrier options with flexible arithmetic averages as measurement instruments, we can simply use the approximation result in (7.12) to normalize the flexible arithmetic averages which include standard arithmetic averages as special cases.

11.10. CORRIDOR OPTIONS

Corridor options are also called dual-barrier options or barrier options with two barriers. They are more often called corridor options in practice. Corridor options, whether knockouts or knock-ins, are cancelled or activated if at any time within the life of the option the underlying asset price hits an upper or lower barrier. Compared to vanilla barrier options with one barrier, corridor options with double barriers have lower premiums, because they impose an additional barrier which restricts the movement of the underlying asset prices and in turn the payments of the options. If we say that the buyer of a vanilla barrier option expresses the view of the underlying asset price, the buyer of a corridor barrier option expresses the view of the underlying asset price more specifically within the time to maturity of the option.

With the concept of range trading being most commonly related to the currency market, corridor options are actively trading in it. They are also popular in index options when buyers want to express their view of the stock market as a whole.

A corridor option is similar to a knockout option in the sense that it is canceled if either one of the two barriers is touched at any time within the effective time of the option. As its name implies, there are two barriers for each dual-barrier option, one above the spot price $U > S$ and the other below the spot price $L < S$. Compared to vanilla barrier options studied in Chapter 10, we can easily find that the meaning of "knock-in" options and "knock-out" options changes. It makes no difference whether the up-barrier or the low-barrier is touched, or which is touched first, an "out" option is knocked-out. Therefore, there are only four kinds of dual-barrier options: out calls, out puts, in calls, and in puts because the direction of approaching the barrier is no longer a factor affecting the option value.

11.10.1. The Density Function with Dual-Barriers and Definition of Corridor Options

Cox and Miller (1965, p. 222) provided a density function with two barriers, one above and the other below the spot price of the underlying asset. The density function is given as follows:

$$P_{db}(x,t) = \sum_{n=-\infty}^{\infty} p_n(x,t), \text{ for } b < x < a, \tag{11.63a}$$

where

$$\begin{aligned} p_n(x,t) &= e^{x'_n v/\sigma^2} f(x - x'_n) - e^{x''_n v/\sigma^2} f(x - x''_n), \qquad (11.63b)\\ x'_n &= 2n(a-b),\\ x''_n &= 2a - x'_n = 2(1-n)a + 2nb,\\ a &= \ln\left(\frac{U}{S}\right) > 0,\\ b &= \ln\left(\frac{L}{S}\right) < 0, \end{aligned}$$

$f(x)$ is the unrestricted density function in (10.10), U and L are the up- and low-barriers, respectively; $L < S < U, S$ is the spot price of the underlying asset, $v = r - g - \sigma^2/2$, and $p(x,t) = 0$ for $x \le b$ or $x \ge a$ and all t.

The density function in (11.63) can be interpreted as a superposition of a source of unit strength at $x'_0 = 0$, $e^{vx'_n/\sigma^2}$ at the points $x'_n = 2n(a-b), n = \pm 1, 2, 3, \ldots$, and $-e^{vx''_n/\sigma^2}$ at the points $x''_n = 2a - x'_n, = \pm 1, 2, 3, \ldots$. The reason that there are an infinite number of terms in (11.63) is that reflections generate reflections.

We can show that the density function with double barriers in (11.63) degenerates to that with single barrier in (10.20) and (10.24) when the lower barrier $L \to 0$ or when the upper barrier $U \to \infty$. This is consistent with our intuition because when $L \to 0$ (resp. $U \to \infty$), the down- (resp. up-) barrier actually disappears and there is only one barrier left. We will use the density function with dual-barriers given in (11.63) to price corrider options in this section and double-digital options in Chapter 15.

Following a similar procedure as to obtain the density function given in (11.58) to price outside barrier options, we can obtain, using the density function given in (11.63), a general conditional density function for the payment asset price conditioned on whether the measurement asset price is within two pre-specified barriers or not within the life of the option:

$$P_{odb}(x,t) = \sum_{n=-\infty}^{\infty} e^{y'_n v_2/\sigma_2^2} f\left(u - \frac{\rho y'_n}{\sigma_2\sqrt{\tau}}\right) NU(n)$$
$$- \; e^{y''_n v_2/\sigma_2^2} f\left(u - \frac{\rho y''_n}{\sigma_2\sqrt{\tau}}\right) NL(n)\,, \tag{11.64}$$

where

$$NU(n) = N\left\{\frac{d(M,L) + (1-\rho^2)y'_n/(\sigma_2\sqrt{\tau}) + \rho u}{\sqrt{1-\rho^2}}\right\}$$
$$- N\left\{\frac{d(M,U) + (1-\rho^2)y'_n/(\sigma_2\sqrt{\tau}) + \rho u}{\sqrt{1-\rho^2}}\right\}, \tag{11.64a}$$
$$NL(n) = N\left\{\frac{d(M,L) + (1-\rho^2)y''_n/(\sigma_2\sqrt{\tau}) + \rho u}{\sqrt{1-\rho^2}}\right\}$$
$$- N\left\{\frac{d(M,U) + (1+\rho^2)y''_n/(\sigma_2\sqrt{\tau}) + \rho u}{\sqrt{1-\rho^2}}\right\}, \tag{11.64b}$$
$$y'_n = 2n(a-b), y''_n = 2a - y'_n = 2(1-n)a + 2nb\,,$$
$$d(M,X) = \frac{\ln(M/X) + v_2\tau}{\sigma_2\sqrt{\tau}},\; u = \frac{x - v\tau}{\sigma\sqrt{\tau}}\,,$$

and other parameters are the same as in (11.63) and (11.58).

We can show that the density function given in (11.64) degenerates to that for "inside" double barrier options given in (11.63) when the two assets become one, or $\sigma_2 \to \sigma$, $g_2 \to g$, $M \to S$, and $\rho \to 1$. This is because the following must be true when the two assets become one:

$$\ln(L/S) < x < \ln(U/S), \text{ or } -d(L) < u < -d(U)\,. \tag{11.65}$$

Substituting (11.65) and $\sigma_2 \to \sigma$, $g_2 \to g$, $M \to S$, $\rho \to 1$, $v = v_2$ into (11.64a) and (11.64b) yields

$$NU(n) = NL(n) = 1\,. \tag{11.66}$$

Substituting (11.66) and $y'_n = x'_n$, $y''_n = x''_n$ into (11.64) yields (11.63).

Substituting $n = 0$ into (11.64), we can also readily find that (11.64) degenerates exactly to the one for single barrier outside barrier options given on (11.58).

The payoff of an out-corridor option (POTCRD) can be given formally as follows:

$$POTCOP = R(T), \text{if } S(T) \leq L \text{ or } S(T) \geq U, \text{ for some } t < T \leq t^*, \tag{11.67a}$$

or

$$POTCOP = \max\{[\omega S(t^*) - \omega K, 0] | L < S(T) < U, \forall < T \leq t^*\}, \tag{11.67b}$$

where all parameters are the same as in Chapter 10 for vanilla barrier options.

And the payoff of an in-corridor option (INCRD) can be given formally as follows:

$$\begin{aligned} PINCOP = \max\{\omega S(t^*) - \omega K, 0] | S(T) \leq L \\ \text{or } S(T) \geq U, \text{ for some } t < T \leq t^*\}, \end{aligned} \tag{11.68a}$$

or

$$PINCOP = Rd(\tau), \text{ if } L < S(T) < U, \; \forall\, t < T \leq t^*\}, \tag{11.68b}$$

where all parameters are the same as in (11.5).

11.10.2. Pricing Corridor Options Without Rebates

Using the density function given in (11.64), we can obtain the price of an outside knocked-out corridor option without rebate (OUT2DB):

$$\text{OUT2DB} = \omega \text{S}e^{-g\tau}\text{Prob1}(\omega) - \omega \text{K}e^{-r\tau}\text{Prob2}(\omega), \tag{11.69}$$

where

$$\begin{aligned} \text{Prob2}(\omega) = \sum_{n=-\infty}^{+\infty} \left(\frac{U}{L}\right)^{2nv_2/\sigma_2^2} & \{N_2[\omega(d(S,K) + \rho A'_n),\ d(M,L) + A'_n,\ \omega\rho] \\ & - N_2[\omega(d(S,K) + \rho A'_n),\ d(M,U) + A'_n,\ \omega\rho]\} \\ & + \left(\frac{U}{M}\right)^{2v_2/\sigma_2^2} \left(\frac{L}{U}\right)^{2nv_2/\sigma_2^2} \\ & \times \{N_2[\omega(d(S,K) + \rho A''_n),\ d(M,L) + A''_n,\ \omega\rho] \\ & \quad - N_2[\omega(d(S,K) + \rho A''_n),\ d(M,U) + A''_n,\ \omega\rho]\}, \end{aligned}$$

$$\text{Prob1}(\omega) = \sum_{n=-\infty}^{+\infty} \left(\frac{U}{L}\right)^{2n\left(\frac{\rho\sigma}{\sigma_2}+v_2/\sigma_2^2\right)}$$

$$\times \{N_2[\omega(d_1(S,K)+\rho A_n'),\ d(M,L)+A_n'+\rho\sigma\sqrt{\tau},\ \omega\rho]$$

$$- N_2[\omega(d_1(S,K)+\rho A_n'),\ d(M,U)+A_n'+\rho\sigma\sqrt{\tau},\ \omega\rho]\}$$

$$+ \left\{\frac{U}{M}\left(\frac{L}{U}\right)^n\right\}^{2\left(\frac{\rho\sigma}{\sigma_2}+v_2/\sigma_2^2\right)}$$

$$\times \{N_2[\omega(d_1(S,K)+\rho A_n''),\ d(M,L)+A_n''+\rho\sigma\sqrt{\tau},\ \omega\rho]$$

$$- N_2[\omega(d_1(S,K)+\rho A_n''),\ d(M,U)+A_n''+\rho\sigma\sqrt{\tau},\ \omega\rho]\}\,,$$

$$W_\omega = U \text{ if } \omega = 1, \text{ and } W_\omega = L \text{ if } \omega = -1\,,$$

$$A_n' = \frac{y_n'}{\sigma_2\sqrt{\tau}},\ A_n'' = \frac{y_n''}{\sigma_2\sqrt{\tau}}\,,$$

$$y_n' = 2n(a-b) = 2n\ln\left(\frac{U}{L}\right)\,,$$

$$y_n'' = 2\ln\left(\frac{U}{M}\right) - 2n\ln\left(\frac{U}{L}\right) = 2\ln\left\{\left(\frac{U}{M}\right)\left(\frac{L}{U}\right)^n\right\}\,,$$

$$d(S,K) = \frac{\ln(S/K)+v\tau}{\sigma\sqrt{\tau}}\,,$$

$$d_1(S,K) = d(S,K) + \sigma\sqrt{\tau}\,,$$

and $N_2(a,b,c)$ for the cumulative function for the standard bivariate normal distribution with upper limits a and b and correlation coeficient c.

The pricing formula given in (11.69) is rather complicated compared to those of all other options so far covered in this book. We can readily obtain those for "inside" corridor options as special cases of (11.69). This is simply because the density function we used to price outside out corridor options given in (11.64) includes the corresponding "inside" double-barrier density function as special cases when the two assets become one, or when $\sigma_2 \to \sigma$, $g_2 \to g$, $M \to S$, and $\rho \to 1$.

Substituting $\sigma_2 \to \sigma$, $g_2 \to g$, $M \to S$, and $\rho \to 1$ into (11.69) yield the price of an out-corridor option (CRDOT) without a rebate:

$$CRDOT = \omega S e^{-g\tau} PD(v+\sigma^2,\omega) - \omega K e^{-r\tau} PD(v,\omega)\,, \qquad (11.70)$$

where

$$PD(v,\omega) = \sum_{n=-\infty}^{+\infty} \left\{ \left(\frac{U}{L}\right)^{2nv/\sigma^2} \left\{ N\left[\omega d_{bs}(S,K,v) + \left(\frac{\omega x'_n}{\sigma\sqrt{\tau}}\right)\right] \right.\right.$$

$$\left. - N\left[\omega d_{bs}(S,W_\omega,v) + \left(\frac{\omega x'_n}{\sigma\sqrt{\tau}}\right)\right]\right\}$$

$$- \left(\frac{U}{S}\right)^{2v/\sigma^2} \left(\frac{L}{U}\right)^{2nv/\sigma^2} \left\{ N\left[\omega d_{bs}(S,K,v) + \left(\frac{\omega x''_n}{\sigma\sqrt{\tau}}\right)\right] \right.$$

$$\left.\left. - N\left[\omega d_{bs}(S,W_\omega,v) + \left(\frac{\omega x''_n}{\sigma\sqrt{\tau}}\right)\right]\right\}\right\}$$

$$W_\omega = U, \ \text{if} \ \omega = 1 \ \text{and} \ W_\omega = L, \ \text{if} \ \omega = -1,$$

$$x'_n = 2n(a-b) = 2n\ln\left(\frac{U}{L}\right),$$

$$x''_n = 2\ln\left(\frac{U}{S}\right) - 2n\ln\left(\frac{U}{L}\right) = 2\ln\left[\left(\frac{L}{U}\right)^n \left(\frac{U}{S}\right)\right],$$

and all other parameters are the same as in (11.69).

The pricing formula in (11.70) is much simplified from (11.69), but it is still more complicated than the ones with only one barrier given in Chapter 10. Actually, the process to obtain (11.70) is rather straightforward. We can simply work with each component of the density function given in (11.63) following the same steps as in Section 7.4 to price standard single-barrier options, and then we obtain (11.70) through summing up all these components.

The identity given in (10.56) also holds for corridor options. In other words, the summation of the prices of an out-corridor option and its corresponding in-corridor option without any rebates equals the corresponding vanilla option price, because the corridor option can be either "in" or "out" during the life of the option and the total result of both an out-corridor and its corresponding in-corridor options is the same as that of the corresponding vanilla option regardless of whether the barriers are touched. We can hence obtain the price of the corresponding in-corridor option (CRDIN) using the identify and the out-corridor option pricing formula given in (11.70): where all parameters are the same as in (11.70).

$$CRDIN = \omega S e^{-r\tau} \left\{ N[\omega d_{bs}(S,K,v+\sigma^2)] - PD(v+\sigma^2,\omega) \right\}$$

$$- \omega K e^{-r\tau} \left\{ N[\omega d_{bs}(S,K,v)] - PD(v,\omega) \right\} \qquad (11.71)$$

We can also check that it degenerates to that for a single barrier outside barrier option when either the up barrier does extremely large or the low barrier becomes sufficiently small.

11.10.3. The Fourier Series Method to Price Corridor Options

The pricing formulas of corridor options in (11.70) and (11.71) are expressed in terms of a series of infinite terms of univariate cumulative normal distributions. Although the univariate cumulative function values can be calculated quickly, it may take significantly more time to calculate the corridor option prices because the number of cumulative function values (eight values are needed to calculate an out-corridor option price and ten for an in-corridor option price for each "n") needed may increase significantly if the converging process is very slow. We will explore an alternative approach to price out-corridor options.

Using the method of separation of variables, Cox and Miller (1965, p. 222) obtained a Fourier series to solve the dual-barrier problem:

$$p_{adb}(x,t) = \sum_{n=1}^{\infty} p_{sn}(x,t), \text{for} b < x < a\,, \tag{11.72}$$

where

$$p_{sn}(x,t) = a_n e^{-\lambda_n t} e^{xv/\sigma^2} \sin\left[\frac{n\pi(x-b)}{a-b}\right], \tag{11.72a}$$

$$\lambda_n = \frac{1}{2}\left[\frac{v^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{(a-b)^2}\right], \tag{11.72b}$$

$$a_n = \frac{-2}{a-b}\sin\left(\frac{n\pi b}{a-b}\right), \tag{11.72c}$$

and all other parameters are the same as in (11.63), and $p_{adb}(x,t) = 0$ for $x \leq b$ or $x \leq a$.

Using the density function for the dual-absorbing barrier problem in (11.72), we can provide an alternative pricing formula for an out-corridor option (see Appendix for an outline of the derivation):

$$OUTC = \omega S e^{-g\tau} Pa(v+\sigma^2, \omega) - \omega K e^{-r\tau} Pa(v,\omega)\,, \tag{11.73}$$

where

$$Pa(v,\omega) = \sigma^2(a-b)\left(\frac{L}{S}\right)^{v/\sigma^2} \sum_{n=1}^{\infty} \frac{a_n e^{-\lambda_n \tau}}{n^2\pi^2\sigma^4 + v^2(a-b)^2}$$

$$\times \left(n\pi \left[\sigma^2(-1)^{n-1}\left(\frac{U}{L}\right)^{v/\sigma^2}\right]^{(1+\omega)/2} \right.$$

$$+\omega \left\{ n\pi\sigma^2 \cos\left[\frac{\ln(K/L)}{a-b} n\pi\right] \right.$$

$$\left.\left. -v(a-b)\sin\left[\frac{\ln(K/L)}{a-b} n\pi\right]\right\} \left(\frac{K}{L}\right)^{v/\sigma^2} \right),$$

and sin(.) and cos(.) are the sine and the cosine functions, and other parameters are the same as in (11.63), (11.70), and (11.72), and ω is the option operator (1 for a call and -1 for a put).

The pricing formula in (11.73) appears as complicated as that given in (11.70), yet the values of the two intermediate functions $Pa(v,\omega)$ and $Pa(v+\sigma^2,\omega)$ are in general smaller than 1, therefore the converging process can be faster than (11.70). Another obvious difference between them is that the terms start from 1 to ∞ in (11.73) and from $-\infty$ to ∞ in (11.70). Numerical testing shows that the first few terms in the sums of the $Pa(v,\omega)$ and $Pa(v+\sigma^2,\omega)$ are sufficient for convergence because the density for all other terms are nearly zero.

Numerical examples show that the convergence is very fast with both methods in (11.70) and (11.73), and the standard image method in (11.70) converges faster for short-term options and the method in (11.73) converges faster for long-term options in general.

11.10.4. Rebates of Corridor Options

So far, we have priced corridor options without rebates. As studied in Chapter 10 for vanilla barrier options, the present values of the rebates for out-barrier options are more difficult to obtain than those for in-barrier options because the time for the options to be knocked out is uncertain in out-barrier options. In the remaining of this section, we will find the present values of the rebates for both in- and out-corridor options.

11.10.4A

As in the case to find the present value of the rebate for a vanilla out barrier option, we need the distribution for the first passage time. Using Anderson's (1960) results of the density of one line touched earlier than the other, we can obtain the density function that the upper barrier is touched first $\pi_u(t)$ and the lower is touched first $\pi_l(t)$:

$$\pi_u(t) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{U^{n+1}}{L^nS}\right)^{2v/\sigma^2}\left(\frac{a_1(n)-vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_1(n)+vt]^2}{2\sigma^2 t}\right\}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{L^n}{U^n}\right)^{2v/\sigma^2}\left(\frac{a_1(n)+vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_1(n)vt]^2}{2\sigma^2 t}\right\}$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{U^{n+1}}{L^{n+1}}\right)^{2v/\sigma^2}\left(\frac{a_2(n)+vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_2(n)-vt]^2}{2\sigma^2 t}\right\}$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{L^{n+1}}{U^nS}\right)^{2v/\sigma^2}\left(\frac{a_2(n)+vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_2(n)+vt]^2}{2\sigma^2 t}\right\}\right], \tag{11.74a}$$

and

$$\pi_l(t) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{L^{n+1}}{U^nS}\right)^{2v/\sigma^2}\left(\frac{-a_3(n)+vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_3(n)+vt]^2}{2\sigma^2 t}\right\}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{U^n}{L^n}\right)^{2v/\sigma^2}\left(\frac{-a_3(n)-vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_3(n)-vt]^2}{2\sigma^2 t}\right\}$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{L^{n+1}}{U^{n+1}}\right)^{2v/\sigma^2}\left(\frac{a_4(n)+vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_1(n)-vt]^2}{2\sigma^2 t}\right\}$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{U^{n+1}}{L^nS}\right)^{2v/\sigma^2}\left(\frac{a_4(u)-vt}{\sigma\sqrt{2\pi t^3}}\right)\exp\left\{-\frac{[a_4(n)+vt]^2}{2\sigma^2 t}\right\}\right], \tag{11.74b}$$

where

$$a_1(n) = 2nln\left(\frac{U}{L}\right) + \ln\left(\frac{U}{L}\right) > 0\,,$$

$$a_2(n) = (2n+1)\ln\left(\frac{L}{U}\right) + \ln\left(\frac{L}{S}\right) < 0\,,$$

$$a_3(n) = 2nln\left(\frac{L}{U}\right) + \ln\left(\frac{L}{S}\right) < 0\,,\ \text{and}$$

$$a_4(n) = (2n+1)\ln\left(\frac{U}{L}\right) + \ln\left(\frac{U}{S}\right) > 0\,.$$

The present value of the rebate if the upper barrier is touched first V_u is given by

$$V_u = \int_0^\tau (R_u e^{\eta t}) e^{-rt} \pi_u(t) dt, \tag{11.75a}$$

and the present value of the rebate if the lower barrier is touched first V_l

$$V_l = \int_0^\tau (R_l e^{\eta t})^{-rt} \pi_l(t) dt, \tag{11.75b}$$

where R_u and R_l stand for the rebates paid as soon as the upper barrier and the lower barrier is touched, $\eta \geq 0$ for the growth rate of the rebate as in Chapter 10.

Substituting (11.74) into (11.75) and carrying out the standard integration steps as in the Appendix of Chapter 10 for knockout barrier options with single barriers, we can obtain

$$\begin{aligned} V_u = \frac{1}{2} R_u \sum_{n=0}^{\infty} \Bigg\{ & \left(\frac{U^{n+1}}{L^n S}\right)^{2v/\sigma^2} \left(1 - \frac{v}{\psi}\right) V[-a_1(n)] \\ & + \left(\frac{L^n}{U^n}\right)^{2v/\sigma^2} \left(1 + \frac{v}{\psi}\right) V[a_1(n)] - \left(\frac{U^{n+1}}{L^{n+1}}\right)^{2v/\sigma^2} \left(1 - \frac{v}{\psi}\right) V[a_2(n)] \\ & - \left(\frac{L^{n+1}}{U^n S}\right)^{2v/\sigma^2} \left(1 + \frac{v}{\psi}\right) V[-a_2(n)] \Bigg\}, \end{aligned} \tag{11.76a}$$

$$\begin{aligned} V_l = \frac{1}{2} R_l \sum_{n=0}^{\infty} \Bigg\{ & \left(\frac{L^{n+1}}{U^n S}\right)^{2v/\sigma^2} \left(1 + \frac{v}{\psi}\right) V[-a_3(n)] \\ & + \left(\frac{U^n}{L^n}\right)^{2v/\sigma^2} \left(1 - \frac{v}{\psi}\right) V[a_3(n)] - \left(\frac{L^{n+1}}{U^{n+1}}\right)^{2v/\sigma^2} \left(1 + \frac{v}{\psi}\right) V[a_4(n)] \\ & - \left(\frac{U^{n+1}}{L^n S}\right)^{2v/\sigma^2} \left(1 - \frac{v}{\psi}\right) V[-a_4(n)] \Bigg\}, \end{aligned} \tag{11.76b}$$

where

$$V(c) = e^{c(v+\psi)/\sigma^2} N\left\{ -\operatorname{sign}(c) \frac{c + \tau\psi}{\sigma\sqrt{\tau}} \right\} + e^{c(v-\psi)/\sigma^2} N\left\{ -\operatorname{sign}(c) \frac{c - \tau\psi}{\sigma\sqrt{\tau}} \right\},$$

$$\psi = \sqrt{v^2 + 2(r - \eta)\sigma^2},$$

and $N(.)$ is the cummulative function of the standard univariate normal distribution and sign(c) is the sign function which gives 1 if c is positive and -1 if it is negative.

The total present value of the rebate V is simply the sum of the two terms given in (11.76)

$$V = V_u + V_l\,.$$

11.10.4B

The present value of the rebate of an in-corridor option (CRDINRBT) can be obtained by discounting the rebate to be paid at maturity of the option and multiplying it by the probability that the option is not knocked in within the life of the option:

$$CRDINRBT = R(\tau)e^{-r\tau}Pa(\upsilon,\omega)\,, \tag{11.77}$$

where $Pa(\upsilon,\omega)$ is the same as in (11.70).

Using the value of an out-corridor option without a rebate in (11.67) or (11.71) and the present value of the rebate of an out-corridor option in (11.76), we can obtain the price of an out-corridor option with a rebate (PCRDOT):

$$PCRDOT = CRDOT + CRDOTRBT\,. \tag{11.78}$$

The price of an in-corridor option with a rebate (PCRDIN) can be found by adding up the value of an in-corridor option without a rebate in (11.71) and the present value of the rebate of the in-corridor option in (11.77):

$$PCRDIN = CRDIN + CRDINRBT. \tag{11.79}$$

We have studied standard corridor options in this section. Although out corridor options are most common, there are other variants, one of which has one of the barriers as a knock-in and the other a knockout. Another variant is that the two barriers can be effective for different time periods, or one barrier is effective within a certain time period and the other within a different time period. Also, since the order of hitting the upper or the lower barrier is irrelevant to standard corridor barrier options, the third variant can be an up-out or down-in or a down-in or up-out. These variants can be analyzed with existing methods, yet they are beyond the scope of this book.

11.11. BARRIER OPTIONS WITH TWO CURVED BARRIERS

We introduced and priced corridor options in the previous section. Although corridor options are general and include vanilla barrier options as special cases, the two barriers in our analysis were assumed to be constant. In some applications, the barriers are time-dependent as in Section 11.2

for single time-dependent barrier options. Corridor options with two time-dependent barriers are more flexible than those with constant barriers we studied in the previous section. Kunitomo and Ikeda (1992) first studied such options and provided pricing formulas for them. They extended the standard corridor options to include barriers that fluctuate with time exponentially. As a matter of fact, the barrier options with two curved barriers are combinations of corridor options studied in Section 11.10 and two floating barriers discussed in Section 11.2.

Kunitomo and Ikeda introduced two floating barriers as the floating barriers we discussed in Section 11.2, both floating barriers having constant rates of change. Specifically, they used two rates δ_1 and δ_2 for the low and up barriers, respectively. The two floating barriers can be expressed as follows in our notations:

$$L(T) = Le^{\delta_2 T} \quad \text{and} \quad U(T) = Ue^{\delta_1 T}, \ 0 \le T \le \tau \tag{11.80}$$

where $L < S < U$, L and U represent constant up and low barriers, respectively.

Assuming that the underlying asset price follows the same geometric Brownian motion given in (2.3), Kunitomo and Ikeda first generalized Levy's (1948) well-known formula [the same as the density function given in (11.63)] and obtained the density function for the underlying asset price within the two floating barriers $[L(T), U(T)]$. The density function can be written as follows after a slight modification from Kunitomo and Ikeda's original result which does not consider the payout rate of the underlying asset:

$$P_{ki}(y) = \sum_{n=-\infty}^{\infty} k_n(y)\,, \tag{11.81}$$

where

$$\begin{aligned} k_n(y) &= \left(\frac{U^n}{L^n}\right)^{c_{1n}} \left(\frac{A}{S}\right)^{c_{2n}} f\left\{\frac{\log(y/S) - 2n\log(U/L) - \nu\tau}{\sigma\sqrt{\tau}}\right\} \\ &\quad - \left(\frac{L^{n+1}}{SU^n}\right)^{c_{3n}} f\left\{\frac{\log(yS/L^2) - 2n\log(L/U) - \nu\tau}{\sigma\sqrt{\tau}}\right\}, \\ c_{1n} &= 2\,\frac{[r - g - \delta_2 - n(\delta_1 - \delta_2)]}{\sigma^2} - 1\,, \\ c_{2n} &= 2\,\frac{n(\delta_1 - \delta_2)}{\sigma^2}\,, \end{aligned}$$

and

$$c_{3n} = 2\,\frac{[r - g - \delta_2 + n(\delta_1 - \delta_2)]}{\sigma^2} - 1\,.$$

The density function given in (11.81) is in terms of the underlying asset price rather than the log-returns as in previous sections in this chapter and all previous chapters. It can be shown that (11.81) includes the density function given in (11.63) as a special case when $\delta_1 = \delta_2 = 0$ (see Exercise 11.44).

Using the density function given in (11.81), we can readily obtain the pricing formulas for both knock-out call and put options with two-curved barriers (TCB) in compacted form after making the modification to include the payout rate of the underlying asset:

$$TCB = \omega S e^{-g\tau} Pb(v + \sigma^2, \omega, 1) - K e^{-r\tau} Pb(v, \omega, 0)\,, \tag{11.82}$$

where

$$\begin{aligned} Pb(v,\omega,q) = {} & \left(\frac{U^n}{L^n}\right)^{2q+c_{1n}} \left(\frac{L}{S}\right)^{c_{2n}} \left\{ N\left[\omega d_{bs}(S,K,v) + \frac{\omega x_n'}{\sigma\sqrt{\tau}}\right] \right. \\ & \left. - N\left[\omega d_{bs}(S,W_\omega^c,v) + \frac{\omega x_n'}{\sigma\sqrt{\tau}}\right] \right\} \\ & - \left(\frac{L^{n+1}}{U^n S}\right)^{2q+c_{3n}} \left\{ N\left[\omega d_{bs}(S,K,v) + \frac{\omega x_n''}{\sigma\sqrt{\tau}}\right] \right. \\ & \left. - N\left[\omega d_{bs}(S,W_\omega^c,v) + \frac{\omega x_n''}{\sigma\sqrt{\tau}}\right] \right\} \end{aligned}$$

$$W_\omega^c = U e^{\delta_1 T} \text{ if } \omega = 1 \text{ and } W_\omega^c = L e^{\delta_2 T} \text{ if } \omega = -1\,,$$

and ω is the same option binary operator (1 for a call and -1 for a put) and other parameters are the same as in (11.63) and (11.70).

The pricing formula given in (11.82) is expressed in terms of our algebra for convenient comparisons. Comparing the pricing formulas given in (11.70) and (11.82), we can readily find that the only differences between the two are in the powers of their corresponding probability functions $Pb(v, \omega, q)$ and $Pb(v, \omega, q)$, the latter being a function of the two curvature parameters through the three intermediate functions c_{1n}, c_{2n}, and c_{3n}, the former being a constant because the two barriers are constant. We can readily show that (11.82) degenerates to the pricing formula given in (11.70) when $\delta_1 = \delta_2 = 0$ (see Exercise 11.45).

The corresponding pricing formulas for in corridor options, rebates for both in and out corridor options with one or two floating barriers can be easily obtained accordingly following the same procedures as in the previous sections of this chapter.

11.12. SUMMARY AND CONCLUSIONS

We have introduced, discussed, and priced all other popular exotic barrier options besides vanilla barrier options in this chapter. After a brief introduction to popular exotic barrier options, we introduced barrier options with floating or time-dependent barriers and provided closed-form solutions for them using the pricing formulas of vanilla barrier options given in Chapter 10. We then introduced flexible geometric Asian barrier options and obtained closed-form solutions using again the pricing formulas for vanilla barrier options. Although closed-form solutions for arithmetic Asian barrier options are difficult to find, for reasons similar to arithmetic Asian options, we can approximate their prices using the approximation results developed in Chapter 6 to lognormalize arithmetic averages with their corresponding geometric averages.

We introduced and analyzed forward-start barrier options with barriers to be effective some time after the initialization of the options to their maturity. We found closed-form solutions for forward-start barrier options in terms of the cumulative functions of bivariate normal distributions using the pricing formulas of vanilla barrier options. Forced forward-start barrier options are special forward-start barrier options. They can be knocked out if the underlying asset price turns out to be on the opposite side of the barrier to the spot price.

Early-ending barrier options are complements to forward-start barrier options. We found a unified pricing formula for early-ending barrier options in a Black-Scholes environment. This formula is applicable to all eight types of early-ending barrier options. To find the value of a particular early-ending barrier option, we need only to specify the binary operator set (ω, θ, ζ) and substitute these binary operator values into the unified formula. We have shown that this unified formula includes all eight types of vanilla barrier options as special cases when the early-ending time approaches the maturity time of the option. With it, we obtained a unified formula for window barrier options. We have also found closed-form solutions for the present values of the time-dependent rebates of early-ending out-barrier options. The rebates of window barrier options were also obtained in closed-form in terms of the cumulative functions of standard bivariate normal distributions. The forward-start, early-ending, and window barrier options covered in this chapter are somewhat similar to the "partial barrier options" studied by Heynen and Kat (1994b), yet the analytical simplicity demonstrated in the unified pricing formulas for these options in this chapter is significantly different from their results.

An outside barrier option is a barrier option with the measurement instrument separated from the payment asset. Following the method to express the prices of all eight types of early-ending barrier options in a unified formula in Section 11.6, we found a unified pricing formula for all eight types of outside barrier options. The unified pricing formula for outside barrier options include all vanilla barrier options as special cases when the measurement instrument and the payment asset are perfectly correlated. A corridor option is also called a dual-barrier or a double-barrier option with two barriers, normally one above and one below the spot price. We analyzed and found solutions for both in- and out-corridor options and their rebates. As these solutions are expressed in sums of an infinite number of terms, convergence of these terms is important for practical use. In order to increase the convergence speed, we analyzed corridor options using the method of variable change which yields solutions normally faster in convergence speed.

The most important characteristic of this chapter is the two unified pricing formulas for early-ending barrier options and outside barrier options. These unified pricing formulas are very convenient not only for theoretical understanding but also for computer implementation, risk-parameter calculation, and so on. Using the unified pricing formulas and the method to find sensitivities involving the cumulative functions of bivariate normal distributions developed in Zhang (1995d), we can obtain closed-form expressions for the deltas, gammas, vegas, and other sensitivities of early-ending barrier options and outside barrier options in a unified manner as the unified pricing formulas.

We have extended vanilla barrier options to many types of exotic barrier options in this chapter. This extension can be further developed to capture how long the barrier is crossed, how far the barrier is crossed, and possibly how many times the barrier is crossed within the life of the option. These extensions are beyond the scope of this book although they can be interesting and useful. To some degree, the feature how far the barrier is crossed is captured by ladder options with appropriate ladders. In particular, the number of crossings within the life of the option may be developed following the mathematical results and the related literature in Slud (1991). Instead of knocked-in-or-out as soon as the barrier is touched, a barrier option can be knocked-in-or-out only when the barrier is broken for a pre-specified period of time, say two days or a week. These kind of barrier options are called parisan options. No closed form solution has been found for parisan options in a Black-Scholes world.

The underlying asset price is measured against the barrier continuously in all barrier options covered in this chapter with the only exception of Asian barrier options. Several institutions have traded barrier options which knock in or out only in discrete time, such as daily, weekly, or monthly intervals. These barrier options are combinations of barrier options and Bermuda options, thus we may call them Bermuda barrier options. Since standard Bermuda options cannot be priced in closed-form, Bermuda barrier options cannot, in general, be priced in closed-form neither.

Another aspect that is worth exploring is local volatilities. We have assumed a constant volatility to price essentially all exotic options so far in this book. As we argued in Chapter 4, volatilities are generally rather volatile in general. For forward-start, early-ending, and window barrier options, barriers are generally effective in different subperiods within the lives of the options, but the volatilities of the underlying assets are very often different in different subperiods because these options are designed to capture some particular events in these subperiods. Most of the analyses developed in this chapter can somehow be extended to capture the local volatilities. This is beyond the scope of this book and interested readers may pursue in these directions according to the analyses in this chapter.

QUESTIONS AND EXERCISES

Questions

11.1. What are exotic barrier options?

11.2. Why are exotic barrier options more flexible than vanilla barrier options?

11.3. What are floating barrier options?

11.4. Why are down-in barrier options with declining (resp. increasing) barriers cheaper (resp. more expensive) than their corresponding vanilla barrier options?

11.5. Is it true that the higher the growth rate of the barrier, the higher the price of a down-in barrier option with an increasing barrier? Why?

11.6. Should the present value of an up-out barrier option with a decreasing barrier be higher or lower than that of its corresponding vanilla out barrier option? Why?

11.7. What are Asian barrier options?

11.8. How many types of Asian barrier options are there?

11.9. Are Asian barrier options always cheaper than their corresponding vanilla barrier options? Why?

11.10. Are Asian barrier options with continuous averaging cheaper than those with discrete averaging? Why?

11.11. Does a closed-form solution exist for arithmetic Asian barrier options in a Black-Scholes environment? Why?

11.12. What are forward-start barrier options?

11.13. How many types of basic forward-start barrier options are there?

11.14. What flexibility can forward-start barrier options provide?

11.15. What are forced forward-start barrier options? Why are they popular in the market?

11.16. How many types of basic forced forward-start barrier options are there?

11.17. What is the most important difference between a forced forward-start barrier option and its corresponding forward-start barrier option?

11.18. What are early-ending barrier options?

11.19. How many types of early-ending barrier options are there?

11.20.* What makes it possible to obtain the unified pricing formula for all types of early-ending barriers?

11.21. When are vanilla barrier options special cases of early-ending barrier options?

11.22. Why is the unified pricing formula for early-ending barrier options attractive?

11.23. Are early-ending barrier options more expensive or cheaper with more time left after the ending time of their barriers? Why?

11.24. What are window barrier options? Why does a window barrier option include at least a forward-start barrier option and an early-ending barrier option?

11.25. What are outside barrier options? Why may we call them correlation barrier options?

11.26. Why do outside barrier options have the potential to be more widely used in the future?

11.27. How many types of outside barrier options are there?

11.28. Under what conditions can outside barrier options become vanilla barrier options?

11.29. What are Asian outside barrier options?

11.30. Why may Asian outside barrier options be more attractive than standard Asian barrier options?

11.31. Give one application of an Asian out-barrier option with the spot price as the measurement price and an average as the payment asset.

11.32. What are dual-barrier or corridor options?

11.33. Under what conditions are vanilla barrier options special cases of corridor options?

11.34. What are barrier options with curved barriers?

11.35. What are the possible ways to extend existing exotic barrier options covered in this chapter?

Exercises

11.1. Find the down-in call and put option prices in Example 11.1 if the barrier increases exponentially 5%.

11.2. Find the down-in call and put option prices in Example 11.1 if the barrier decreases exponentially 3%.

11.3. Find the price of a down-in barrier call option to mature in one year in Exercise 10.1.

11.4. Find the up-in call and put option prices in Example 10.4 if the barrier increases exponentially 5%

11.5. Find the corresponding up-out call and put option prices in Example 10.4 if the barrier increases exponentially 5%

11.6. Find the effective yield and volatility of a flexible geometric average with all information the same as in Example 11.3 for standard geometric averages.

11.7. Find the prices of up-out Asian barrier at-the-money options with the strike $100 and the barrier $105 to expire in one year and the effective yield and volatility are the same as in (11.6).

11.8. Find the corresponding effective yield and volatility of the arithmetic average in Example 11.6.

11.9. Find the prices of down-out at-the-money arithmetic Asian barrier options with strike $100 and the barrier $95 to expire in half a year.

11.10. Find the effective yield and volatility of a geometric average with continuous averaging and other parameters are the same as in Example 11.3.

11.11. Find the price of an up-out Asian barrier at-the-money option with the strike $100 and the barrier $105 to expire in one year and the effective yield and volatility are the same as in (11.10).

11.12. Find the price of a down-in forward-start barrier call option to start in two months, given the spot price \$100, the strike price \$104, the barrier \$95, the time to maturity of the option half a year, the interest rate 6%, the payout rate of the underlying asset 2%, and the volatility of the underlying asset 25%.

11.13. Find the present value of a down-in forward-start barrier option if the rebate is paid \$1.5 and other information is the same as in Exercise 11.12.

11.14. Find the present value of a down-in forward-start barrier option to start in four months if the rebate is paid \$1.5 and other information is the same as in Exercise 11.12.

11.15. Find the price of the down-in forward-start barrier option in Exercise 11.12 with the rebate as given in Exercise 11.13.

11.16. Find the price of the down-in forward-start barrier option in Exercise 11.12 with the rebate as given in Exercise 11.14.

11.17. Find the present value of the rebate of an out forward-start barrier option if the rebate increases 4% from \$1 and other information is the same as in Example 11.12.

11.18. Find the prices of the corresponding forced forward-start down-in options in Exercise 11.12.

11.19. Find the price of early-ending up-in barrier call option with the early-ending time three months before the maturity of the option, the strike price \$103, the barrier \$105, the spot price \$100, and other parameters remain the same as in Example 10.11.

11.20. Find the price of the corresponding early-ending up-out barrier call option in Example 10.18 with the ending time one month earlier than the option maturity, other parameters remain unchanged.

11.21. Find the price of the corresponding down-in put option in Example 11.14.

11.22. Find the price of the corresponding down-out put option in Exercise 11.15.

11.23. Find the price of the corresponding up-in barrier put option in Exercise 11.19.

11.24. Find the price of the corresponding up-out barrier put option in Exercise 11.20.

11.25.* Show that the unified pricing formula for early-ending barrier options degenerates to the pricing formula of vanilla options when the early-ending time is zero.

11.26.* Find the present value of the rebate of a forward-start out option in terms of cumulative functions of bivariate normal distributions using the present-value formula for vanilla out options given in (A10.13) and the expressions in Appendix of this chapter.

11.27.* Show that the density function in (11.32) for the log-return of the underlying asset price at maturity includes the density functions of vanilla up-barrier options as special cases.

11.28. Find the present value of the rebate of the down-out barrier option in Example 10.15 if the early-ending time is four months before the maturity of the option, and other parameters remain the same as in Example 10.15.

11.29. Find the present value of the rebate of the down-out barrier option in Examples 10.15 and 11.15 if the forward-start time is two months, and the ending time is one month before the maturity of the option.

11.30.* Show that (11.25) degenerates to the present value of a vanilla out-barrier option in (10.48a) when the forward-start time approaches zero.

11.31.* Show that the unified pricing formula in (11.36) includes the pricing formula of vanilla up-out call options as a special case.

11.32.* Show that the unified pricing formula in (11.36) includes the pricing formula of vanilla down-out put options as a special case.

11.33.* Show that the unified pricing formula in (11.36) includes the pricing formula of vanilla up-in put options as a special case.

11.34.* Show that the unified pricing formula for outside barrier options degenerates to the pricing formula of vanilla options when the volatility of the measurement asset is zero and the yield of the measurement is the same as the interest rate.

11.35. Find the prices of the corresponding down-in and down-out outside barrier put options in Example 11.17.

11.36. Find the prices of the down-in and down-out outside barrier call options with the correlation coefficient 50%, and other parameters remain unchanged as in Example 11.17.

11.37. Find the prices of up-in and up-out outside barrier call options with the barrier \$105 and other parameters are the same as in Example 11.17.

11.38. Find the prices of down-in Asian outside-barrier call options with the barrier \$98 and other parameters remain unchanged as in Example 11.18.

11.39. Find the prices of down-out Asian outside-barrier call options with the barrier \$98 and other parameters remain unchanged as in Example 11.18.

11.40.* Show that $W(a,0) = W(a,2a) =$ the present value of the rebate of a vanilla up-out barrier option with the growth rate $\eta \leq r + \psi^2/(2\sigma^2)$.

11.41.* Derive the pricing formula of in-corridor options in (11.67) without using the identity that the sum of the prices of an out-corridor option and its corresponding in option equals the price of their corresponding vanilla option [Hint: find the density function for an in-corridor option using the density function given in (11.63) or (11.72).]

11.42.* Show that the following is always true for any real number a: $\int_a^\infty f(u)f(\alpha+\beta u)du = N_2(-a, \frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\beta}{\sqrt{1+\beta^2}})$ (Hint: use the results in A11.4).

11.43.* Show the identity given in (A11.26) in Appendix of this chapter.

11.44. Show the density function given in (11.81) includes the density function given in 11.43 as a special case when $\delta_1 = \delta_2 = 0$.

11.45. Show that the pricing formula given in (11.82) includes the pricing formula given 11.66 as a special case when $\delta_1 = \delta_2 = 0$.

APPENDIX

A11.1. DOUBLE INTEGRATION WITH BIVARIATE NORMAL DENSITY FUNCTIONS

The cumulative function of a standard bivariate normal distribution can be written as follows

$$N_2(a,b,\rho) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(u,v)dudv, \tag{A11.1}$$

where a and b are the integration bounds for the two variables, $f(.,.)$ is the function of the standard bivariate normal distribution and is given as

$$f(u,v) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right]. \tag{A11.2}$$

In order to express the following double integration in terms of the standard cumulative function of a bivariate normal distribution given in (A11.1)

$$\int_{-\infty}^{s} f(u)N(\alpha+\beta u)du,$$

where α and β are real numbers, we need to find another upper bound b_n and a correlation coefficient ρ such that

$$\int_{-\infty}^{a} f(u)N(\alpha+\beta u)du = \int_{-\infty}^{a} f(u)N\left(\frac{b_n-\rho u}{\sqrt{1-\rho^2}}\right)du. \qquad \text{(A11.3)}$$

Solving the following equation $\frac{b_n}{\sqrt{1-\rho^2}} = \alpha$ and $\frac{-\rho}{\sqrt{1-\rho^2}} = \beta$ for ρ and b_n and substituting the values of ρ and b_n into (A11.3) using the definition of the standard bivariate normal distribution given in (A11.1) yields

$$\int_{-\infty}^{a} f(u)N(\alpha+\beta u)du = N_2\left(a, \frac{\alpha}{\sqrt{1+\beta^2}}, \frac{-\beta}{\sqrt{1+\beta^2}}\right). \qquad \text{(A11.4)}$$

Since the pricing formulas of all eight types of vanilla options in Chapter 10 contain the Black-Scholes formula $C_{bs}(S,K,\tau)$, the pricing formulas of up and in options in (11.13) and (11.14) contain

$$C_{bs}[S(\tau_1),K,\tau] = C_{bs}[Se^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}},K,\tau-\tau_1].$$

We will show shortly how the integration of the above Black-Scholes formula can be expressed in terms of the cumulative functions of a standard bivariate normal distribution as a simple example to illustrate how the pricing formulas of forward-start barrier options given in (11.13) and (11.14) can be expressed in terms of the cumulative functions of a standard bivariate normal distribution. Substituting the Black-Scholes formula into the integration yields the integration of the Black-Scholes formula starting from τ_1

$$e^{-\tau_1 r}\int_{-d_{bs}}^{\infty} C_{bs}[Se^{\upsilon\tau_1+u\sigma\sqrt{\tau_1}},K,\tau-\tau_1]f(u)du$$

$$= Se^{-\tau_1 r}\int_{-d_{bs}}^{\infty} e^{\upsilon\tau_1+u\sigma\sqrt{\tau-\tau_1}}N\left[\frac{\ln(S/K)+\psi\tau+\sigma^2(\tau-\tau_1)+u\sigma\sqrt{\tau_1}}{\sigma\sqrt{\tau-\tau_1}}\right]f(u)du.$$

$$-Ke^{-r\tau_1 r}\int_{-d_{bs}}^{\infty} e^{-r(\tau-\tau_1)}N\left[\frac{\ln(S/K)+\upsilon\tau+u\sigma\sqrt{\tau_1}}{\sigma\sqrt{\tau-\tau_1}}\right]f(u)du. \qquad \text{(A11.5)}$$

The second term of (A11.5) can be expressed in terms of the cumulative function of the bivariate normal distribution using the formula given in (A11.4):

$$\int_{-d_{bs}}^{\infty} N\left[\frac{\ln(S/K)+\upsilon\tau+u\sigma\sqrt{\tau_1}}{\sigma\sqrt{\tau-\tau_1}}\right]f(u)du = N_2\left[d_{bs}(H,\tau_1), d_{bs}(K,\tau), \sqrt{\frac{\tau_1}{\tau}}\right], \qquad \text{(A11.6)}$$

where $d_{bs}(X, s)$ is the same argument as in the Black-Scholes formula with the strike price X and the time to maturity s, respectively.

The first term of (A11.5) can be expressed in terms of the cumulative function of the bivariate normal distribution using the formula given in (A11.4) after making the substituting $v = u - \sigma\sqrt{\tau_1}$:

$$\int_{-d_{bs}}^{\infty} e^{u\sigma\sqrt{\tau_1}} N\left[\frac{\ln(S/K) + v\tau + \sigma^2(\tau - \tau_1) + u\sigma\sqrt{\tau_1}}{\sigma\sqrt{\tau - \tau_1}}\right] f(u)du$$
$$= N_2\left[d_{1bs}(H, \tau_1), d_{1bs}(K, \tau), \sqrt{\frac{\tau_1}{\tau}}\right], \tag{A11.7}$$

where $d_{1bs}(X, s) = d_{bs}(X, s) + \sigma\sqrt{s}$ is the same argument as in the Black-Scholes formula with the strike price X and the time to maturity s, respectively.

Substituting (A11.6) and (A11.7) into (A11.5) yields

$$e^{-\tau_1 r}\int_{-d_{bs}}^{\infty} C_{bs}[Se^{v\tau_1 + u\sigma\sqrt{\tau_1}}, K, \tau - \tau_1] f(u)du$$
$$= Se^{-g\tau} N_2\left[d_{1bs}(H, \tau_1), d_{1bs}(K, \tau), \sqrt{\frac{\tau_1}{\tau}}\right]$$
$$- Ke^{-r\tau} N_2\left[d_{bs}(H, \tau_1), d_{bs}(K, \tau), \sqrt{\frac{\tau_1}{\tau}}\right]. \tag{A11.8}$$

Similarly, the integration of the Black-Scholes pricing formula from $-\infty$ to $-d_{bs}(S, X, s)$ can be obtained as follows

$$e^{-\tau_1 r}\int_{-\infty}^{-d_{bs}} C_{bs}[Se^{v\tau_1 + u\sigma\sqrt{\tau_1}}, K, \tau - \tau_1] f(u)du$$
$$= Se^{-g\tau} N_2\left[-d_{1bs}(S, H, \tau_1), d_{1bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}}\right]$$
$$- Ke^{-r\tau} N_2\left[-d_{bs}(S, H, \tau_1), d_{bs}(S, K, \tau), -\sqrt{\frac{\tau_1}{\tau}}\right]. \tag{A11.9}$$

The integrations of the Black-Scholes pricing formula

$$\left\{\frac{H}{S(\tau_1)}\right\}^{2v/\sigma^2} C_{bs}\left[\frac{H^2}{S(\tau_1)}, K, \tau - \tau_1\right]$$

from $-d_{bs}(X,s)$ to ∞ and from $-d_{bs}(X,s)$ can be obtained accordingly

$$
\begin{aligned}
&\mathrm{e}^{-\tau_1 r}\int_{-d_{bs}}^{\infty}\left[\frac{H}{S(\tau_1)}\right]^{2v/\sigma^2} C_{bs}\left(\frac{H^2}{S}\mathrm{e}^{-v\tau_1-u\sigma\sqrt{\tau_1}},K,\tau-\tau_1\right)f(u)du\\
&=\left(\frac{H}{S}\right)^{2v/\sigma^2}\left\{\frac{H^2}{S}\mathrm{e}^{-g\tau}N_2\bigg[-d_{bs}(H,\tau_1)\right.\\
&\quad+2\left(\frac{\sigma^2+v}{\sigma}\right)\sqrt{\tau_1},-d_{1bs}(K,\tau),-\sqrt{\frac{\tau_1}{\tau}}\bigg]\\
&\quad-K\mathrm{e}^{-r\tau}N_2\bigg[-d_{bs}(S,H,\tau_1)+\frac{2v}{\sigma}\sqrt{\tau_1},-d_{bs}(S,K,y\tau),-\sqrt{\frac{\tau_1}{\tau}}\bigg].
\end{aligned}
\tag{A11.11}
$$

A11.2. THE DERIVATION OF THE UNIFIED DENSITY FUNCTION FOR EARLIER-ENDING BARRIER OPTIONS

Let $g(x,y)$ be the joint density function of the two random variables x and y. The two random variables x and y are independent because there is no overlapping time for the two variables, the joint density function $g(x,y)$ can be obtained as follows:

$$g(x,y)=[ENDN(x)]\mathrm{f}(y), \tag{A11.12}$$

where $ENDN(x)$ is the density function of the underlying asset price at the early-ending time given in (11.27) and (11.28), and $f(y)$ is given in (11.31).

Making the transformation $z=x+y$ and $z'=y$, or $x(z,z')=z-z'$ and $y(z,z')=z'$; and since the Jacobian of the transformation is always 1, we can find the joint density function for z and z'

$$\xi(z,z')=g[x(z,z'),y(z,z')]=[ENDN(z-z')]\mathrm{f}(z'). \tag{A11.13}$$

For a down-out ending ending barrier option, $\theta=\xi=1$. The density function of the underlying asset price at the option maturity z is the marginal density function of $\xi(z,z')$ for z. The marginal density function of $\xi(z,z')$ for z can be obtained by integrating $\xi(z,z')$ for all z'. Because $x>a$ implies $z'<z-a$, and $x<a$ implies $z'>z-a$, we can find the density function of the underlying asset price at the option maturity z by integrating

$$[ENDN(z-z')]f(z')=[f(z-z')-\mathrm{e}^{2va/\sigma^2}f(z-z'-2a)]f(z') \tag{A11.14}$$

from $-\infty$ to $z-a$. Carrying out the integration yields

$$\int_{-\infty}^{z-a} f(z-z')f(z')dz' = f(z)N\left(\frac{z\tau_e - a\tau}{\sigma\sqrt{\tau\tau_e(\tau-\tau_e)}}\right), \qquad \text{(A11.15a)}$$

and

$$\int_{-\infty}^{z-a} f(z-z'-2a)f(z')dz' = f(z)N\left[\frac{z\tau_e - a(\tau-2\tau_e)}{\sigma\sqrt{\tau\tau_e(\tau-\tau_e)}}\right]. \qquad \text{(A11.15b)}$$

Substituting (A11.15a) and (A11.15b) into the integration of (A11.14) yields the density function of the underlying asset price at maturity for $\theta = \zeta = 1$ given in (11.33). The density function for the other three situations can be similarly obtained using the corresponding functions given in (11.27) and (11.28) and appropriate integration domains for z'.

A11.3. PROOF OF THE THREE IDENTITIES

A11.3.1. $N_2[A, B, 1] = M[\min(A, B)]$

Using the definition of the cumulative function of the bivariate normal distribution given in (A11.1), we can obtain

$$\begin{aligned}
N_2[A,B,1] &= \lim_{\rho\to 1}\{N_2[A,B,\rho]\} \\
&= \lim_{\rho\to 1}\int_{-\infty}^{\min(A,B)} f(s)N\{[\max(A,B)-\rho s]/\sqrt{1-\rho^2}\}ds \\
&= \int^{\min(A,B)} -\infty f(s)\left\{\lim_{\rho\to 1}[\max(A,B)-\rho s]/\sqrt{1-\rho^2}\right\}ds \\
&= \int_{-\infty}^{\min(A,B)} f(s)N(\infty)ds \\
&= \int_{-\infty}^{\min(A,B)} f(s)ds = N[\min(A,B)].
\end{aligned}$$

11.3.2. $N_2(A, B, -1) = 0, A + B \leq 0$

Similarly, we obtain the following:

$$\begin{aligned}
N_2(A,B,-1) &= \lim_{\rho\to 1-1}\left\{\int_{-\infty}^{a} f(u)N\left[(B-\rho u)/\sqrt{1-\rho^2 du}\right]du\right\} \\
&= \lim_{\rho\to 1-1}\left\{\int_{-\infty}^{a} f(u)N\left[(B+\theta u)/\sqrt{1-\theta^2}\right]du\right\}.
\end{aligned}$$

As $u < A, B + u \leq A + B = A + B \leq 0, (B + \theta u)/\sqrt{1-\theta^2} = (A + B)/\sqrt{1-\theta^2} \to -\infty$, thus, $N\left\{(B + \theta u)/\sqrt{1-\theta^2}\right\} du = 0$. Therefore

$$N_2(A, B, -1) = \int_{-\infty}^{a} f(u) N\left[(B + \theta u)/\sqrt{1-\theta^2}\right] du = 0\,.$$

A11.3.3. $N_2(A, B, -1) = N[\max(A, B)] - N[-\min(A, B)]$, for $A + B > 0$

$$N_2(A, B, \rho) = \int_{\min}^{\max} f(z) N\left\{[\min(A, B) - \rho z]/\sqrt{1-\rho^2}\right\} dz$$

$$= \int_{\min}^{\max} f(z) N\left\{\left[\min(A, B) - \rho z/\sqrt{1-\rho^2}\right]/\sqrt{1-\rho^2}\right\} dz$$

$$+ \int_{\min}^{\max} f(z) N\left\{[\min(A, B) - \rho z]/\sqrt{1-\rho^2}\right\},$$

$$N_2(A, B, -1) = \lim_{\rho \to -1}\left(\int_{-\infty}^{\min} f(z) N\left[(\min - \rho z)/\sqrt{1-\rho^2}\right] dz\right.$$

$$\left. + \int_{\max}^{\min} f(z) N\left[(\min - \rho z)/\sqrt{1-\rho^2}\right] dz\right)$$

$$= \lim_{\theta \to 1}\left(\int_{-\infty}^{\min} f(z) N\left[(\min + \theta z)/\sqrt{1-\theta^2}\right]\right.$$

$$\left. + \int_{\min}^{\max} f(z)\left[(\min + \theta z)/\sqrt{1-\theta^2}\right] dz\right)$$

$$= \lim_{\theta \to 1}\left(\int_{-\infty}^{\min} f(z) N\left[(\min + z)/\sqrt{1-\theta^2}\right] dz\right.$$

$$\left. + \int_{-\min}^{\max} f(z) N\left[(\min + z)/\sqrt{1-\theta^2}\right] dz\right)$$

$$= \int_{-\infty}^{-\min} f(z) N(-\infty)\, dz + \int_{-\min}^{\max} f(z) N(+\infty)\, dz$$

$$= 0 + \int_{-\min}^{\max} f(z) dz = N(\max) - N(-\min)\,,$$

where $\max = \max(A, B)$ and $\min = \min(A, B)$.

The above results can be proven alternatively. Since $\rho(X,Y) = -1, X = -Y$ must be true. Thus if $A+B>0, A>-B$,

$$\begin{aligned} N_2(A,B,-1) =& P[X \le A \& Y \le B] = P[X \le A \& -X \le B] \\ =& P[X \le A \& X \le -B] \\ =& P[-B \le X \le A] = N(A) - N(-B) \\ =& N[\max(A,B)] - N[-\min(A,B)] \,. \end{aligned}$$

A11.4. THE DERIVATION OF THE UNIFIED DENSITY FUNCTION OUTSIDE BARRIER OPTIONS

Since x and y are correlated with the correlation coefficient ρ, we cannot find the joint density function between x and y directly by multiplying the density functions of the two parts as in early-ending barrier options in (A11.12). However, this difficulty can be overcome. Let $v = (y - \upsilon_2\tau)/(\sigma_2\sqrt{\tau})$ stand for the standardized variable for y. As x and y are correlated with the correlation coefficient ρ, their corresponding standardized variables u and v are also correlated with the correlation coefficient ρ. It can be shown that the variable $z = u - \rho v$ is normally distributed and independent of v, and its density function is given as:

$$f(z) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[-\frac{(u-\rho v)^2}{2(1-\rho^2)}\right] . \tag{A11.16}$$

Since we know the distribution function of z in (A11.16) and the density function v in (11.56) and (11.57), we can find the joint density function between z and v by multiplying (A11.16) by (11.56) or (11.57) because z and v are independent:

$$\mathrm{J}(z,v) = f(z) ENDN(y,\zeta,\tau) . \tag{A11.17}$$

where $ENDN(y,\zeta,\tau)$ is the density function of the measurement asset price at maturity given in (11.56) if $\zeta = 1$, and in (11.57) if $\zeta = -1$, and f(z) is given in (A11.16).

The joint distribution between x and $y, G(x,y)$, can be obtained immediately from the joint distribution given in (A11.17):

$$\mathrm{G}(x,y) = \mathrm{J}[z(x,y),v] = f[z(x,y)] ENDN(y,\zeta,\tau) . \tag{A11.18}$$

The marginal density function of x can be obtained by integrating the joint density function $G(x, y)$ with appropriate ranges for y. In the case of a down-out barrier option, the integration domain for y is from a to $+\infty$ for the marginal density function $ENDN(y, \zeta, \tau)$. Integrating $G(x, y)$ given in (A11.18), we can find the density function of the underlying asset price at the option maturity z by integrating

$$f[z(x,y)ENDN(z-z') = f[z(x,y)]f(y) - e^{2v_2 a/\sigma_2^2} f[z(x,y)]f(y-2a)\,, \tag{A11.19}$$

from $-\infty$ to $z - a$. Carrying out the integration yields

$$\int_a^{+\infty} f[z(x,y)]f(y)dy = f(x)N\left\{\frac{[d_{bs}(M,H,\sigma_2)+\rho u]}{\sqrt{1-\rho^2}}\right\}, \tag{A11.20a}$$

and

$$\begin{aligned}&\int_a^{-\infty} f[z(x,y)]f(y-2a)dy \\ &= f\left(u - \frac{2\rho a}{\sigma_2\sqrt{\tau}}\right)N\left\{\frac{d_{bs}(M,H,\sigma_2)+\rho u}{\sqrt{1-\rho^2}} + \frac{2a}{\sigma_2\sqrt{\tau}}\sqrt{1-\rho^2}\right\}.\end{aligned} \tag{A11.20b}$$

Substituting (A11.20a) and (A11.20b) into the integration of (A11.19) yields the density function of the underlying asset price at maturity for $\theta = \zeta = 1$ given in (11.59). The density functions for the other three situations can be similarly obtained using the corresponding functions given in (11.56) and (11.57) and the appropriate integration domains for y.

A11.5. THE DERIVATION OF THE PRICING FORMULA WITH FOURIER SERIES

Following the similar procedure in deriving (11.66), we can integrate each component of (11.68). Integrating by part using some basic trigonometric function properties, we can obtain the following necessary indefinite integration

$$\int e^{\alpha y}\sin(y)dy = \frac{\alpha\sin(y) - \cos(y)}{1+\alpha^2}e^{\alpha y}\,, \tag{A11.21}$$

where α is any given real number. Making the integration substitution $y = nu\pi(x-b)/(a-b)$ and using the integration result given in (A11.21), we can obtain the pricing formula given in (11.69) after simplifications.

A11.6. THE DERIVATION OF THE PRESENT VALUES OF REBATES OF OUT-CORRIDOR OPTIONS

Making the same substitutions as to derive (10.48) shown in Appendix of Chapter 10, we obtain the following integration

$$SQ = \int_{1/\sqrt{\tau}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{y^2} \exp\left\{\frac{[(a-x_n')y - \psi(r-\eta)/y]^2}{2\sigma^2}\right\} dy\,. \qquad \text{(A11.22)}$$

Making the substitution $u = (a-x_n')y - \psi(r-\eta)/y$, we can solve for y in terms of u (choose the positive root if $x_n' < a$, and the negative root if $x_n' > a$):

$$y = \frac{1}{2(a-x_n')}\left\{u + \sqrt{u^2 \pm 4(a-x_n')\psi(r-\eta)}\right\}, \qquad \text{(A11.23a)}$$

therefore

$$dy = \frac{1}{2(a-x_n')}\left\{1 + \frac{u}{\sqrt{u^2 \pm 4(a-x_n')\psi(r-\eta)}}\right\} du\,. \qquad \text{(A11.23b)}$$

Substituting (A11.23) into (A11.22) yields

$$SQ = \frac{1}{2(a-x_n')}\int_{lowx'}^{\infty}\left[\frac{2(a-x_n')}{u+\sqrt{u^2 \pm 4(a-x_n')\psi(r-\eta)}}\right]^2 \times \left[1 + \frac{u}{\sqrt{u^2 \pm 4(a-x_n')\psi(r-\eta)}}\right] f(u)du\,, \qquad \text{(A11.24)}$$

where $lowz' = [a - x_n' - \tau\psi(r-\eta)]/(\sigma\sqrt{\tau})$ and $f(u)$ is the density function of a standard normal distribution.

Making the substitution $v = \sqrt{u^2 + 4(a-x_n')\psi(r-\eta)}$, we can find the lower bound for the integration given in (A11.24) $lowx' = lowvx' = [a - x_n' + r\psi(r-\eta)]/(\sigma\sqrt{\tau})$ and we can obtain the following integrations in order to carry out the integration in (A11.24):

$$\int_{lowvx'}^{\infty} v^2 f(u)dv = lowvx' f(lowvx') + N(-lowvx'), \quad \text{(A11.25a)}$$

$$\int_{lowvx'}^{\infty} \frac{u}{\sqrt{u^2 + 4(a - x_n')\psi(r - \eta)}} f(u)du = e^{2a\psi(r-\eta)/\sigma^2} N(-lowvx'), \quad \text{(A11.25b)}$$

$$\int_{lowvx'}^{\infty} u\sqrt{u^2 + 4(a - x_n')\psi(r - \eta)} f(u)du$$

$$= e^{2a\psi(r-\eta)/\sigma^2} [lowvx' f(lowvx') + N(lowvx')], \quad \text{(A11.25c)}$$

$$\int_{lowvx'}^{\infty} \frac{u^3}{\sqrt{u^2 + 4(a - x_n)\psi(r - \eta)}} f(u)du$$

$$= e^{2a\psi(r-\eta)/\sigma^2} \{lowvx' f(lowvx')$$

$$+ [1 - 4(a - x_n')\psi(r - \eta)] N(-lowvx')\}. \quad \text{(A11.25d)}$$

Substituting (A11.25b), (A11.25c), and (A11.25d) into (A11.24) completes the integration given in (A11.24). Using the results given in (A11.24) and the procedures to obtain (10.48) in Chapter 10, we can obtain (11.72) after simplifications.

A11.7. APPROXIMATING THE BIVARIATE NORMAL CUMULATIVE FUNCTION VALUES

In expressing prices of one-clique options in Chapter 9, forward-start barrier options, early-ending barrier options, outside barrier options in Chapter 11, and most correlation options in Part IV, we need to use the cumulative function of the standard bivariate normal distribution $N_2(a, b, \rho)$. Although these cumulative function values can be calculated using numerical double integration, the double integration is not very convenient to carry out. Following the idea to approximate the integration in the cumulative function of a standard univariate normal distribution in Appendix of Chapter 2, we try to introduce some ways to approximate the cummulative function of the standard bivariate normal distribution.

A11.7.1. To Express the Bivariate Function in Terms of Univariate Functions

Using the bivariate density function given in (A11.2), we can readily find the bivariate cumulative function (we leave this as an exercise):

$$N_2(a,b,\rho) = \int_{-\infty}^{a} f(u)N\Big\{(b-\rho u)/\sqrt{1-\rho^2}\Big\}du\,, \qquad \text{(A11.26)}$$

where $f(.)$ and $N(.)$ stand for the density and accumulative functions of the standard univariate normal distribution, respectively.

As $N(.)$ can be approximated readily using the method described in Appendix of Chapter 2, the expression given in (A11.26) can be obtained easily using univariate numerical integration.

A11.7.2. Drezner's Approximations

Hull (1993) corrected a typo in Drezner (1978) approximation. The approximation can be expressed in our algebra if $a < 0$, $b \leq 0$, and $-1 < \rho \leq 0$:

$$N_2(a,b,\rho) = 2(1-\rho^2)\sum_{i=1}^{4}\sum_{j=1}^{4} A_i A_j f(a,b)\mathrm{e}^{\rho B_i B_j + a(B_i - \rho B_j) + b(B_j - \rho B)}\,, \qquad \text{(A11.27)}$$

where

$$f(a) = \frac{1}{2\pi\sqrt{1-\rho^2}}\,\mathrm{e}^{-(a^2-2\rho ab+b^2)/[2(1-\rho^2)]}$$

is the density function of a standard bivariate normal distribution,

$$A_1 = 0.3253030,\ A_2 = 0.4211071,\ A_3 = 0.1334425,\ A_4 = 0.006374323,$$

$$B_1 = 0.1337764,\ B_2 = 0.6243247,\ B_3 = 1.3425378,\ B_4 = 2.2626645\,.$$

If the product of a, b, and ρ is negative or zero, one of the following identities can be used (see Appendix of Chapter 21 for the proofs):

$$N_2(a,b,\rho) = N(a) - N_2(a,-b,-\rho)\,, \qquad \text{(A11.28a)}$$

$$N_2(a,b,\rho) = N(b) - N_2(-a,b,-\rho)\,, \qquad \text{(A11.28b)}$$

and

$$N_2(a,b,\rho) = N(a) + N(b) - 1 + N_2(-a,-b,\rho)\,. \qquad \text{(A11.28b)}$$

If the product of a, b, and ρ is positive, the following identity can be used:

$$N_2(a,b,\rho) = N_2(a,0,-\rho_1) + N_2(0,b,-\rho_1) - \frac{1-\text{sgn}(a)\text{sgn}(b)}{4} \qquad \text{(A11.29)}$$

where

$$\rho_1 = \frac{(\rho a - b)\text{sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}, \ \rho_2 = \frac{(\rho b - a)\text{sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}},$$

and $\text{sgn}(x) = 1$ if $x \geq 0$ and -1 if $x < 0$.

Chapter 12

LOOKBACK OPTIONS

12.1. INTRODUCTION

A lookback option is an option with payoff determined not only by the settlement price but also by the maximum or minimum price of the underlying asset within the life of the option. Since a lookback option can yield the best possible payoff of the underlying asset can be achieved with a lookback option, it somehow minimizes the regret of investors. Lookback options have become attractive because they keep track of past events and allow their holders to take advantage of anticipated market movements without knowing the exact dates of their occurences. These options may also provide psychological comforts to holders by minimizing regrets.

There are several kinds of lookback options — floating strike lookback options, fixed strike lookback options, partial lookback options, American lookback options, and so on. The payoff of a floating strike lookback call option is the difference between the settlement price and the minimum price of the underlying asset within the option lifetime, and the payoff of a floating strike lookback put option is the difference between the maximum price of the underlying asset within the option lifetime and the settlement price of the underlying asset. Floating strike lookback options are true "no regret" options because they provide the largest possible payoffs for each type of options. Lookback options can somehow capture investors' fantasy of buying low, selling high, and minimize regrets, as Goldman, Sosin, and Gatto (1979) argued. However, the "no-free-lunch" principle guarantees that these options are expensive to buy. The high premiums of lookback options prevent them from being widely used.

A fixed strike lookback option is similar to a vanilla option in which the underlying price at maturity is replaced with the maximum or the minimum of the underlying asset price within the life of the option. The payoff of a fixed strike lookback call (resp. put) option is the difference between the maximum price of the underlying asset within the life of the option and the fixed strike price (resp. the difference between the fixed strike and the minimum price).

To overcome the limitation of high premiums of standard lookback options, partial lookback options came into existence. Partial lookback options or fractional lookback options are similar to standard lookback options but only a percentage of the extreme values are in effect, or the extremal values are monitored during a subset of the lives of the options, thus keeping the memory of the extreme values in standard lookback options whereas making them less expansive. All the lookback options discussed above are European-style, and most often lookback options are of the above two kinds. However, these lookback options can also be American, or they can be exercised before maturity.

Following the seminal work of Goldman, Sosin, and Gatto (1979) who first studied European-style lookback options, there have been several studies on lookback options. Garman (1989) extended lookback options to currency options and discussed possible applications of currency lookback options. Conze and Viswanathan (1991) analyzed European-style, American-style, and both European and American partial lookback options. They also connected their approach with Merton's (1973) study of "down-and-out" options. More recently, Heynen and Kat (1994) discussed a new type of partial lookback options in which the monitored period is only a subset of the lives of the options. In this chapter, we will provide pricing formulas for floating strike, fixed strike, and partial lookback options within a Black-Scholes environment.

12.2. DISTRIBUTIONS OF EXTREME VALUES

Since all kinds of lookback options depend on the maximum or the minimum values of the underlying asset prices, we need to use the two variables defined in (10.11) and (10.12). They are actually the maximum and minimum of all underlying asset prices within the life of an option. Figures 12.1 and 12.2 depict how the maximum and minimum values change with time. The maximum (resp. minimum) value is somewhat similar to the cumulative function of a standard normal distribution because the maximum value is also somewhat "cumulative" in the sense that the maximum (resp. minimum) value remains unchanged even if the new observation of the underlying asset price is smaller (resp. larger).

To price lookback options, we need the distributions of the maximum and the minimum values. The density functions of these extreme values can

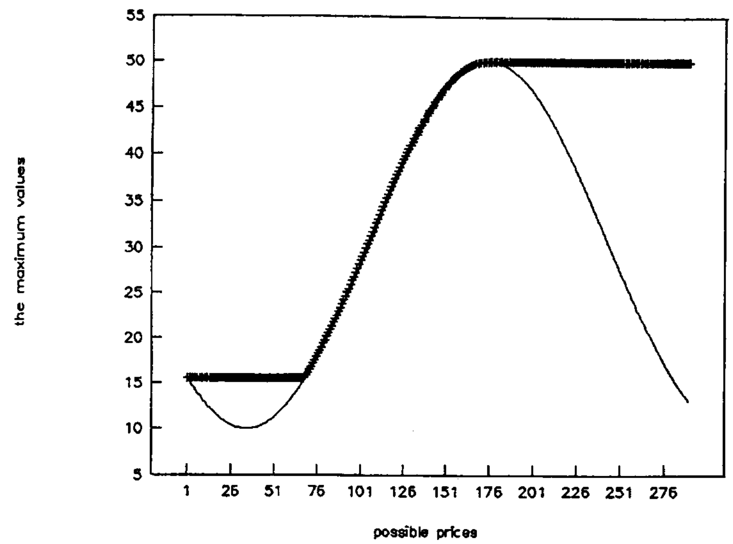

Fig. 12.1. Distribution of maximum prices.

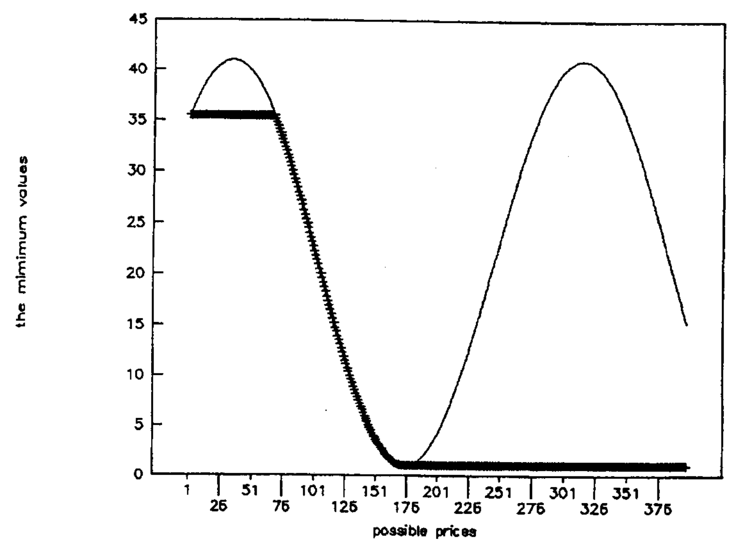

Fig. 12.2. Distribution of minimum prices.

be derived from the joint cumulative functions given in (10.17) and (10.23). For $y \geq 0$, the cumulative function of the log-return of the maximum value Y_T can be obtained by sustituting $x = y$ into the joint cumulative function between the maximum value and the log-return of the underlying asset given in (10.17)

$$P(Y_\tau \le y) = N\Big(\frac{y - v\tau}{\sigma\sqrt{\tau}}\Big) - e^{2yv/\sigma^2} N\Big(\frac{-y - v\tau}{\sigma\sqrt{\tau}}\Big). \tag{12.1}$$

Similarly, the cumulative function of the log-return of the minimum value y_τ can be obtained by sustituting $x = y$ into the joint cumulative function between the minimum value and the log-return of the underlying asset given in (10.23) for $y \le 0$,

$$P(y_\tau \ge y) = N\Big(\frac{-y + v\tau}{\sigma\sqrt{\tau}}\Big) - e^{2yv/\sigma^2} N\Big(\frac{y + v\tau}{\sigma\sqrt{\tau}}\Big). \tag{12.2}$$

The density functions of the maximum and the minimum can thus be obtained by taking the first-order derivative to (12.1) and (12.2) with respect to y, respectively:

$$\begin{aligned} g_{\max}(y) &= \frac{1}{\sigma\sqrt{\tau}} f\Big(\frac{y - v\tau}{\sigma\sqrt{\tau}}\Big) - \frac{2v}{\sigma^2} e^{2yv/\sigma^2} N\Big(-\frac{y + v\tau}{\sigma\sqrt{\tau}}\Big) \\ &\quad + \frac{1}{\sigma\sqrt{\tau}} e^{2yv/\sigma^2} f\Big(\frac{y + v\tau}{\sigma\sqrt{\tau}}\Big), \end{aligned} \tag{12.3}$$

for $y \ge 0$ and

$$\begin{aligned} g_{\min}(y) &= \frac{1}{\sigma\sqrt{\tau}} f\Big(\frac{y - v\tau}{\sigma\sqrt{\tau}}\Big) + \frac{2v}{\sigma^2} e^{2yv/\sigma^2} N\Big(\frac{y + v\tau}{\sigma\sqrt{\tau}}\Big) \\ &\quad + \frac{1}{\sigma\sqrt{\tau}} e^{2yv/\sigma^2} f\Big(\frac{y + v\tau}{\sigma\sqrt{\tau}}\Big), \end{aligned} \tag{12.4}$$

for $y \le 0$.

With these density functions, we can price lookback options in the following sections.

12.3. FLOATING STRIKE LOOKBACK OPTIONS

In their seminal work, Goldman, Sosin, and Gatto (1979) studied the hedging and valuation of lookback options with floating strike prices prespecified as either the maximum or the minimum of the underlying asset prices depending on whether the options are calls or puts. They illustrated that a put option on the maximum ($P_{\max}$) and a call option on the minimum ($C_{\min}$) can be perfectly hedged, and provided closed-form valuation expressions for these options without considering the payout of the underlying asset. Conze and Viswanathan (1991) did not consider the payout of the underlying asset either. We will generalize the floating strike European lookback options in this section.

Assume that the underlying asset price is distributed as in the extended Black-Scholes model with the payout rate of the underlying asset g given in (3.1). The solution of the underlying asset price using the initial condition $S(t) = S$ is given in (5.3). As their name implies, floating strike lookback options are options with floating strike prices. The payoff of a European call option on the minimum (PC_{min}) of the underlying asset price within the life of the option can be formally expressed as

$$PC_{\text{min}} = \max[S(t^*) - m_t^{t^*},\ 0] = S(t^*) - m_t^{t^*}\ , \tag{12.5}$$

where t and t^* are the current and maturity time of the option, respectively, and $m_t^{t^*}$ stands for the minimum value of the underlying asset price from t to t^* (both t and t^* are included) given in (10.12).

Similarly, the payoff of a European put option on the maximum (PP_{max}) of the underlying asset price within the life of the option can be formally expressed as

$$PP_{\text{max}} = \max[M_t^{t^*} - S(t^*),\ 0] = S(t^*) - M_t^{t^*}\ , \tag{12.6}$$

where $M_t^{t^*}$ stands for the maximum value of the underlying asset price from t to t^* (both t and t^* are included) given in (10.11) and others are the same as in (12.5).

In the payout expressions in (12.5) and (12.6), the minimum value $m_t^{t^*}$ and the maximum value $M_t^{t^*}$ are in the same positions as the strike prices in the payout of a standard European call option given in (2.1) and that of a standard European put option given in (2.2), respectively. That is why the European-style lookback options described in (12.5) and (12.6) are called floating strike lookback options because these extreme values $m_t^{t^*}$ and $M_t^{t^*}$ are not fixed as the strike prices in (2.1) and (2.2). The payoffs of floating strike lookback options given in (12.5) and (12.6) are the greatest possible payoffs of a call option and a put option based on historical events within the lives of the options. Thus, floating lookback options are true "no regret" options.

Using the density function for the minimum value of the underlying asset price within the life of the option given in (12.4), we can obtain the price of a European floating strike lookback call option (C_{min}) for $r \neq g$:[1]

[1]There are two terms in the expectation of the minimum of the underlying asset price because $E(m_t^{t^*}) = E[\min(m_\tau^0, m_0^\tau)] = m_\tau^0 \text{Prob}(m_t^{t^*} \geq m_\tau^0) + E(m_t^{t^*} | m_t^{t^*} < m_\tau^0)$.

$$
\begin{aligned}
C_{\min} &= \mathrm{e}^{-r\tau} E[S(t^*) - m_t^{t^*}] = \mathrm{e}^{-r\tau}\{E[S(t^*)] - E(m_t^{t^*})\} \\
&= C_{bs}(S, m_{\tau 1}^0) + \frac{S\sigma^2}{2(r-g)}\Big\{ - \mathrm{e}^{-g\tau} N\Big[- d_{bs1}(S, m_{\tau 1}^0)\Big] \\
&\quad + \mathrm{e}^{-r\tau}\left(\frac{S}{m_{\tau 1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\Big[d_{bs}(m_{\tau 1}^0, S)\Big]\Big\}, \qquad (12.7)
\end{aligned}
$$

where $C_{bs}(S, m_{\tau 1}^0)$ is the extended Black-Scholes pricing formula for a call option ($\omega = 1$) in (10.31) with strike price $K = m_{\tau 1}^0$, and $m_{\tau 1}^0$ is the current minimum value of the underlying asset price or the minimum price of the underlying asset from the initiation time of the option $\tau 1$ to the present.

Formula (12.7) involves one factor that has not appeared earlier in this book: the current minimum price of the underlying asset or the minimum price of the underlying asset from time $\tau 1$ to the present. The current minimum price depends on how far back in the past or how long the option has been valid. If $\tau 1 = 0$, the current minimum price is the same as the current spot price S. The further back in the past, the smaller the value of the current minimum price. The pricing formula in (12.7) indicates that the price of a European floating strike lookback call option $C_{\min}$ is always greater than the corresponding vanilla call option price.

Example 12.1. Find the price of a floating lookback call option to expire in one year, given the spot price \$100, volatility of the underlying stock 12%, yield on the underlying stock 2%, interest rate 8%, and the minimum value of the underlying asset \$95.

Substituting $S = \$100$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, $m_{\tau 1}^0 = \$95$ into (12.7) yields

$$
\begin{aligned}
d_{bs}(S, m_{\tau 1}^0) &= \frac{\ln(S/m_{\tau 1}^0) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
&= \frac{\ln(100/95) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}} = 0.8674, \\
d_{bs1}(S, m_{\tau 1}^0) &= d_{bs}(S, m_{\tau 1}^0) + \sigma\sqrt{\tau} = 0.8674 + 0.12 \times 1 = 0.9874, \\
d_{bs}(m_{\tau 1}^0, S) &= \frac{\ln(m_{\tau 1}^0/S) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
&= \frac{\ln(95/100) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}} = 0.0126, \\
d_{bs1}\left(m_{\tau 1}^0, S\right) &= d_{bs}\left(m_{\tau 1}^0, S\right) + \sigma\sqrt{\tau} = 0.0126 + 0.12 \times 1 = 0.1326,
\end{aligned}
$$

$$\begin{aligned}C_{\min} &= Se^{-g\tau}N\Big[d_{bs1}\Big(S, m_{\tau1}^0\Big)\Big] - m_{\tau1}^0 e^{-r\tau}N\Big[d_{bs}\Big(S, m_{\tau1}^0\Big)\Big] \\ &\quad + \frac{S\sigma^2}{2(r-g)}\left\{e^{-r\tau}\left(\frac{S}{m_{\tau1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\Big[d_{bs}\Big(m_{\tau1}^0, S\Big)\Big]\right. \\ &\quad \left. - e^{-g\tau}N\Big[-d_{bs1}\Big(S, m_{\tau1}^0\Big)\Big]\right\} \\ &= 100e^{-0.02\times1}N[d_{bs1}(100, 95)] - 95e^{-0.08\times1}N[d_{bs}(100, 95))] \\ &\quad + \frac{100\times 0.12^2}{2(0.08-0.02}\left[e^{-0.08\times1}\left(\frac{100}{95}\right)^{-\frac{2(0.06)}{0.12^2}} N(0.0126)\right. \\ &\quad \left. - e^{-0.02\times1}N(-0.9874)\right] \\ &= 100e^{-0.02\times1}N(0.9874) - 95e^{-0.08\times1}N(0.8674) \\ &\quad + \frac{1.44}{0.12}\left[e^{-0.08\times1}\left(\frac{20}{19}\right)^{-\frac{0.12}{0.12^2}} N(0.0126) - e^{-0.02\times1}N(-0.9874)\right] \\ &= \$13.120\,.\end{aligned}$$

The pricing formula given in (12.7) is for $r \neq g$. When $r = g$, it can not be used. For most applications in fixed-income derivatives, both r and g are set to be zero. Fortunately, we can find the corresponding formula for $r = g$ using (12.7) (see Exercise 12.17):

$$C_{\min}(r=g) = C_{bs}(S, m_{\tau1}^0) + \frac{\sigma}{\sqrt{\tau}}\, Se^{-r\tau} f[d_{bs1}(S, m_{\tau_1}^0)]\,, \tag{12.8}$$

where $f(.)$ is the density function for the standard normal distribution, and all parameters are the same as in (12.7).

Similarly, we can obtain the price of a European floating strike lookback put option $P_{\max}$ using the density function for the maximum given in (12.3):

$$\begin{aligned}P_{\max} &= e^{-r\tau}E\Big[M_t^{t^*} - S(t^*)\Big] = e^{-r\tau}E\Big(M_t^{t^*}\Big) - E[S(t^*)] \\ &= P_{bs}\Big(S, M_{\tau1}^0\Big) + \frac{S\sigma^2}{2(r-g)}\left\{e^{-g\tau}N\Big[d_{bs1}\Big(S, M_{\tau1}^0\Big)\Big]\right. \\ &\quad \left. -e^{-r\tau}\left(\frac{S}{M_{\tau1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\left[-d_{bs}\left(M_{\tau1}^0, S\right)\right]\right\}\,,\end{aligned} \tag{12.9}$$

where $P_{bs}(S, M_{\tau1}^0)$ is the extended Black-Scholes pricing formula for a put option ($\omega = -1$) in (10.31) with strike price $K = M_{\tau1}^0$, and $M_{\tau1}^0$ is the current maximum value of the underlying asset price or the maximum price of the underlying asset from $\tau 1$ to the present.

Example 12.2. Find the price of the corresponding floating lookback put option with the maximum value of the underlying asset \$110 and other parameters the same as in Example 10.1.

Substituting $S = \$100$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, $M_{\tau 1}^0 = \$110$ into (12.9) yields

$$d_{bs}\left(S, M_{\tau 1}^0\right) = \frac{\ln(100/110) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}} = 0.3543\,,$$

$$d_{bs1}\left(S, M_{\tau 1}^0\right) = d_{bs}\left(S, M_{\tau 1}^0\right) + \sigma\sqrt{\tau} = -0.3543 + 0.12 \times 1 = -0.2343\,,$$

$$d_{bs}\left(M_{\tau 1}^0, S\right) = \frac{\ln(110/100) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}} = 1.2343\,,$$

$$d_{bs1}\left(M_{\tau 1}^0, S\right) = d_{bs}\left(M_{\tau 1}^0, S\right) + \sigma\sqrt{\tau} = 0.2343 + 0.12 \times 1 = 1.3543\,,$$

$$\begin{aligned} P_{\max} &= -Se^{-g\tau}N\left[-d_{bs1}\left(S, M_{\tau 1}^0\right)\right] - M_{\tau 1}^0 e^{r\tau}N\left[-d_{bs}\left(S, M_{\tau 1}^0\right)\right] \\ &\quad + \frac{S\sigma^2}{2(r-g)}\left\{e^{-g\tau}N\left[d_{bs1}\left(S, M_{\tau 1}^0\right)\right]\right. \\ &\quad \left. - e^{-r\tau}\left(\frac{S}{M_{\tau 1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\left[-d_{bs}\left(M_{\tau 1}^0, S\right)\right]\right\} \\ &= -100e^{-0.02\times 1}N[-d_{bs1}(100, 110)] + 110e^{-0.08\times 1}N[-d_{bs}(100, 110)] \\ &\quad + \frac{100 \times 0.12^2}{2(0.08 - 0.02)}\left\{e^{-0.02\times 1}N\left[d_{bs1}(100, 110)\right]\right. \\ &\quad \left. - e^{-0.08\times 1}\left(\frac{100}{110}\right)^{-\frac{2(0.06)}{0.12^2}} N\left[-d_{bs}(110, 100)\right]\right\} \\ &= \$10.748\,. \end{aligned}$$

The pricing formula floating strike lookback put options given in (12.9) is valid for $r \neq g$. When $r = g$, we cannot use it directly. However, we can find the corresponding pricing formula for $r = g$ using (12.9) (see Exercise 12.9):

$$P_{\max}(r = g) = P(S, m_{\tau 1}^0) + \frac{\sigma}{\sqrt{\tau}}\, Se^{-r\tau}f[d_{bs1}(S, m_{\tau_1}^0)]\,, \qquad (12.10)$$

where $f(.)$ is the density function of the standard normal distribution and other parameters are the same as in (12.9).

12.4. FIXED STRIKE LOOKBACK OPTIONS

We studied floating strike lookback options in the previous section. Unlike floating strike lookback options with strike prices specified as the extreme values of the underlying asset prices within the lives of options, fixed strike lookback options are options with fixed strike prices and the underlying asset prices at maturity are substituted with the maximum or minimum of the underlying asset prices. More specifically, the payoff of a European lookback call option with a fixed strike K (PLCK) is given as:

$$PLCK = \max\left(M_t^{t^*} - K,\ 0\right), \tag{12.11}$$

where $M_t^{t^*}$ is the maximum value of the underlying asset price from t to t^* given in (10.11).

Similarly, the payoff of a European lookback put option with a fixed strike K (PLPK) is given as

$$PLPK = \max\left(K - m_t^{t^*},\ 0\right), \tag{12.12}$$

where $m_t^{t^*}$ stands for the minimum value of the underlying asset price from t to t^* given in (10.12).

Using the density function for the maximum value of the underlying asset price within the life of the option given in (12.3), we can obtain the price of a European fixed strike lookback call option (LCK) for $r \neq g$:

$$\begin{aligned} LCK &= \mathrm{e}^{-r\tau} E[\max(M_t^{t^*} - K,\ 0)] \\ &= C_{bs}(S, K) + \frac{S\sigma^2}{2(r-g)}\left\{\mathrm{e}^{-g\tau} N[d_{bs1}(S, K)] \right. \\ &\quad \left. - \mathrm{e}^{-r\tau}\left(\frac{S}{K}\right)^{-\frac{2(r-g)}{\sigma^2}} N[-d_{bs}(K, S)]\right\} \end{aligned} \tag{12.13}$$

in the case of $K \geq M_{\tau 1}^0$, where $C_{bs}(S, K)$ is the extended Black-Scholes pricing formula for a call option ($\omega = 1$) given in (10.31) with spot price S and strike price K, and

$$\begin{aligned} LCK &= \mathrm{e}^{-r\tau} E[\max(M_t^{t^*} - K,\ 0)] \\ &= \mathrm{e}^{-r\tau}(M_{\tau 1}^0 - K) + C_{bs}(S, M_{\tau 1}^0) + \frac{S\sigma^2}{2(r-g)}\left\{\mathrm{e}^{-g\tau} N\left[d_{bs1}\left(S, M_{\tau 1}^0\right)\right] \right. \\ &\quad \left. - \mathrm{e}^{-r\tau}\left(\frac{S}{M_{\tau 1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\left[-d_{bs}\left(M_{\tau 1}^0, S\right)\right]\right\}, \end{aligned} \tag{12.14}$$

when $K < M^0_{\tau 1}$, where $C_{bs}(S, M^0_{\tau 1})$ is the extended Black-Scholes pricing formula for a call option ($\omega = 1$) given in (10.31) with spot price S and strike price $K = M^0_{\tau 1}$.

Example 12.3. Find the prices of the fixed strike call options to expire in one year with strike prices \$100 and \$110, given the spot price \$100, volatility of the underlying stock 12%, yield on the underlying stock 2%, interest rate 8%, and the maximum of the underlying asset price \$105.

Substituting $S = K = \$100$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the maximum of the underlying asset $M^0_{\tau 1} = \$105$ into (12.14) yields

$$d_{bs}\left(S, M^0_{\tau 1}\right) = d_{bs}\left(K, M^0_{\tau 1}\right) = \frac{\ln(100/105) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}}$$
$$= 0.0334\,,$$
$$d_{bs1}\left(S, M^0_{\tau 1}\right) = d_{bs1}\left(S, M^0_{\tau 1}\right) = d_{bs}\left(S, M^0_{\tau 1}\right) + \sigma\sqrt{\tau} = 0.0334 + 0.12 \times 1$$
$$= 0.1534\,,$$
$$d_{bs}\left(M^0_{\tau 1}, S\right) = \frac{\ln(105/100) + (0.08 - 0.02 - 0.12^2/2)1}{0.12\sqrt{1}} = 0.8466\,,$$

$$LCK(K < M^0_{\tau 1})$$
$$= e^{-r\tau}(M^0_{\tau 1} - K) + Se^{-g\tau}N[d_{bs1}(S, M^0_{\tau 1})] - M^0_{\tau 1}e^{-r\tau}N[d_{bs}(S, M^0_{\tau 1})]$$
$$+\frac{S\sigma^2}{2(r-g)}\left\{e^{-g\tau}N\left[d_{bs1}\left(S, M^0_{\tau 1}\right)\right] - e^{-r\tau}\left(\frac{S}{M^0_{\tau 1}}\right)^{-\frac{2(r-g)}{\sigma^2}}N\left[-d_{bs}\left(M^0_{\tau 1}, S\right)\right]\right\}$$
$$= e^{-0.08\times 1}(105 - 100) + 100e^{-0.02\times 1}N(0.1534) - 105e^{-0.08\times 1}N(0.0334)$$
$$+\frac{100 \times 0.12^2}{2(0.08 - 0.02)}\left[e^{-0.02\times 1}N(0.1534) - e^{-0.08\times 1}\left(\frac{100}{105}\right)^{-\frac{2(0.06)}{0.12^2}}N(-0.8466)\right]$$
$$= 19.668\,.$$

Substituting $S = \$100$, $K = \$110$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the maximum of the underlying asset $M^0_{\tau 1} = \$105$ into (12.14) yields

$$LCK(K > M^0_{\tau 1})$$
$$= Se^{-g\tau}N[d_{bs1}(S, K)] - Ke^{-r\tau}N[d_{bs}(S, K)]$$
$$+\frac{S\sigma^2}{2(r-g)}\left\{e^{-g\tau}N\left[d_{bs1}(S, K)\right] - e^{-r\tau}\left(\frac{S}{K}\right)^{-\frac{2(r-g)}{\sigma^2}}N[-d_{bs}(K, S)]\right\}$$

$$= 100e^{-0.02\times1}N(-0.3543) - 110e^{-0.08\times1}N(-0.2343)$$
$$+\frac{100\times0.12^2}{2(0.08-0.02)}\left[e^{-0.02\times1}N(-0.3543) - e^{-0.08\times1}\left(\frac{100}{110}\right)^{-\frac{2(0.06)}{0.12^2}}N(-1.2343)\right]$$
$$= \$1.624\,.$$

The pricing formulas given in (12.13) and (12.14) are for $r \neq g$. We can find the corresponding formulas for $r = g$ as follows [see Exercise 12.7]:

$$LCK(r=g) = C_{bs}(S,K) + \frac{\sigma}{\sqrt{\tau}}\,Se^{-r\tau}f[d_{bs1}(S,K)]\,, \tag{12.15}$$

in the case $K \geq m^0_{\tau1}$, and

$$LCK(r=g) = e^{-r\tau}(m^0_{\tau1}, -K) + C_{bs}(S, m^0_{\tau1}) + \frac{\sigma}{\sqrt{\tau}}\,Se^{-r\tau}f[d_{bs1}(S, m^0_{\tau1})]\,, \tag{12.16}$$

when $K < m^0_{\tau1}$, and all parameters are the same as in (12.13) and (12.14).

Similarly, we can obtain the price of a European fixed strike lookback put option (LPK):

$$\begin{aligned} LPK &= e^{-r\tau}E[\max(K - m_t^{t^*},\ 0)] \\ &= P_{bs}(S,K) + \frac{S\sigma^2}{2(r-g)}\Bigg\{ - e^{-g\tau}N[-d_{bs1}(S,K)] \\ &\quad + e^{-r\tau}\left(\frac{S}{K}\right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(K,S)]\Bigg\}\,, \end{aligned} \tag{12.17}$$

for $K < m^0_{\tau1}$, where $P_{bs}(K)$ is the extended Black-Scholes pricing formula for a put option ($\omega = -1$) given in (10.31) with strike price K, and

$$\begin{aligned} LPK &= e^{-r\tau}E[\max(K - m_t^{t^*},\ 0)] \\ &= e^{-r\tau}(K - m^0_{\tau1}) + P_{bs}(S, m^0_{\tau1}) + \frac{S\sigma^2}{2(r-g)}\Bigg\{ - e^{-g\tau}N[-d_{bs1}(S, m^0_{\tau1})] \\ &\quad + e^{-r\tau}\left(\frac{S}{m^0_{\tau1}}\right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(m^0_{\tau1}, S)]\Bigg\}\,, \end{aligned} \tag{12.18}$$

for $K > m^0_{\tau1}$, where all parameters are the same as in (12.17).

Example 12.4. Find the prices of the fixed strike put options to expire in one year with strike prices \$90 and \$110, given the spot price \$100, volatility of the underlying stock 12%, yield on the underlying stock 2%, interest rate 8%, and the minimum of the underlying asset price \$95.

Substituting $S = \$100$, $K = \$90$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the minimum of the underlying asset $m^0_{\tau 1} = \$95$ into (10.17) yields

$$
\begin{aligned}
&LPK(K < m^0_{\tau 1}) \\
&= -Se^{-g\tau}N[-d_{bs1}(S,K)] + Ke^{-r\tau}N[-d_{bs}(S,K)] \\
&\quad + \frac{S\sigma^2}{2(r-g)}\left\{ -e^{-g\tau}N[-d_{bs1}(S,K)] + e^{-r\tau}\left(\frac{S}{K}\right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(K,S)]\right\} \\
&= -100e^{-0.02\times 1}N(-1.438) + 90e^{-0.08\times 1}N(-1.318) \\
&\quad + \frac{100\times 0.12^2}{2(0.08-0.02)}\left[-e^{-0.02\times 1}N(-1.438) + e^{-0.08\times 1}\left(\frac{100}{90}\right)^{-\frac{2(0.06)}{0.12^2}} N(-0.438)\right] \\
&= \$16.093\,,
\end{aligned}
$$

and substituting $S = \$100$, $K = \$110$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the minimum of the underlying asset $m^0_{\tau 1} = \$95$ into (12.18) yields

$$
\begin{aligned}
&LPK(K > m^0_{\tau 1} \\
&= e^{-r\tau}(K - m^0_{\tau 1}) - Se^{-g\tau}N[-d_{bs1}(S, m^0_{\tau 1})] + m^0_{\tau 1}e^{-r\tau}N[-d_{bs}(S, m^0_{\tau 1})] \\
&\quad + \frac{S\sigma^2}{2(r-g)}\left\{ -e^{-g\tau}N[-d_{bs1}(S, m^0_{\tau 1})] + e^{-r\tau}\left(\frac{S}{m^0_{\tau 1}}\right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(m^0_{\tau 1}, S)]\right\} \\
&= e^{-0.08\times 1}(110-95) - 100e^{-0.02\times 1}N(-0.9874) + 95e^{-0.08\times 1}N(-0.8674) \\
&\quad + \frac{100\times 0.12^2}{2(0.08-0.02)}\left[-e^{-0.02\times 1}N(-0.9874) + e^{-0.08\times 1}\left(\frac{100}{95}\right)^{-\frac{2(0.06)}{0.12^2}} N(-0.0126)\right] \\
&= \$26.906\,.
\end{aligned}
$$

The pricing formulas given in (12.17) and (12.18) are for $r \neq g$ (Readers may find the corresponding pricing formulas for $r = g$ by comparing (12.8), (12.10), (12.15), and (12.16) with their corresponding formulas (12.7), (12.9), (12.13), and (12.14), respectively. We will leave this as an Exercise.

12.5. "PARTIAL LOOKBACK" OPTIONS

"Partial lookback" or "fractional lookback" options are designed to reduce the high premiums of standard lookback options. They are similar to standard lookback options and yet remain cheaper than them. In this section, we will explain how "partial lookback" options are possibly designed and how they are priced in a Black-Scholes environment. We will first consider partial floating strike lookback options and then discuss partial fixed strike lookback options.

Similar to the payoff of a European floating strike lookback call option in (12.5), the payoff of a European partial floating strike lookback call option ($PPC_{\min}$) can be expressed as

$$PPC_{\min} = \max[S(t^*) - \lambda m_t^{t^*},\ 0]\,, \tag{12.19}$$

where $\lambda \geq 1$ is a constant representing the degree of partiality.

Similar to the payoff of a European floating strike lookback put option in (12.6), the payoff of a European partial floating strike lookback put option ($PPP_{\max}$) can be expressed as

$$PPP_{\max} = \max[\lambda M_t^{t^*} - S(t^*),\ 0]\,, \tag{12.20}$$

where $0 < \lambda \leq 1$ is a constant representing the degree of partiality.

It is obvious that if the constant partiality parameter $\lambda = 1$, the payoffs in (12.19) and (12.20) become exactly the same as the payoffs of standard floating strike lookback options given in (12.5) and (12.6), respectively. The greater (resp. smaller) the value of the partiality parameter λ in (12.19) [resp. (12.20)], the smaller the payoffs of the partial lookback call (resp. put) options in (12.19) [resp. (12.20)], and in turn the lower their expected payoffs. Therefore, the smaller (greater) a partiality parameter λ implies a higher (lower) price of the partial lookback options.

Using the density function for the minimum value of the underlying asset price within the life of the option given in (10.10), we can obtain the price of a partial floating strike lookback call option $PC_{\min}$ (the derivation is rather long because many steps are involved in the integration process, so we simply provide the results here. Interested readers may check the proof in Appendix of this chapter) for $r \neq g$:

$$\begin{aligned} PC_{\min} &= e^{-r\tau} E[S(t^*) - \lambda m_t^{t^*}] \\ &= C_{bs}(S, \lambda m_{\tau 1}^0) + \frac{\lambda S \sigma^2}{2(r-g)} \left\{ - e^{-g\tau} (\lambda)^{\frac{2(r-g)}{\sigma^2}} N[-d_{bs1}(\lambda S, m_{\tau 1}^0)] \right. \\ &\quad \left. + e^{-r\tau} \left(\frac{S}{m_{\tau 1}^0} \right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(m_{\tau 1}^0, \lambda S)] \right\}, \end{aligned} \tag{12.21}$$

where all other parameters are the same as in (10.14).

Substituting $\lambda = 1$ into (12.21), we can readily find that the partial lookback call option pricing formula becomes exactly the same as the pricing formula for a standard floating lookback call option given in (12.7). Thus, the pricing formula in (12.21) indicates that standard floating lookback call options are special cases of partial lookback options when the partiality parameter $\lambda = 1$.

Example 12.5. Find the price of the partial floating lookback call option to expire in one year with the partial parameter $\lambda = 1.1$, given the spot price \$100, volatility of the underlying stock 12%, yield on the underlying stock 2%, interest rate 8%, and the minimum of the underlying asset price \$95.

Substituting $S = \$100$, $\lambda = 1.1$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the minimum of the underlying asset $m_{\tau 1}^0 = \$95$ into (12.21) yields

$$\begin{aligned}
PC_{\min} &= e^{-r\tau} E[S(t^*) - \lambda m_t^{t^*}] \\
&= Se^{-g\tau} N[d_{bs1}(S, \lambda m_{\tau 1}^0)] - \lambda m_{\tau 1}^0 e^{-r\tau} N[d_{bs}(S, \lambda m_{\tau 1}^0)] \\
&\quad + \frac{\lambda S \sigma^2}{2(r-g)} \left\{ -e^{-g\tau} (\lambda)^{\frac{2(r-g)}{\sigma^2}} N[-d_{bs1}(\lambda S, m_{\tau 1}^0)] \right. \\
&\quad \left. + e^{-r\tau} \left(\frac{S}{m_{\tau 1}^0} \right)^{-\frac{2(r-g)}{\sigma^2}} N[d_{bs}(m_{\tau 1}^0, \lambda S)] \right\} \\
&= 100e^{-0.02\times 1} N(0.1932) - 1.10 \times 95e^{-0.08\times 1} N(0.0732) \\
&\quad + \frac{1.1 \times 100 \times 0.12^2}{2(0.08 - 0.02)} \left[-e^{-0.02\times 1} (1.1)^{\frac{2(0.06)}{0.12^2}} N(-1.7817) \right. \\
&\quad \left. + e^{-0.08\times 1} \left(\frac{100}{95} \right)^{-\frac{2(0.06)}{0.12^2}} N(0.7817) \right] \\
&= \$10.621\,,
\end{aligned}$$

which is obviously lower than the standard floating strike call option price \$13.120 in Example 12.1.

Similarly, we can obtaine the price of a partial floating strike lookback put option $PP_{\max}$:

$$\begin{aligned}
PP_{\max} &= e^{-r\tau} E[\lambda M_t^{t^*} - S(t^*)] \\
&= P_{bs}(S, \lambda M_{\tau 1}^0) + \frac{\lambda S \sigma^2}{2(r-g)} \left\{ e^{-g\tau} (\lambda)^{\frac{2(r-g)}{\sigma^2}} N[d_{bs1}(\lambda S, M_{\tau 1}^0)] \right. \\
&\quad \left. - e^{-r\tau} \left(\frac{S}{M_{\tau 1}^0} \right)^{-\frac{2(r-g)}{\sigma^2}} N[-d_{bs}(M_{\tau 1}^0, \lambda S)] \right\}, \qquad (12.22)
\end{aligned}$$

where all parameters are the same as in (12.8).

Example 12.6. Find the price of the partial floating lookback put option to expire in one year with the partial parameter $\lambda = 0.9$, given the spot price \$100, volatility of the underlying stock 12%, yield on the underlying stock 2%, interest rate 8%, and the maximum of the underlying asset price \$110.

Substituting $S = \$100$, $\lambda = 0.9$, $\tau = 1$, $\sigma = 0.12$, $g = 0.02$, $r = 0.08$, and the minimum of the underlying asset $M_{\tau 1}^{0} = \$110$ into (12.22) yields

$$\begin{aligned} PP_{\max}(\lambda = 0.9) &= -Se^{-g\tau}N[d_{bs1}(S, \lambda M_{\tau 1}^{0})] - \lambda M_{\tau 1}^{0} e^{-r\tau} N[d_{bs}(S, \lambda M_{\tau 1}^{0})] \\ &\quad + \frac{\lambda S\sigma^2}{2(r-g)}\left\{ e^{-g\tau}(\lambda)^{\frac{2(r-g)}{\sigma^2}} N[d_{bs1}(\lambda S, M_{\tau 1}^{0})] \right. \\ &\quad \left. - e^{-r\tau}\left(\frac{S}{M_{\tau 1}^{0}}\right)^{-\frac{2(r-g)}{\sigma^2}} N[-d_{bs}(M_{\tau 1}^{0}, \lambda S)] \right\} \\ &= -100\,e^{-0.02\times 1}N(0.6438) - 0.9 \times 110e^{-0.08\times 1}N(0.5238) \\ &\quad + \frac{0.9 \times 100 \times 0.12^2}{2(0.08 - 0.02)}\left[e^{-0.02\times 1}(0.9)^{\frac{2(0.06)}{0.12^2}} N(-1.1123) \right. \\ &\quad \left. - e^{-0.08\times 1}\left(\frac{100}{110}\right)^{-\frac{2(0.06)}{0.12^2}} N(-2.1123) \right] \\ &= \$8.886\,, \end{aligned}$$

which is obviously lower than the standard floating strike call option price \$10.748 in Example 12.2.

The pricing formulas in (12.21) and (12.22) are for "partial lookback" floating strike options. The corresponding results for "partial lookback" fixed strike options are easier to obtain. For example, the payoff of a partial fixed strike lookback call option can be given as

$$PLCK = \max(\lambda M_t^{t^*} - K,\ 0)\,, \tag{12.23}$$

where $0 < \lambda \leq 1$ is the same partiality parameter as in partial floating strike lookback options studied above and other parameters are the same as in (12.9).

Using the density function for the maximum value of the underlying asset price within the life of the option given in (12.3), we can obtaine the price of a European "partial lookback" fixed strike call option (PLCK) for

$r \neq g$:

$$\begin{aligned} PLCK &= \mathrm{e}^{-r\tau} E[\max(\lambda M_t^{t^*} - K,\ 0)] \\ &= \lambda \mathrm{e}^{-r\tau} E[\max(M_t^{t^*} - K/\lambda,\ 0)] \\ &= \lambda C_{bs}(S, K/\lambda) + \frac{\lambda S \sigma^2}{2(r-g)} \left\{ \mathrm{e}^{-g\tau} N[d_{bs1}(S, K/\lambda)] \right. \\ &\quad \left. - \mathrm{e}^{-r\tau} \left(\frac{\lambda S}{K}\right)^{-\frac{2(r-g)}{\sigma^2}} N[-d_{bs}(K/\lambda, S)] \right\} \end{aligned} \tag{12.24}$$

in the case of $K \geq \lambda M_{\tau 1}^0$, where $C_{bs}(S, K/\lambda)$ is the extended Black-Scholes pricing formula for a call option ($\omega = 1$) given in (10.31) with spot price S and strike price K/λ, and

$$\begin{aligned} PLCK &= \mathrm{e}^{-r\tau} E[\max(\lambda M_t^{t^*} - K,\ 0)] \\ &= \mathrm{e}^{-r\tau}(\lambda M_{\tau 1}^0 - K) + \lambda C_{bs}(S/\lambda, M_{\tau 1}^0) \\ &\quad + \frac{\lambda S \sigma^2}{2(r-g)} \left\{ \mathrm{e}^{-g\tau} N\left[d_{bs1}\left(\frac{S}{\lambda},\ M_{\tau 1}^0\right)\right] \right. \\ &\quad \left. - \mathrm{e}^{-r\tau} \left(\frac{S}{\lambda M_{\tau 1}^0}\right)^{-\frac{2(r-g)}{\sigma^2}} N\left[-d_{bs}\left(M_{\tau 1}^0,\ \frac{S}{\lambda}\right)\right] \right\}, \end{aligned} \tag{12.25}$$

for $K < \lambda M_{\tau 1}^0$.

It is easy to check that the pricing formula of a fixed strike lookback option is a special case of (12.25) when $\lambda = 1$. The corresponding formula for a partial lookback fixed strike put option is given as an exercise of this chapter.

We can have only provided pricing formulas for partial lookback options for $r \neq g$ in this section. Their corresponding expression with $r = g$ can be obtained by taking limit as we did in previous sections of this chapter. We will leave this as an exercise.

12.6. "PARTIAL" VS "FULL" LOOKBACK OPTIONS

The partial lookback options discussed in the previous section can also be called "fractional" lookback options beause only part of the extreme values are in effect in the payoff functions of the options. Heynen and Kat (1994b) discussed and analyzed partial lookback options which are "partial" because the extreme values of the underlying asset prices are within a subset of the lives of the options. More specifically, the partial lookback options studied by Heynen and Kat are lookback options with payoffs determined by the extreme values of the underlying asset prices with the observation period specified during the early part of the lives of the options. When the observation period is the same as the time to maturity of the options, these partial lookback options become the standard lookback options or full lookback options. It is not difficult to price such partial lookback options because we can use the correlation coefficients between the underlying asset prices and the extreme values of the underlying asset within the observation period, as given in Proposition 5.1. Although the idea is straightforward, it takes a lot of space to cover such pricing formulas. We choose not to list these results and interested readers may go to Heynen and Kat (1994b). We will leave this issue to price partial lookback options as an Exercise by the end of this chapter.

The partial lookback options studied by Heynen and Kat are similar to early-ending barrier options because the extreme values are monitored from the current time to some time before maturity. These partial lookback options can be extended to those with monitoring periods to start some time in the future within the lives of the options as in forward-start barrier options. In the general case, we may borrow the idea of windows from window barrier options to structure partial lookback options with monitoring periods as subperiods within the lives of the options. We may call these partial lookback options window lookback options. Window lookback options can be similarly priced with the techniques developed in Chapters 10 and 11 for barrier options and in the previous sections in this chapter.

12.7. AMERICAN LOOKBACK OPTIONS

Holders of American options have the right to exercise their options prior to their maturity. The American character can be implemented to all European lookback options so far covered in this chapter. There are floating strike American lookback options, fixed strike American lookback options, and partial lookback American options. Holders of American lookback options have the right to exercise their options prior to their maturity as standard American options. They would exercise their floating lookback put (resp. call) options earlier if they believe the recent high (resp. low) price of the underlying asset is not to repeat, since there will be no incentive to wait longer if the extrema value is believed not to repeat until the maturity of the options. Therefore, American lookback options can better capture what investors desire to achieve, since they have the flexibility of being exercised at the optimal time. American lookback options are more expensive than their corresponding European lookback options, just as vanilla American options are more expensive than the corresponding European options.

As we discussed in Chapter 4, there is no closed-form solution for standard American options. It is thus generally impossible to obtain closed-form solutions for American lookback options. The binomial tree method can be used to price American lookback options. However, if we can find reasonably tight bounds for American options, it would be very useful.[2] Conze and Viswanathan (1991) provided bounds for American-style standard lookback, fixed strike lookback, and partial lookback options. We will not go too far in this direction.

12.8. SUMMARY AND CONCLUSIONS

We have studied floating strike lookback options, fixed strike lookback options, partial floating strike lookback options, and partial fixed strike lookback options in this chapter. Floating strike lookback options are true "no regret" options because they can provide the largest payoffs with either call or put options. Yet, their obvious shortcoming is that they are generally rather expensive. To reduce the high premiums of floating strike lookback options, partial lookback options are designed to cut the payoffs of either floating strike or fixed strike lookback options. Partial lookback options remain attractive in the sense that their payoffs are still connected to the extreme values of the underlying asset prices, yet their premiums can be significantly lower than their corresponding standard lookback options with approperiate partial parameters.

To some extent, lookback options are true path-dependent options because their values depend on the maximum or the minimum values of the underlying assets within the lives of the options which in turn depend on the whole paths of the underlying asset prices.

QUESTIONS AND EXERCISES

Questions

12.1. What are lookback options?

12.2. How many types of popular lookback options are there?

[2]Zhang (1993) studied bounds for vanilla option prices. Zhang (1994b) provided bounds for vanilla option prices by elminating some assumptions of the Black–Scholes model listed in Chapter 2. Broadie and Detemple (1995) provided bounds for American options. Interested readers may find useful literature in this area.

12.3. What are the important differences between a floating lookback and a fixed lookback options?

12.4. What are partial lookback options?

12.5. Why are lookback options generally expensive?

12.6. Why are floating strike lookback options "no regret" options?

12.7. What are "fractional" lookback options? Why are they popular in the market?

12.8. What are window lookback options? Why are they atractive to many buyers?

12.9. In order to price lookback options in the case of $r = g$, can we simply use the results of the corresponding formulas of $r \neq g$? Why?

12.10. Is it possible to combine "fractional" lookback options and window lookback options? How?

Exercises

12.1. Find the floating strike lookback call option price if the minimum value is changed to \$90 and other parameters remain unchanged as in Example 12.1.

12.2. Find the floating strike lookback call option price if the time to maturity is changed to 9 months and other parameters remain unchanged as in Example 10.1.

12.3. Find the floating strike lookback call option price if the time to maturity is changed to 9 months and the minimum value is changed to \$90 and other parameters remain unchanged as in Example 12.1.

12.4. Find the floating strike lookback put option price if the maximum value is changed to \$105 and other parameters remain unchanged as in Example 12.2.

12.5. Find the floating strike lookback put option price if the maximum value is changed to \$115 and other parameters remain unchanged as in Example 12.2.

12.6. Find the price of the fixed strike lookback call option on the S&P-500 Index to expire in half a year with strike price \$540, given the volatility of the Index 15%, interest rate 8.5%, yield on the Index is 3%, the spot Index is \$535, and the current minimum level of the Index is \$510.

12.7. Find the price of the fixed strike lookback call option if the minimum level of the S&P Index is changed to \$500 and other parameters remain unchanged as in Exercise 12.6.

12.8. Find the price of the fixed strike lookback put option on the S&P 500 Index to expire in half a year with strike price \$540, given the volatility of the Index 15%, interest rate 8.5%, yield on the Index is 3%, the spot Index is \$535, and the current maximum level of the index is \$542.

12.9. Find the price of the fixed strike lookback put option if the maximum level of the Index is changed to \$545 and other parameters remain unchanged as in Exercise 12.8.

12.10. Find the price of the partial floating lookback call option on the US dollar/Japanese yen exchange rate to expire in three months with the partial parameter $\lambda = 1.08$, given the spot exchange rate is ¥84.5 per dollar, the US interest rate 7.5%, the Japanese interest rate 3%, the volatility of the dollar-yen exchange rate is 20%, and the highest level of dollar-yen rate is ¥88 per dollar (remenber the exchange rate has to be converted into US dollar per yen).

12.11. Find the price of the corresponding partial floating lookback put option in Exercise 12.10 given the lowest level of the dollar-yen exchange rate ¥80.50 per dollar.

12.12. Show that the pricing formulas given in (12.13) and (12.14) for fixed strike lookback options are special cases of the pricing formulas for partial lookback fixed strike options given in (12.24) and (12.25) when $\lambda = 1$.

12.13. Find the price of the partial lookback fixed strike call option on IBM stock with strike \$85 to expire in nine months, given the partial parameter $\lambda = 0.96$, spot price \$82, the maximum price of the stock \$87, interest rate 8.25%, yield on IBM stock 3.5%, and the volatility of the stock 18%.

12.14. Answer the same question in Exercise 12.13 for $\lambda = 0.98$ and 1, respectively.

12.15. Find the corresponding pricing formulas for partial lookback fixed strike put options.

12.16. Find the price of the corresponding partial lookback fixed strike put option in Exercise 12.13 using the pricing formula obtained in Exercise 10.15 if the partial parameter is $\lambda = 1.04$ and the minimum stock price is \$78.

12.17.* Show that (12.7) becomes (12.8) for $r = g$ [Hint: taking limit to Equation (12.7) for $r - g = 0$].

12.18. Find the price of the corresponding floating lookback call option with $r = g = 8\%$, other parameters remain unchanged as in Example 12.1

12.19.* Show that Equation (12.19) becomes (12.20) for $r = g$.

12.20. Find the price of the corresponding floating lookback put option with $r = g = 8\%$, other parameters remain unchanged as in Example 12.2.

12.21. Find the prices of the corresponding fixed-strike lookback call options in Example 12.3 with $r = g = 8\%$, and other parameters are the same as in Example 12.3.

12.22. Find the corresponding pricing formulas for (12.17) and (12.18) with $r = g$.

12.23.* Find the corresponding pricing formulas for (12.21), (12.22), (12.24), and (12.25) with $r = g$.

12.24.* Find the corresponding pricing formulas for early-ending lookback options with early-ending time $0 \leq \tau_e \leq \tau$.

APPENDIX

The partiality parameter $\lambda > 1$ for partial lookback options makes the derivation more difficult than $\lambda = 1$ for floating strike lookback options because in the latter case the payoff $S(t^*) - m_t^{t^*}$ is always positive so that the expected payoff of a floating strike lookback call option can be obtained simply by taking the expectation of the two terms using the density function. However, $S(t^*) - \lambda m_t^{t^*}$ may be negative for some $\lambda > 1$, therefore the payoff of a partial lookback call option $E[S(t^*) - \lambda m_t^{t^*},\ 0]$ cannot be obtained directly following the procedure for floating lookback options. Double integration has to be carried out because both the spot price at maturity and the minimum value change over time.

The joint cumulative distribution function between the log-return of the underlying asset and the log-return of the minimum value y_τ is given as follows [see Harrison (1985), p. 13 for a proof]:

$$F(X_\tau > x,\ y_\tau \geq y) = N\left(\frac{-x + v\tau}{\sigma\sqrt{\tau}}\right) - e^{2yv/\sigma^2} N\left(\frac{-x + 2y + v\tau}{\sigma\sqrt{\tau}}\right), \quad \text{(A12.1)}$$

where $v = r - g - \sigma^2/2$. The joint density function between the log-return of the underlying asset and the log-return of the minimum value $\nu(x, y)$ can be obtained by differentiating the cumulative function given in (A12.1) twice with respect to the two variables x and y, respectively

$$\nu(x,y) = \frac{2(x-2y)}{(\sigma\sqrt{\tau})^3} e^{2yv/\sigma^2} f\left(\frac{x-2y-v\tau}{\sigma\sqrt{\tau}}\right). \qquad \text{(A12.2)}$$

Using the joint density function in (A12.2), we can show the following after simplifications:

$$\begin{aligned} I_1(y) &= \int_{y+\ln\lambda}^{\infty} \nu(x,y)dx \\ &= 2e^{2yv/\sigma^2}\left[\frac{v}{\sigma^2}N\left(\frac{y-\ln\lambda+v\tau}{\sigma\sqrt{\tau}}\right) + \frac{1}{\sigma\sqrt{\tau}}f\left(\frac{u-\ln\lambda+v\tau}{\sigma\sqrt{\tau}}\right)\right] \end{aligned} \qquad \text{(A12.3)}$$

and

$$\begin{aligned} I_2(y) &= \int_{y+\ln\lambda}^{\infty} e^{x}\nu(x,y)dx \\ &= 2e^{2yv/\sigma^2}\left\{\left(\frac{v+\sigma^2}{\sigma^2}\right)N\left[\frac{y-\ln\lambda+(v+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right]\right. \\ &\quad \left. + \frac{1}{\sigma\sqrt{\tau}}f\left[\frac{y-\ln\lambda+(v+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right]\right\}. \end{aligned} \qquad \text{(A12.4)}$$

The expected payoff of the partial lookback call option $PPC_{\min}$ in (12.15) can be alternatively expressed as:

$$\begin{aligned} E(PPC_{\min}) &= E(PPC_{\min}|m_t^{t^*} > m_{\tau 1}^0) + E(PPC_{\min}|m_t^{t^*} \le m_{\tau 1}^0) \\ &= E\{max[S(t^*) - \lambda m_{\tau 1}^0],\ 0|m_t^{t^*} > m_{\tau 1}^0\} \\ &\quad + E(PPC_{\min}|m_t^{t^*} \le m_{\tau 1}^0). \end{aligned} \qquad \text{(A12.5)}$$

The first part on the right-hand side of (A12.5) is actually the expected payoff of a down-out call option with the lower barrier $H = m_{\tau 1}^0 < S$, the strike price $K = \lambda m_{\tau 1}^0 > H = m_{\tau 1}^0$, and the rebate zero. The price of this down-out call (PDOUTC) option can be found immediately from (10.44):

$$PDOUTC = C_{bs}(S,\ \lambda_{\tau 1}^0) - \left(\frac{m_{\tau 1}^0}{S}\right)^{2v/\sigma^2} C_{bs}\left[\frac{(m_{\tau 1}^0)^2}{S},\ K\right], \qquad \text{(A12.6)}$$

where $C_{bs}(S,K)$ stands for the Black-Scholes call option pricing formula given in (10.31) with $\omega = 1$.

With the two integrations given in (A12.3) and (A12.4), we can obtain the second part of (A12.5) when $m_t^{t^*} \le m_{\tau 1}^0$:

$$
\begin{aligned}
E(PPC_{\min}|m_t^{t^*} \le m_{\tau 1}^0) &= \int_{-\infty}^{a} [S\mathrm{e}^x - \lambda S\mathrm{e}^y]\nu(x,y)dxdy \\
&= S\int_{-\infty}^{a} [\mathrm{e}^x - \lambda \mathrm{e}^y]\nu(x,y)dxdy \\
&= S\int_{-\infty}^{a} I_1(y)dy - \lambda S\int_{-\infty}^{a} \mathrm{e}^y I_2(y)dy\,,
\end{aligned} \tag{A12.7}
$$

where $a = \ln(m_{\tau 1}^0/S)$ and $I_1(y)$ and $I_2(y)$ are given in (A12.3) and (A12.4), respectively.

Substituting (A12.3) and (A12.4) into (A12.7) and carrying out the necessary integrations in (A12.7), we obtain the expected payoff of the lookback option when $m_t^{t^*} \le m_{\tau 1}^0$ after simplifications:

$$
\begin{aligned}
&E(PPC_{\min}|m_t^{t^*} \le m_{\tau 1}^0) \\
&\quad = \mathrm{e}^{(r-g)\tau}\left(\frac{m_{\tau 1}^0}{S}\right)^{\frac{2(r-g)}{\sigma^2}+1} N\left[d_{bs1}(m_{\tau 1}^0, S) - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right] \\
&\quad - \lambda\left[1 - \frac{\sigma^2}{2(r-g)}\right]\left(\frac{M_{\tau 1}^0}{S}\right)^{\frac{2(r-g)}{\sigma^2}} N\left[d_{bs}(m_{\tau 1}^0, S) - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right] \\
&\quad - \frac{\sigma^2}{2(r-g)}\mathrm{e}^{(r-g)\tau}(\lambda)^{\frac{2(r-g)}{\sigma^2}+1} N\left[-d_{bs1}(S, m_{\tau 1}^0) - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right].
\end{aligned} \tag{A12.8}
$$

Substituting (A12.8) into (A12.5) and discounting (A12.5) by the risk-free rate of return r yields the partial lookback call option price given in (12.17) after simplifications.

The corresponding formula for a partial lookback put option can be obtained following a similar approach using the joint-cumulative function between the log-return of the underlying asset and the log-return of the maximum value X_τ given in (10.23).

PART IV:

CORRELATION/MULTIASSETS OPTIONS

INTRODUCTION AND ORGANIZATION

There are two trends in financial markets: cross-market integration and globalization. The first trend has stimulated the growing development of cross-market products and the other has accelerated investment across national boundaries around the globe. The New York Stock Exchange (NYSE) has been considering trading foreign stocks in different currencies in an effort to maintain its prestige in the global marketplace. These efforts are being made to prevent the market from becoming a regional exchange in a global marketplace. The National Association of Securities Dealers Automated Quotations (NASDAQ), the largest stock market in the world measured by dollar trading volume (it surpassed that of NYSE for the first time in history in 1994), is also taking efforts in this direction. With further development of cross-market integration and globalization, the need to hedge cross-market and global positions will certainly increase. Various correlation options and other kinds of cross-market products have been created to meet this need.

Correlation options are also called multiassets options. Generally speaking, correlation options are options with payoffs affected by at least two underlying instruments. These instruments can be assets such as stocks, bonds, currencies, commodities, indices such as S&P-100, S&P-500, Nikkei 225, exchange rates, and so on. The instruments can be of the same or of different asset classes. If the two underlying assets are from different asset classes, the correlation option is often called a cross-asset option. Since there are two or more underlying assets which determine the values of correlation options, the correlation coefficients among these assets play an important role in pricing these options besides all the underlying assets involved. Because of the important role of the correlation coefficients, options with at least two underlying instruments are called correlation options. We will analyze most of the popular correlation options existing in the OTC marketplace and show how to price and use them in this part.

Correlation options can be divided into first- and second-order, according to the ways correlation affects option payoffs. Correlation has first-order or primary effects if it directly influences option payoffs as in spread options and out-performance options. Thus spread options and out-performance options are first-order correlation options. Quanto options are second-order correlation options because correlation merely modifies the option payoffs here. An option can reflect both first- and second-order correlation effects. Take, for example, an out-performance option on the DAX and CAC-40 denominated in Sterling. The first-order effect is on the covariance of the indices. The second-order effect comes from the degree of relationship between movements in both of these indices (and their covariance) and changes in the French franc/Sterling and German mark/Sterling exchange rates.

Part IV covers 14 chapters. The order of the chapters largely follows the degree of the complexity of the products. Chapter 13 introduces and prices exchange options and finds an alternative method to rate investment choices using the exchange option pricing fromula. Chapter 14 discusses options paying the best/worst of two risky assets and cash. Chapter 15 reviews standard digital options and introduces and prices correlation digital options which include the standard digital options as special cases. Chapter 16 studies ratio options or quotient options. Chapter 17 covers product options and prices foreign equity options with domestic strikes using the product option pricing formula. Chapter 18 introduces options on the maximum or minimum of two or more than two underlying assets. Chapter 19 studies quanto options. Chapter 20 prices foreign equity options. Chapter 21 discusses spread options. Chapter 22 covers options written on the spread between two rainbows, i.e., the maximum and minimum of two asset prices. Chapter 23 deals with out-performance options. Chapter 24 studies alternative options. Chapter 25 prices dual-strike options. Chapter 26 points out the limitations of using constant correlation coefficients in pricing all correlation options and estimates the errors of constant correlation coefficients when they are non-deterministic.

CORRELATION COEFFICIENTS

Before we start to introduce correlation options, it is necessary for us to review some statistical concepts involved in all correlation options — covariance and correlation coefficients. To have a better understanding of covariance and correlation coefficients, we need to review the concepts of means and standard deviations. A mean or mathematical expectation for a continuous random variable is defined as an integral of the product of the variable and its corresponding density function over a given domain

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

and its corresponding variance is defined as the integral of the product of the square of the difference between the variable and its mean and the corresponding density function

$$Var(X) = E\{[x - E(X)]^2\} = \int_{-\infty}^{\infty} [x - E(X)]^2 f(x) dx .$$

Variance as defined above is always positive for any random variable. It is always zero for any deterministic variable. The standard deviation of a random variable is simply the square-root of its variance. It is most often represented by the Greek letter σ. Thus, we can obtain the standard deviation of any random variable: $\sigma = \sqrt{Var(X)}$, and we can obtain the variance by squaring the standard deviation: $Var(X) = \sigma^2$.

The covariance between two random variables X and Y is defined as follows:

$$\begin{aligned} Cov(X,\ Y) &= E[X - E(X)][Y - E(Y)]\} \\ &= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} [x - E(X)][y - E(Y)] g(x, y) dx dy , \end{aligned}$$

where $g(x, y)$ represents the joint density function of the two variables X and Y.

Covariance as defined above is somewhat similar to variance. If the two variables X and Y are the same, the above covariance becomes exactly the variance of X or Y. Thus, we can consider variance as a special case of covariance. The covariance defined above can be shown to be equivalent to

$$Cov(X,\ Y) = E[XY] - [E(X)][E(Y)] ,$$

which implies that the covariance between two random variables equals the difference of the mean of the product of the two variables and the product of the means of the two random variables.

Covariance between any two random variables can be either positive, zero, or negative because the mean of the product of two variables can be either greater than, equal to, or smaller than the product of the two means, depending on the degree of dependence between the two variables.

With the above definition of covariance between any two random variables, we can now define the correlation coefficient ρ between any two random variables X and Y

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{Cov(X, Y)}{\sigma_x \sigma_y}.$$

As defined above, the correlation coefficient between any two random variables is the ratio of their covariance and the product of the two standard deviations. It can be either positive, zero, or negative because covariance can be either positive, zero, or negative. If it is zero, we say that the two random variables are stochastically independent. If positive (resp. negative), we say that the two random variables are positively (resp. negatively) stochastically dependent. The correlation coefficient between any two random variables is always between negative one and positive one, or $-1 \leq \rho \leq 1$. If it is positive (resp. negative) one, we say that the two random variables are positively (resp. negatively) perfectly correlated.

CORRELATION COEFFICIENTS AND COINTEGRATION

Francis Galton (1822–1911), an English anthropologist and eugenist, is generally regarded as the founder of correlation analysis. The concept of correlation was developed in late nineteenth century. Alexander and Johnson (1994) argued that cointegration, a form of dynamic correlation, is likely to have some advantages over the statistical correlation coefficient, or unconditional correlation coefficient. Whereas their argument is true because cointegration captures the time factor for various observations and the standard statistical correlation coefficient does not, we will still focus on the unconditional correlation coefficient because, to my best knowledge, all existing financial theories and all correlation option pricing models are based on the unconditional correlation coefficient instead of cointegration.

BIVARIATE NORMAL DISTRIBUTION

With the above review on correlation coefficients between two random variables, we can introduce the density function between two underlying risky instruments. The density function is used to price all correlation options in Part IV and some options in Part V. There are two instruments involved in most popular correlation options. In a Black-Scholes environment, the two asset prices are assumed to be bivariate lognormally distributed. Specifically, suppose that the two instruments are I_1 and I_2, both following the standard geometric Brownian motion as in (3.1):

$$dI_i = (\mu_i - g_i)I_i dt + \sigma_i I_i dz_i(t),\ i = 1 \text{ and } 2\,, \tag{IV1}$$

where $z_i(t)$, and $i = 1$ and 2 are two standard Gauss-Wiener processes with the correlation coefficient ρ, μ_i and σ_i stand for the instantaneous mean and standard deviation of the two assets or indices, respectively, and g_i is the payout rate of the ith underlying asset.

In a risk-neutral world, the stochastic process in (IV1) will become

$$dI_i = (r - g_i)I_i dt + \sigma_i I_i dz_i(t),\ i = 1 \text{ and } 2\,.$$

Solving the two equations given in (IV1) using the standard method (see Appendix at the end of Chapter 2) yields:

$$I_i(\tau) = I_i \exp[(\mu_i - g_i - \frac{1}{2}\sigma_i^2)\tau + \sigma_i z_i(\tau)],\ i = 1 \text{ and } 2\,, \tag{IV2}$$

where $\tau = t^* - t$, and t^* are the current and the expiration time of the option, respectively, and I_1 and I_2 represent the current prices of the two assets.

Let $x = \ln[I_1(\tau)/I_1]$ and $y = \ln[I_2(\tau)/I_2]$. It can be proven that both x and y are normally distributed with means $\mu_x = (-g_1 - \sigma_1^2/2)\tau$ and $\mu_2 - g_2 - \sigma_2^2/2)\tau$ and variances $\sigma_x^2 = \sigma_1^2\tau$ and $\sigma_y^2 = \sigma_2^2\tau$, respectively. It can also be shown that x and y are joint normally distributed with the correlation coefficient ρ. The joint density function can be expressed as follows:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right], \tag{IV3}$$

where

$$u = \frac{x - \mu_x}{\sigma_x} \text{ and } v = \frac{y - \mu_y}{\sigma_y}\,.$$

The bivariate density function in (IV3) can be expressed alternatively:

$$f(x,y) = f(y)f(x|y)\,, \tag{IV4}$$

where

$$f(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right),$$

$$f(x|y) = \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{(u-\rho v)^2}{2(1-\rho^2)}\right],$$

and u and v are the same as in (IV3).

The bivariate density function $f(x, y)$ in (IV3) can also be expressed

$$f(x,y) = f(x)f(y|x)\,, \qquad \text{(IV5)}$$

where

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right),$$

$$f(y|x) = \frac{1}{\sigma_y\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left[-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right],$$

and u and v are the same as in (IV3).

The bivariate density functions in (IV4) and (IV5) are used to derive pricing formulas for nearly all popular correlation options in a Black-Scholes environment in this book. The choice in using (IV4) or (IV5) depends on the specific integration order involved in each kind of correlation options. If u is integrated first, we use (IV4), otherwise (IV5) is used. We will refer to these two expressions of the bivariate density function in every chapter in Part IV.

Rubinstein (1994) described the difficulties in pricing American correlation options and provided some general steps to approach the problems using binomial trees. We will largely concentrate on European-style correlation options because of the transparency of these products. The closed-form solutions for European-style correlation options can be used as control variables for their corresponding American options.

Chapter 13

EXCHANGE OPTIONS

13.1. INTRODUCTION

Exchange options are options which give their holders the right to exchange one asset for another. The holder of an exchange option is entitled to receive at maturity one underlying asset in return for paying for the other underlying asset. Exchange options are the basic type of correlation options because many other correlation options can be transferred into exchange options and therefore analyzed in terms of them. Although these options were studied by William Margrabe (1978), their applications were not studied until many years later. In general, there is no distinction between a call option and a put option for exchange options. However, a standard exchange option can be interpreted as a call option on asset one with the strike price the same as the future price of asset two at the option maturity, or as a put option on asset two with the strike price the same as the future price of asset one at maturity.

Since exchange options are the basic type of correlation options and they are also the simplest one, we start Part IV with exchange options.

13.2. EXCHANGE OPTIONS

Assume that the two underlying asset prices $I_1(\tau)$ and $I_2(\tau)$ follow the stochastic process given in (IV1) and the returns of the assets are correlated with the correlation coefficient ρ. The payoff of a European-style exchange option to pay the second asset in exchange for the first can be expressed:

$$PFEX = \max\,[I_1(\tau) - I_2(\tau),\ 0]\,, \tag{13.1}$$

where max (. , .) is a function that gives the larger of two numbers.

The payoff in (13.1) is called the payoff of the exchange option to exchange the second asset for the first. It is the same as that of a vanilla call option in (2.1) if we substitute the strike price K with the second asset price $I_2(\tau)$, therefore the exchange option can be considered as a call option written on the first asset with the strike price the same as the future price of the second asset. The payoff in (13.1) is also the same as that of a vanilla put option in (2.2) if we consider the future price of the first asset as the strike price, thus the exchange option can also be considered as a put option with the strike price the same as the future price of the first asset.

The payoff of the exchange option given in (13.1) can be alternatively expressed as follows:

$$\max\left[I_1(\tau) - I_2(\tau),\ 0\right] = \max\left[I_1(\tau), I_2(\tau)\right] - I_2(\tau)\,, \tag{13.2}$$

or

$$\max\left[I_1(\tau) - I_2(\tau),\ 0\right] = I_1(\tau) - \min\left[I_1(\tau), I_2(\tau)\right]. \tag{13.3}$$

The two expressions in (13.2) and (13.3) will be used to price options written on the better or worse of two underlying risky assets in Chapter 21.

13.3. PRICING EXCHANGE OPTIONS

As we will show in the following chapters, the complexity of correlation options depends on that of their integration domains to calculate their expected payoffs in a Black-Scholes environment. Exchange options are the simplest correlation option because their integration domain is the simplest in shape. If the first and the second asset prices are the horizontal and vertical axes in a two-dimensional plane, respectively, the integration domain, or the area where an exchange option has a positive payoff, is simply the area below the forty-five degree line starting from the origin. The integration domain for exchange options to exchange the second asset for the first is given in Figure 13.1. Using the density functions given in (IV4) and (IV5), we can obtain the expected payoff of the exchange option in (13.1) by double integration:

$$E(PFEX) = I_1 e^{(\mu_1 - g_1)\tau} A_{e1} - I_2 e^{(\mu_2 - g_2)\tau} A_{e2}\,, \tag{13.4}$$

where

$$A_{e1} = \int_{-\infty}^{\infty} f(u - \sigma_x) N\left\{\frac{\ln(I_1/I_2) + (\mu_x - \mu_y)]/\sigma_y - [\rho - (\sigma_x/\sigma_y)]u}{\sqrt{1-\rho^2}}\right\} du\,, \tag{13.4a}$$

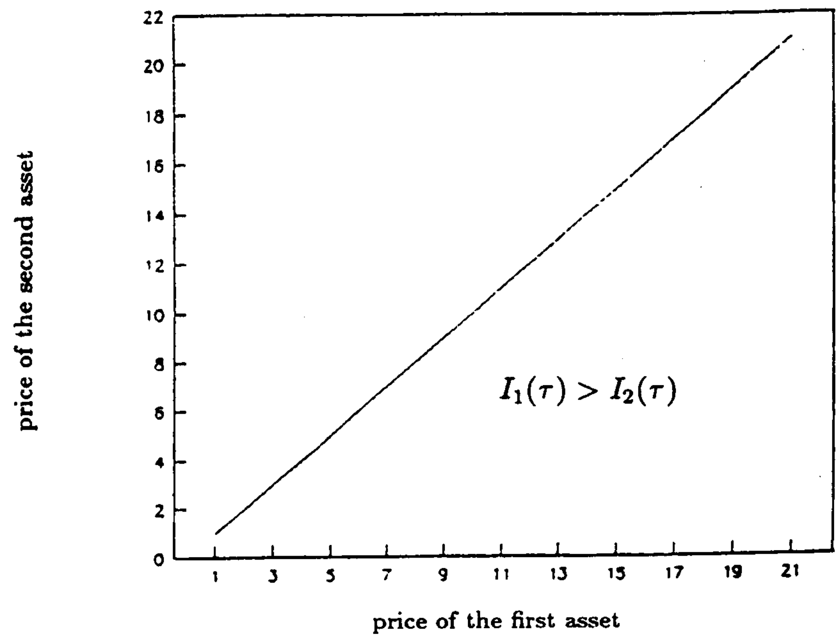

Fig. 13.1. The integration domain of an exchange option.

and

$$A_{e2} = \int_{-\infty}^{\infty} f(u - \rho_y) N \times \left\{ \frac{\ln(I_1/I_2) + (\mu_x - \mu_y)]/\sigma_y - (1 - \rho^2)\sigma_y - [\rho - (\sigma_x/\sigma_y)]u}{\sqrt{1 - \rho^2}} \right\} du\,. \tag{13.4b}$$

The two coefficients A_{e1} and A_{e2} are in terms of univariate integrations. They can be calculated easily with any numerical methods using a computer because the density function $f(.)$ and the cumulative function of the standard normal distribution $N(.)$ can be calculated readily. However, these two coefficients can be simplified in closed-form using some mathematical identities. We first change the integration variable $z = u - \sigma_x$ or $u = z + \sigma_x$ into (13.4a) and can simplify it to

$$A_{el} = \int_{-\infty}^{\infty} f(z) N \times \left\{ \frac{\ln(I_1/I_2) + (\mu_x - \mu_x)]/\sigma_y - [\rho - (\sigma_x/\sigma_y)]\sigma_x[\rho - (\sigma_x/\sigma_y)]z}{\sqrt{1 - \rho^2}} \right\} dz\,. \tag{13.5}$$

As the argument in the cumulative functions in (13.5) is a linear function of its corresponding integration variable z, we use the following mathematical identity to simplify (13.5) further:

$$\int_{-\infty}^{\infty} f(z)N(A+Bz)dz = N\left(\frac{A}{\sqrt{1+B^2}}\right), \tag{13.6}$$

where A and B are constant real numbers.

The method shown above using (13.6) to simplify double integrations in terms of the univariate cumulative function $N(.)$ of the standard normal distribution will be used very often throughout this book. As an univariate cumulative function is much easier to calculate than the bivariate cumulative normal distribution, and analytical expressions for sensitivities can be obtained easily with the univariate cumulative function, we will employ this method to simplify double integrations as often as possible.

Using (13.6), we can simplify (13.5) to $A_{e1} = N(d_{e1})$ and following the similar method we can simplify $A_{e2} = \mathrm{N}\ (d_{e2})$. Therefore (13.5) becomes

$$E(PFEX) = I_1 e^{(\mu_1-g_1)\tau} N(d_{e1}) - I_2 e^{(\mu_2-g_2)\tau} N(d_{e2}), \tag{13.7}$$

where

$$d_{e2} = \left\{\ln(\frac{I_1}{I_2}) + [(\mu_1 - g_1) - (\mu_2 - g_2)]\tau - \frac{1}{2}\sigma_a^2\tau\right\} \Big/ (\sigma_a\sqrt{\tau}),$$

$$d_{e1} = d_{e2} + \sigma_a\sqrt{\tau},\ \sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

and ρ is the correlation coefficient between the returns of the two underlying assets.

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate r. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate $\mu_1 = \mu_2 = r$, we can obtain the price of an exchange option to exchange the second asset for the first (PEXOP12) by discounting the expected payoff given in (13.7) by the risk-free rate r,

$$PEXOP12 = I_1 e^{-g_1\tau} N(d_{e1}) - I_2 e^{-g_2\tau} N(d_{e2}), \tag{13.8}$$

where

$$d_{e2} = \left[\ln(\frac{I_1}{I_2}) + (g_2 - g_1 - \frac{1}{2}\sigma_a^2)\tau\right] \Big/ (\sigma_a\sqrt{\tau}),$$

$$d_{e1} = d_{e2} + \sigma_a\sqrt{\tau},$$

and σ_a is the same as in (13.7).

The pricing formula in (13.8) is of the Black-Scholes type as the price is expressed in cumulative functions of the standard univariate normal distributions. The functions d_{e1} and d_{e2} are very similar to the arguments in the cumulative functions in the Black-Scholes formula. The important difference between (13.8) and the Black-Scholes formula is the volatility function σ_a. We may call σ_a the aggregate volatility because it is the effective volatility used in the pricing formula. The aggregate volatility function is determined not only by the volatilities of the two underlying assets but also by the degree these two assets are correlated. This aggregate volatility expression appears in most other correlation option pricing formulas in the following chapters.

Example 13.1. Suppose that there are two stocks with the spot prices I_1 = \$100, I_2 = \$100, the volatilities $\sigma_1 = 20\%$ and $\sigma_2 = 15\%$, the yields on the two stocks are $g_1 = 5\%$ and $g_2 = 4\%$, and the two stock returns are correlated with the correlation coefficient $\rho = 75\%$, what is the price of the exchange option to exchange the second stock for the first stock in one year?

Substituting the given parameters into (13.8) yields:

$$\begin{aligned}
\sigma_a &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \\
&= \sqrt{0.20^2 - 2 \times 0.75 \times 0.20 \times 0.15 + 0.15^2} \\
&= 0.1323\,, \\
d_{e2} &= \left[\ln\left(\frac{I_1}{I_2}\right) + \left(g_2 - g_1 - \frac{1}{2}\sigma_a^2\right)\tau\right] \Big/ \left(\sigma_a\sqrt{\tau}\right) \\
&= \left\{\ln\left(\frac{100}{100}\right)\left(+ 0.05 - 0.04 - \frac{1}{2} \times 0.1323^2\right) \times 1\right\} \Big/ \left(0.1323 \times \sqrt{1}\right) \\
&= -0.1417\,, \\
d_{e1} &= d_{e2} + \sigma_a\sqrt{\tau} = -0.4441 + 0.1323 = -0.0094\,.
\end{aligned}$$

Thus, the price of the exchange option to exchange the second stock for the first is obtained directly from (13.8)

$$\begin{aligned}
PEXOP12 &= I_1 e^{-g_\tau} N(d_{e1}) - I_2 e^{-g_2\tau} N(d_{e2}) \\
&= 100e^{-0.05} N(-0.0094) - 100e^{-0.04} N(-0.1417) \\
&= 100 \times 0.9608 \times 0.4962 - 100 \times 0.9512 \times 0.4436 = \$4.578.
\end{aligned}$$

Formula (13.8) gives us the price of the exchange option to pay the second asset in exchange for the first. We can readily obtain the price of the option to pay the first asset in exchange for the second using the following identity:

$$\max\left[I_1(\tau) - I_1(\tau)\right] = I_2(\tau) - I_1(\tau) + \max\left[I_1(\tau) - I_2(\tau)\right]. \qquad (13.9)$$

The identity in (13.9) indicates that the payoff of the option to exchange the first asset for the second can be expressed in terms of that of the exchange option to exchange the second asset for the first, and the prices of the two assets at the option maturity. Taking mathematical expectation to both sides of (13.9) and discounting the expected payoff at the risk-free rate yields the price of the option to exchange the first asset for the second (PEXOP21):

$$\begin{aligned} PEXOP21 &= I_2 \mathrm{e}^{-g_2\tau} - I_1 \mathrm{e}^{-g_1\tau} \\ &\quad + \left[I_1 \mathrm{e}^{-g_1\tau} N(d_{e1}) - I_2 \mathrm{e}^{-g_2\tau} N(d_{e2})\right]. \end{aligned} \qquad (13.10)$$

The first term on the right-hand side of (13.10) is obtained by discounting the expected price of the first asset $I_1 \mathrm{e}^{(r-g)\tau}$ at the risk free r,[1] or multiplying the expected price by the continuous discounting factor $\mathrm{e}^{-r\tau}$ using (13.8).

Using the identity $N(-z) = 1 - N(z)$ for any real number z, we can immediately simplify (13.10) to

$$PEXOP21 = I_2 \mathrm{e}^{-g_2\tau} N(-d_{e2}) - I_1 \mathrm{e}^{-g_1\tau} N(-d_{e1}), \qquad (13.11)$$

where all the parameters are the same as in (13.8).

The pricing formula in (13.11) is consistent with the intuition that the exchange option to exchange the first asset for the second can be considered as a put option written on the first asset with the strike price as the future price of the second asset. Thus, (13.11) looks very similar to the Black-Scholes put option pricing formula in (3.24). It can be readily obtained by changing the positions of the two spot prices and the signs of the arguments in their corresponding cumulative functions.

Example 13.2. What is the price of the option to exchange the first stock for the second in one year in Example 13.1?

Substituting $I_1 = I_2 = \$100$, the volatilities $\sigma_1 = 20\%$ and $\sigma_2 = 15\%$, the yields $g_1 = 5\%$ and $g_2 = 4\%$, the correlation coefficient $\rho = 0.75$, and $\sigma_a = 0.1323$, $d_{e2} = -0.1417$, $d_{e1} = -0.0094$ (from Example 13.1) into (13.11)

[1] **Using the solution given in (IV2) and the moment-generating function of a normal distribution $N(\mu, \sigma^2)$, $M(m) = \exp[\mu + \sigma^2 m^2/2)]$, we can obtain the expected value of $I_1(\tau) = I_1 \exp[(\tau_1 - g_1)\tau]$ in the risk-neutral world.**

yields the price of the exchange option to exchange the first asset for the second

$$\begin{aligned} PEXOP21 &= I_2 e^{-g_2\tau +} N(-d_{e2}) - I_1 e^{-g_1\tau} N(-d_{e1}) \\ &= 100e^{-0.04} N(0.1417) - 100e^{-0.05} N(0.0094) \\ &= \$5.534\,. \end{aligned}$$

The results of Examples 13.1 and 13.2 indicate that the option to pay the first asset in exchange for the second is more expensive than that to pay the second asset in exchange for the first, implying that the second asset is more valuable than the first. These results are consistent with the fact that the expected value of the second asset is indeed greater than that of the first asset in a risk-neutral world because their expected returns should be the same but the payout rate of the first is higher. The dominance of the second asset over the first may also result from the higher volatility of the first asset return.

13.4. SENSITIVITIES

Using the two arguments d_{e1} and d_{e2} in (13.8), we can obtain the following identity:

$$I_1 e^{-g_1\tau} f(d_{e1}) = I_2 e^{-g_2\tau} \mathrm{f}(d_{e2})\,. \tag{13.12}$$

One obvious difference between a vanilla option and a correlation option is that there is only one underlying asset for the former but at least two for the latter. When there are two underlying assets, there will be two deltas. We may call these delta's, partial delta's, compared to deltas of vanilla options because they are obtained by taking partial derivative of the pricing formula in (13.8) with respect to the two spot prices and simplifying the results using (13.12) yields the following two deltas for the exchange option to exchange the second asset for the first:

$$\frac{\partial PEXOP12}{\partial I_1} = e^{-g_1\tau} \mathrm{N}(d_{e1})\,,$$

and (13.13)

$$\frac{\partial PEXOP21}{\partial I_2} = -e^{-g_2\tau} N(-d_{e2})\,,$$

which indicate that the delta of the exchange option with respect to the first asset price is the same functionally as that of a vanilla call option, and that the delta of the exchange option to exchange the first asset for the second, with respect to the second asset price is also the same functionally as that of a vanilla call option but the sign is opposite. As we will see in the following chapters, the deltas of most other correlation options are very different from those of their corresponding vanilla options.

Before we start to analyze the vegas of an exchange option, it is necessary for us to analyze the sensitivity of the exchange option price with respect to the aggregate volatility σ_a. The sensitivity of the exchange option price with respect to the aggregate volatility σ_a can be obtained using the identity given in (13.12):

$$\frac{\partial PEXOP}{\partial \sigma_a} = \sqrt{\tau} I_1 e^{-g_1 \tau} f(d_{e1}) > 0\,, \tag{13.14}$$

which is precisely the same as the vega of a vanilla options given in (3.33), implying that the exchange option is more valuable when the aggregate volatility σ_a is higher.

Since there are two underlying assets for exchange options, there are two underlying volatilities, therefore, there are two vegas for any exchange options. The two vegas can be obtained by taking partial derivatives of (13.8) with respect to the two volatilities σ_1 and σ_2 and simplifying the partial derivatives using (13.12) and (13.14):

$$\frac{\partial PEXOP12}{\partial \sigma_1} = \sqrt{\tau} I_1 e^{-g_1 \tau} f(d_{e1}) \frac{\sigma_1 - \rho\sigma_2}{\sigma_a}\,,$$

and

$$\frac{\partial PEXOP12}{\partial \sigma_2} = \sqrt{\tau} I_1 e^{-g_1 \tau} f(d_{e1}) \frac{\sigma_2 - \rho\sigma_1}{\sigma_a}\,. \tag{13.15}$$

Although the two deltas of an exchange option are similar to that of a vanilla option, the two vegas are rather different. They are always positive like that of a vanilla option when the correlation coefficient is negative or zero, yet they become in general uncertain when the correlation coefficient between the two assets are positive, depending on the relative volatilities of the two assets and the degree the two asset returns are correlated. This is because the exchange option price is always positively related to the aggregate volatility σ_a which may decline as the individual volatility σ_1 or σ_2 increases with certain positive correlation coefficients.

The sensitivities of vanilla options with respect to various option parameters have been given names in Greek letters and these names have become very popular. Yet, there is no corresponding name for the sensitivity of a correlation option value with respect to the correlation coefficient which is so important in determining the values of all kinds of correlation options.

As the Greek alphabet χ is pronounced as "chi" in English which has the same first letter c as the phrase correlation coefficient, we may simply use chi to stand for the sensitivity of the correlation option value with respect to its correlation coefficient. Taking partial derivative of (13.8) with respect to the correlation coefficient ρ yields the chi of the exchange option:

$$\frac{\partial PEXOP12}{\partial \rho} = -\sqrt{\tau} I_1 e^{-g_1 \tau} \left(\frac{\sigma_1 \sigma_2}{\sigma_a}\right) \mathrm{f}(d_{e1}) < 0\,, \qquad (13.16)$$

which indicates that the exchange option price decreases monotonically with the correlation coefficient. The reason for the inverse relationship is that a higher correlation coefficient reduces the aggregate volatility σ_a which is positively related to the exchange option value and thus leads to a lower option premium.

From (13.14) and (13.16) we know that an exchange option has the maximum value when the aggregate volatility reaches the maximum $\sigma_a = \sigma_1 + \sigma_2$ with the correlation coefficient $\rho = -1$, and it has the minimum value when the aggregate volatility reaches the minimum $\sigma_a = |\sigma_1 - \sigma_2|$ with the correlation coefficient $\rho = 1$.

Example 13.3. What are the maximum and minimum prices of the option to exchange the first stock for the second in one year in Example 13.1 for various possible correlation coefficients?

Since the effective voaltility has the maximum (resp. minimum) value when the correlation coefficient is -1 (resp. 1), the two extrema volatility values are $\sigma_1 + \sigma_2$ and $\sigma_1 - \sigma_2$, or 35% and 5%. Substituting $\rho = -1$ and 1 and other parameters into (13.8) and following the same procedure as in Example 13.1 yields the two prices as \$12.808 and \$1.467, respectively. The maximum price is nearly nine times as large as the minimum price.

There is another interesting property of exchange options. If $I_1 = I_2$, $\sigma_1 = \sigma_2$, $g_1 = g_2$, and the correlation coefficient between these two assets is zero, then the prices of exchange options to exchange one asset for the other are zero. These results can be proven by taking limits to the pricing formulas in (13.8) and (13.11). These results confirm our intuition that option to exchange its underlying asset with itself should have no value at all.

We can find other sensitivities for exchange options with respect to other factors such as the time to maturity, the interest rate, the payout rates of the two underlying assets, and so on. They can be similarly obtained by taking partial derivatives to (13.8) and using (13.12) to simplify the results.

13.5. AN APPLICATION

Exchange options can be used in many situations. Margrabe (1978) discussed four applications using the exchange option pricing formula performance incentive fees, margin accounts, an exchange offer, and the standby commitment. We do not want to repeat these applications here. We will give one example to show how we can use the pricing formula (13.8) and illustrate how the pricing formula can be used to solve some practical problems.

In Margrabe's first application, an adviser receives a performance incentive fee $(R_1 - R_2)$ multiplied by a fixed percentage of the total managed portfolio, where R_1 and R_2 stand for the returns of the managed portfolio and a standard portfolio against which the performance is measured, respectively. If the adviser has the protection of limited liability in case the fee became negative, the portfolio management fee is exactly the value of the exchange option to exchange the standard return for the managed return. We will figure out the management fee if the current managed portfolio and the standard portfolio returns are 10% and 5%, respectively, the volatilities of the two returns are both 10%, the fee arrangement lasts for one year, the two portfolios are correlated with a correlation coefficient 50%, and the fixed percentage of the total managed portfolio 10 million dollars is 15%.

We can translate the above given conditions as: $R_1 = 0.10$, $R_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.10$, $g_1 = g_2 = 0$, $\rho = 0.50$, $\tau = 1$ year. Substituting these values into formula (13.8) yields

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.10\,,$$
$$d_{e2} = \left[\ln\left(\frac{I_1}{I_2}\right) + \left(g_2 - g_1 - \frac{1}{2}\sigma_a^2\right)\tau\right] \Big/ (\sigma_a\sqrt{\tau}) = 0.688\,,$$
$$d_{e1} = d_{e2} + \sigma_a\sqrt{\tau} = 0.788\,,$$

and the unit exchange option price

$$\begin{aligned} &R_1 e^{-g_1\tau} N(d_{e1}) - R_2 e^{-g_2\tau} N(d_{e2}) \\ &\quad = 0.10 \times N(0.788) - 0.05 \times N(0.688) = \$0.1162\,. \end{aligned}$$

Therefore, the management fee becomes $0.1162 \times 0.15 = 0.01743$ million dollars or \$17430.

13.6. SUMMARY AND CONCLUSIONS

Exchange options are the basic correlation options which can be used to analyze and price many other correlation options in the following chapters.

They are also the simplest correlation options because their payoff patterns are the simpliest. Because of that, their prices can be expressed in terms of univariate normal cumulative functions. We priced exchange options and analyzed their major sensitivity measures. The prices of exchange options to exchange one for another are negatively correlated with the correlation coefficient between the two assets.

QUESTIONS AND EXERCISES

Questions

13.1. What are exchange options?

13.2. Are there exchange call options or exchange put options?

13.3. Why can an exchange option to exchange the second asset for the first be considered as a call option on the first asset with the price of the second asset at the option maturity as the strike price?

13.4. Why can an exchange option to exchange the second asset for the first be considered as a put option on the second asset with the price of the first asset at the option maturity as the strike price?

13.5. Is there any strike price in an exchange option?

13.6. Why are exchange options considered as simple correlation options?

13.7. Why are exchange options regarded as a basic type of correlation options?

13.8. How many deltas are there in an exchange option?

13.9. Are vegas of exchange options always positive?

13.10. Does the price of an exchange option always increase with the volatilities of the two assets? Why?

Exercises

13.1. Show the identity in (13.12).

13.2. If the spot prices of the assets are \$50 and \$65, the yields on the two underlying assets are 2% and 3%, the volatilities of the two assets are 12% and 18%, and the correlation coefficient between the two assets is 65%, then what is the price of the exchange option to exchange the first asset for the second?

13.3. What is the price of the exchange option to exchange the second asset for the first in Exercise 13.2?

13.4. Show that the prices of options to exchange one for the other are the same if $I_1 = I_2$, $\sigma_1 = \sigma_2$, and $g_1 = g_2$. Why?

13.5. What are the prices of the two exchange options in Exercises 13.2 and 13.3 if the payout rates of the two underlying assets are zero, other things being equal?

13.6. Find the deltas of the exchange option in Exercise 13.2?

13.7. Find the vegas of the exchange option in Exercise 13.2?

13.8. Find the performance ratio of the two assets using the results in Exercises 13.2 and 13.3.

13.9.* Show that A_{e2} given in (13.4b) can be simplified to $N(d_{e2})$ using (13.6).

13.10. What is the price of the exchange option in Exercise 13.2 if the volatility of the first asset is changed to 25% and other parameters remain unchanged?

13.11. What is the price of the exchange option in Exercise 13.2 if the volatility of the second asset is changed to 10% and other parameters remain unchanged?

13.12.* Show that the prices of exchange options to exchange one asset for the other are zero if $I_1 = I_2$, $\sigma_1 = \sigma_2$, $g_1 = g_2$, and the correlation coefficient between these two assets is zero.

13.13.* Show the expected payoff of an exchange option in (13.4).

13.14.* Show that the price of an exchange option to exchange one asset for another is zero if the two assets are completely the same.

Chapter 14

OPTIONS PAYING THE BEST/WORST AND CASH

14.1. INTRODUCTION

Options paying the best or worst and cash are also called options delivering the best or worst and cash. An option paying the best (resp. worst) of two assets entitles its holder the right to receive the maximum (minimum) of the two underlying assets at maturity. An option paying the best (resp. worst) of two risky assets and cash entitles its holder the right to receive the maximum (resp. minimum) of the two underlying assets and a fixed amount of cash at maturity. Since the payoffs of these options depend on the maximum or minimum of two assets, there is no distinction between a call and a put for these options. The purpose of this chapter is to show how to price these options and analyze their basic properties within a Black-Scholes environment.

The payoff of an option paying the best and cash and the worst and cash are given as follows

$$\mathrm{max}c(\tau) = \max[I_1(\tau),\ I_2(\tau),\ K]\,, \tag{14.1}$$

and

$$\min c(\tau) = \min[I_1(\tau),\ I_2(\tau),\ K]\,, \tag{14.2}$$

where max (. , .) and min (. , .) are functions that give the larger and smaller of the two prices involved, respectively, and K is a prespecified amount of cash.

14.2. PRICING OPTIONS PAYING THE BEST OR WORST OF TWO ASSETS

An option paying the best (resp. worst) of two assets without any cash payment is among the simplest correlation option. In order to illustrate the

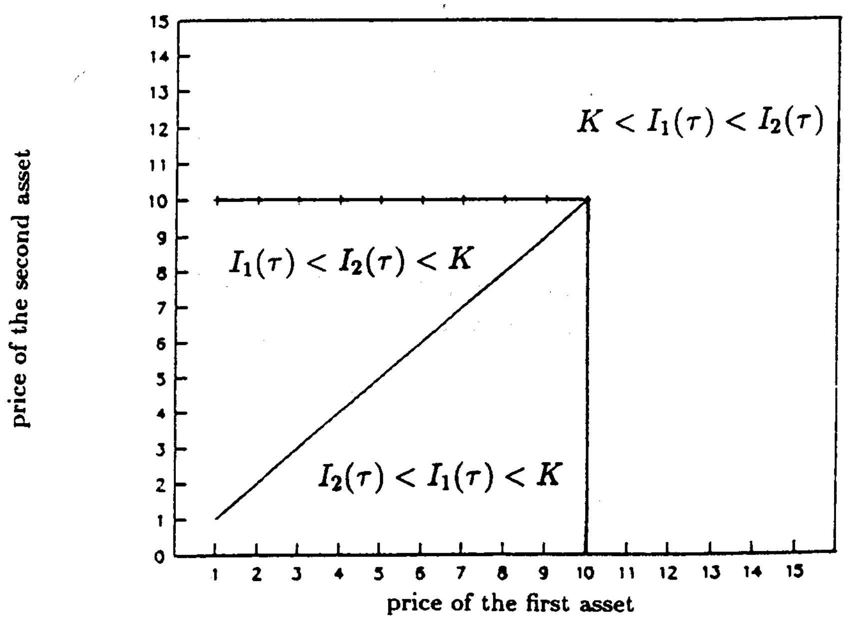

Fig. 14.1. The integration domain for an option paying the best of two assets.

procedures to price an option paying the best/worst and cash, we consider the simple case without cash payment in this section and then extend the results to $K > 0$ in Section 14.5.

Assume that the two underlying asset prices follow the stochastic process in (IV1). Figure 14.1 depicts the integration domain or areas in which an option paying the best or worst of two assets takes specific values. For any point below the forty-five degree line and above the horizontal axis which represents the first asset price, the first asset price at maturity is greater than that of the second asset, and for any point above the forty-five degree line and to the right of the vertical axis which represents the second asset price, the first asset price at maturity is smaller than that of the second asset. Using the bivariate normal density function given in (IV3) and (IV4), we can obtain the expected payoff of an option paying the best (14.1) for $K = 0$ by double integration:

$$E\{\max\,[I_1(\tau),\ I_2(\tau)]\} = I_1 e^{(\mu_1 - g_1)\tau} A_{m1} + I_2 e^{(\mu_2 - g_2)\tau} A_{m2}\,, \qquad (14.3)$$

where

$$A_{m1} = \int_{-\infty}^{\infty} f(u - \sigma_x) N\left\{ \frac{[\ln(I_1/I_2) + \mu_x - \mu_y]/\sigma_y - (\rho - \sigma_x/\sigma_y)u}{\sqrt{1-\rho^2}} \right\} du \qquad (14.3a)$$

and

$$A_{m2} = \int_{-\infty}^{\infty} f(v - \sigma_y) N\left\{ \frac{[\ln(I_2/I_1) + \mu_y - \mu_x]/\sigma_x - (\rho - \sigma_y/\sigma_x)v}{\sqrt{1-\rho^2}} \right\} dv\,. \tag{14.3b}$$

The two coefficients A_{m1} and A_{m2} are in terms of univariate integrations. They can be calculated easily with any numerical methods because the density function $f(.)$ and the cumulative function of the standard normal distribution $N(.)$ can be calculated easily. However, these two coefficients can be expressed in closed-form using the mathematical identity given in (13.6). We first change the integration variable $z = u - \sigma_x$ or $u = z + \sigma_x$ into (14.3a) and

$$A_{m1} = \int_{-\infty}^{\infty} f(z) N\left\{ \frac{[\ln(I_1/I_2) + \mu_x/\mu_y]/\sigma_y - (\rho - \sigma_x/\sigma_y)\sigma_x - (\rho - \sigma_x/\sigma_y)z}{\sqrt{1-\rho^2}} \right\} dz\,. \tag{14.4a}$$

As the argument in the cumulative functions in (14.4a) is a linear function of its corresponding integration variable z, we can use the identity in (13.6) to simplify (14.4a) to $A_{m1} = N(d_m)$, and following a similar method we can simplify $A_{m2} = N(-d_m + \sigma_a\sqrt{\tau})$. Therefore, (14.3) becomes

$$E\{\max\,[I_1(\tau),\ I_2(\tau)]\} = I_1 e^{(\mu_1 - g_1)\tau} N(d_m) + I_2 e^{(\mu_2 - g_2)\tau} N(-d_m + \sigma_a\sqrt{\tau})\,, \tag{14.5}$$

where

$$d_m = \left\{ \ln\left(\frac{I_1}{I_2}\right) + [(\mu_1 - g_1) - (\mu_2 - g_2)]\tau + \frac{1}{2}\sigma_a^2 \right\} \Big/ (\sigma_a\sqrt{\tau})\,,$$

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,,$$

and ρ is the correlation coefficient between the returns of the two underlying assets.

Similarly, we can obtain the expected payoff of the option paying the worst of the two underlying assets in (14.2) for $K = 0$ with the same method:

$$E\{\min\,[I_1(\tau),\ I_2(\tau)]\} = I_1 e^{(\mu_1 - g_1)\tau} N(-d_m) + I_2 e^{(\mu_2 - g_2)\tau} N(d_m - \sigma\sqrt{\tau})\,, \tag{14.6}$$

where all parameters are the same as in (14.5).

Using the arbitrage-free argument or the risk-neutral valuation relationship, we can obtain the option prices by discounting their expected payoffs at the risk-free interest rate r. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate, or $\mu_1 = \mu_2 = r$, we can obtain the prices of the options paying the best (BP) and worst (WP) of two assets by discounting the expected payoffs in (14.5) and (14.6) by the risk-free rate r

$$BP = I_1 e^{-g_1\tau} N(d_m) + I_2 e^{-g_2\tau} N(-d_m + \sigma_a\sqrt{\tau}), \qquad (14.7)$$

$$WP = I_1 e^{-g_1\tau} N(-d_m) + I_2 e^{-g_2\tau} N(d_m - \sigma_a\sqrt{\tau}), \qquad (14.8)$$

where

$$d_m = \left\{\ln\left(\frac{I_1}{I_2}\right) + (g_2 - g_1)\tau + \frac{1}{2}\sigma_a^2\tau\right\} \Big/ (\sigma_a\sqrt{\tau}) \qquad (14.9)$$

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,. \qquad (14.10)$$

The pricing formulas in (14.7) and (14.8) are of the Black-Scholes type as the prices are expressed in cumulative functions of the standard univariate normal distribution. The function d_m is very similar to d in the Black-Scholes formula. The important difference between (14.7) and the Black-Scholes formula is the aggregate volatility function σ_a which is the same as in Chapter 13 for exchange options. The aggregate volatility function is not only determined by the volatilities of the two underlying assets but also by the degree these two assets are correlated. This aggregate volatility appears in most other correlation option pricing formulas.

Example 14.1. Find the prices of options paying the best and worst of two assets to expire in three months with the spot prices $I_1 = \$20$, $I_2 = \$15$, the volatilities $\sigma_1 = 18\%$ and $\sigma_2 = 15\%$, the yields on the two stocks are $g_1 = 5\%$ and $g_2 = 4\%$, and the two stock returns are correlated with correlation coefficient $\rho = 85\%$.

Substituting $I_1 = \$20$, $I_2 = \$15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.15$, $g_1 = 0.05$, $g_2 = 0.04$, $\tau = 0.25$, and $\rho = 0.85$ into (14.7) yields:

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.0949\,,$$

$$d_m = \left[\ln\left(\frac{I_1}{I_2}\right) + (g_2 - g_1 + \frac{1}{2}\sigma_a^2)\tau\right] \Big/ (\sigma_a\sqrt{\tau})$$

$$= 6.0359\,.$$

Thus, the price of the option paying the best of two assets is

$$\begin{aligned} BP &= I_1 \mathrm{e}^{-g_1\tau} N(d_m) + I_2 \mathrm{e}^{-g_2\tau} N(-d_m + \sigma_a\sqrt{\tau}) \\ &= 20\mathrm{e}^{-0.05\times 0.25} N(6.0359) + 15\mathrm{e}^{-0.04\times 0.25} N(-6.0359 + 0.0474) \\ &= = \$19.752\,, \end{aligned}$$

and the price of the option paying the worst of two assets is

$$\begin{aligned} WP &= I_1 \mathrm{e}^{-g_1\tau} N(-d_m) + I_2 \mathrm{e}^{-g_2\tau} N(d_m - \sigma_a\sqrt{\tau}) \\ &= 20\mathrm{e}^{-0.05\times 0.25} N(-6.0359) + 15\mathrm{e}^{-0.04\times 0.25} N(6.0359 - 0.0474) \\ &= \$14.851\,. \end{aligned}$$

Using the simple identity $N(z) + N(-z) = 1$ for any real number z, (14.7) and (14.8), we can obtain the following identity:

$$BP + WP = I_1 \mathrm{e}^{-g_1\tau} + I_2 \mathrm{e}^{-g_2\tau}\,, \tag{14.11}$$

which indicates that the sum of the prices of options paying the best and the worst of two assets equals the sum of the two current prices of the two underlying assets discounted at their corresponding payout rates.

14.3. OPTIONS PAYING THE BEST OR WORST OF TWO ASSETS AND EXCHANGE OPTIONS

Now, let us show how the option prices in (14.7) and (14.8) can be obtained by using the exchange option pricing formula in (13.8). The two identities given in (13.2) and (13.3) can be alternatively expressed:

$$\max\,[I_1(\tau),\; I_2(\tau)] = I_2(\tau) + \max\,[I_1(\tau) - I_2(\tau), 0]\,, \tag{14.10a}$$

and

$$\min\,[I_1(\tau),\; I_2(\tau)] = I_1(\tau) - \max\,[I_1(\tau) - I_2(\tau), 0]\,. \tag{14.10b}$$

The two identities in (14.10a) and (14.10b) indicate that the future value of an option paying the best or worst of two risky assets can be expressed in terms of the payoff of an exchange option and the future prices of the two assets at the option maturity. Following a similar procedure to derive (13.11), taking mathematical expectation to both sides of (14.10a) and (14.10b) and discounting the expected payoffs at the risk-free rate yield the prices of options paying the best and the worst of two risky assets:

$$PPMX = I_2 \mathrm{e}^{-g_2\tau} + [I_1 \mathrm{e}^{-g_1\tau} N(d_m) - I_2 \mathrm{e}^{-g_2\tau} N(-d_m + \sigma_a\sqrt{\tau})]\,, \tag{14.11a}$$

and

$$PPMN = I_1 e^{-g_1\tau} - [I_1 e^{-g_1\tau} N(-d_m) - I_2 e^{-g_2\tau} N(d_m - \sigma_a\sqrt{\tau})], \quad (14.11b)$$

where $PPMX$ and $PPMN$ are the prices of options paying the maximum and the minimum of two risky assets, respectively.

Using the identity $N(-z) = 1 - N(z)$ for any real number z, we can immediately simplify (14.11a) and (14.11b) as follows:

$$PPMX = I_1 e^{-g_1\tau} N(d_{e1}) + I_2 e^{-g_2\tau} N(-d_{e2}), \quad (14.12a)$$

$$PPMN = I_1 e^{-g_1\tau} N(-d_{e1}) + I_2 e^{-g_2\tau} N(d_{e2}), \quad (14.12b)$$

which are exactly the same as BP and WP given in (14.7) and (14.8), respectively, because $d_{e1} = d_m$ and $d_{e2} = d_{m-\sigma_a\sqrt{\tau}}$.

The above method to derive $PPMX$ and $PPMN$ is obviously more efficient than the integration method to derive BP and WP in Section 14.2. Therefore, we will always price some new options using the pricing formulas of known options. We will show how to price other correlation options using the exchange option formula in the following chapters.

14.4. SENSITIVITY TO THE CORRELATION COEFFICIENT

The prices of the options paying the best and the worst of two assets given in (14.7) and (14.8) are affected by the spot prices, the volatilities, the yields of the two assets, and importantly, the correlation coefficient between the two assets. We will find the chi of options paying the best or worst of two assets, or the sensitivity of the prices of such options with respect to the correlation coefficient between the two assets.

From the parameters of the pricing formulas in (14.7) and (14.8), we can readily show

$$\frac{f(d_m - \sigma_a\sqrt{\tau})}{f(d_m)} = \frac{I_1}{I_2} e^{(g_1 - g_2)\tau} = \frac{I_1 e^{-g_1\tau}}{I_2 e^{-g_2\tau}}, \quad (14.13)$$

where $f(z)$ is the density function of a standard normal distribution.

Taking partial derivatives of BP and WP in (14.7) and (14.8) with respect to ρ and simplifying the results using (14.3) yield

$$\frac{\partial BP}{\partial \rho} = -\sqrt{\tau} I_2 e^{-g_2\tau} \frac{\sigma_1\sigma_2}{\sigma_a} f(d_m - \sigma_a\sqrt{\tau}) < 0, \quad (14.14a)$$

and

$$\frac{\partial WP}{\partial \rho} = \sqrt{\tau} I_2 e^{-g_2\tau} \frac{\sigma_1\sigma_2}{\sigma_a} f(d_m - \sigma_a\sqrt{\tau}) > 0. \quad (14.14b)$$

The derivatives in (14.14) indicate that the price of the option paying the best (resp. worst) of two assets decreases (resp. increases) with the correlation coefficient. This is because the maximum (resp. minimum) of two assets given in (14.10) decreases (resp. increases) with the correlation coefficient because the two prices tend to move more in the same direction and the difference between the two becomes smaller.

Example 14.2. Given all other parameters the same as in Example 14.1, find the maximum and minimum prices of the options paying the best and the worst of two assets with all possible correlation coefficients.

Since the price of the option paying the best of two assets decreases monotonically with the correlation coefficient, we can obtain the maximum and the minimum prices with the correlation coefficient $\rho = -1$ and 1, respectively. Substituting $I_1 = \$20$, $I_2 = \$15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.15$, $g_1 = 0.05$, $g_2 = 0.04$, $\tau = 0.25$, and $\rho = -1$ and 1 into (14.7) and following the same procedure as in Example 14.1 yield:

$$\begin{aligned}
\sigma_a(\rho = 1) &= \sqrt{\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2} = \sigma_1 - \sigma_2 = 0.03\,, \\
\sigma_a(\rho = -1) &= \sqrt{\sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2} = \sigma_1 + \sigma_2 = 0.33\,, \\
d_m(\rho = 1) &= \left[\ln\left(\frac{I_1}{I_2}\right) + \left(g_2 - g_1 + \frac{1}{2}\sigma_a^2\right)\tau\right] \Big/ (\sigma_a\sqrt{\tau}) \\
&= 19.0196\,, \\
d_m(\rho = -1) &= \left[\ln\left(\frac{I_1}{I_2}\right) + \left(g_2 - g_1 + \frac{1}{2}\sigma_a^2\right)\tau\right] \Big/ (\sigma_a\sqrt{\tau}) \\
&= 1.8109\,.
\end{aligned}$$

Thus, the prices of the option paying the best of two assets are

$$\begin{aligned}
BP(\rho = 1) &= I_1 e^{-g_1\tau} N(d_m) + I_2 e^{-g_2\tau} N(-d_m + \sigma_a\sqrt{\tau}) \\
&= 20e^{-0.05\times 0.25} N(19.0196) \\
&\quad + 15e^{-0.04\times 0.25} N(-19.0196 + 0.015) = \$19.752
\end{aligned}$$

and

$$\begin{aligned}
BP(\rho = -1) &= I_1 e^{-g_1\tau} N(d_m) + I_2 e^{-g_2\tau} N(-d_m + \sigma_a\sqrt{\tau}) \\
&= 20e^{-0.05\times 0.25} N(1.8109) \\
&\quad + 15e^{-0.04\times 0.25} N(-1.8109 + 0.015) = \$19.800\,.
\end{aligned}$$

Following the similar procedure, we can obtain the maximum and minimum prices of the option paying the worst of two assets with the correlation coefficient $\rho = 1$ and -1, respectively:

$$\begin{aligned} WP(\rho = 1) &= 20e^{-0.05\times 0.25}\mathcal{N}(-19.0196) \\ &\quad + 15e^{-0.04\times 0.25}\mathcal{N}(19.0196 - 0.015) = \$14.851\,, \end{aligned}$$

and

$$\begin{aligned} WP(\rho = -1) &= 20e^{-0.05\times 0.25}\mathcal{N}(-1.8109) \\ &\quad + 15e^{-0.04\times 0.25}\mathcal{N}(1.8109 - 0.015) = \$14.803\,. \end{aligned}$$

14.5. OPTIONS PAYING THE BEST/WORST AND CASH

Now we can work with options delivering the best/worst of two assets and nonzero cash. Assume that the two underlying asset prices follow the same stochastic process in (IV1). Figure 14.2 depicts the integration domain for such options. It shows that there are three regions in which $I_1(\tau)$, $I_2(\tau)$, and K are the largest of the three values. For any combination of the two asset prices within or at the square from the origin, K is always the maximum of the three variables. In the area below (resp. above) the forty-five degree line, $I_1(\tau)$ [resp. $I_2(\tau)$] is the largest of the three variables.

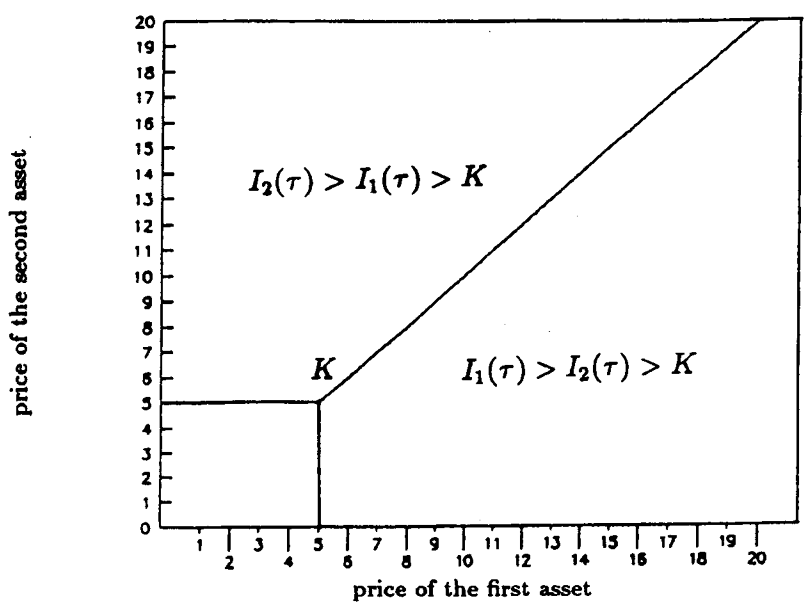

Fig. 14.2. The integration domain for an option paying the best of two assets and nonzero cash.

Using the bivariate normal density function in (IV4) and (IV5) and the integration domain shown in Figure 14.2, we can obtain the expected payoffs of options paying the best or worst of two assets and cash $K > 0$ by integrating (14.1) twice:

$$E\{\max\,[I_1(\tau),\ I_2(\tau),\ K]\} = I_1 e^{(\mu_1 - g_1)\tau} A_{mk1} + I_2 e^{(\mu_2 - g_2)\tau} A_{mk2} + K N_2(-d_{kl},\ -d_{k2},\ \rho)\,, \tag{14.15}$$

where

$$A_{mk1} = \int_{-d_{k1}}^{\infty} f(u - \sigma_x) N\left\{\frac{[\ln(I_1/I_2) + \mu_x - \mu_y]/\sigma_y - (\rho - \sigma_x/\sigma_y)u}{\sqrt{1-\rho^2}}\right\} du\,, \tag{14.15a}$$

$$A_{mk2} = \int_{-d_{k2}}^{\infty} f(v - \sigma_y) N\left\{\frac{[\ln(I_2/I_1) + \mu_y - \mu_x]/\sigma_x - (\rho - \sigma_y/\sigma_x)v}{\sqrt{1-\rho^2}}\right\} dv\,, \tag{14.15b}$$

$$d_{k1} = \left[\ln\left(\frac{I_1}{K}\right) + \left(\mu_x - \frac{1}{2}\sigma_x^2\right)\right] \Big/ \sigma_x,$$

$$d_{k2} = \left[\ln\left(\frac{I_2}{K}\right) + \left(\mu_y - \frac{1}{2}\sigma_y^2\right)\right] \Big/ \sigma_y\,,$$

and $N_2(\alpha, \beta, \theta)$ is the standard bivariate cumulative function with two upper bounds α and β and the correlation coefficient θ. The expression $N_2(\alpha, \beta, \theta)$ is given in Appendix of Chapter 11.

The two coefficients A_{mk1} and A_{mk2} in (14.15a) and (14.15b) are also in terms of univariate integrations. They can be calculated easily with any numerical methods because the density function $f(.)$ and the cumulative function of a standard normal distribution $N(.)$ can be calculated readily. However, we can express these two integrations in terms of normal cumulative functions. We need to change the integration variable $z = -(u - \sigma_x)$ or $u = \sigma_x - z$ into (14.14a) and

$$A_{mk1} = \int_{-\infty}^{d_{1k1}} f(z) \times N\left\{\frac{[\ln(I_1/I_2) + \mu_x - \mu_y]/\sigma_y - (\rho - \sigma_x/\sigma_y)\sigma_x + (\rho - \sigma_x/\sigma_y)z}{\sqrt{1-p^2}}\right\} dz\,, \tag{14.16}$$

where $d_{1k1} = d_{k1} + \sigma_x$.

Although the argument in the cumulative function in (14.16) is also a linear function of its corresponding integration variable z, we cannot use the identity in (13.6) to simplify (14.16) because the upper bound of the integration of z is not infinity but d_{1k1} which is smaller than infinity in general. Although we cannot express (14.16) in terms of univariate normal cumulative function, we can express it in terms of bivariate normal cumulative function. Using a similar method as the one explained in Appendix of Chapter 11, we can simplify $A_{mk1} = N_2(d_{1k1}, d_m, \rho_1)$, and $A_{mk2} = N_2(d_{1k2}, -d_m + \sigma_a\sqrt{\tau}, \rho_2)$. Thus, (14.14) becomes

$$\begin{aligned} E\{\max\,[I_1(\tau),\ I_2(\tau),\ K]\} = I_1 e^{\mu_{xp}} N_2(d_{1k1},\ d_m,\ \rho_1) \\ + I_2 e^{\mu_{yp}} N_2(d_{1k2},\ -d_m + \sigma_a\sqrt{\tau},\ \rho_2) \\ + K N_2(-d_{k1},\ -d_{k2},\ \rho)\,, \end{aligned} \tag{14.17}$$

where

$$\begin{aligned} \mu_{xp} &= \mu_x + \frac{1}{2}\sigma_x^2 = (\mu_1 - g_1)\tau, \\ \mu_{yp} &= \mu_y + \frac{1}{2}v\sigma_y^2 = (\mu_2 - g_2)\tau\,, \\ d_{1k1} &= d_{k1} + \sigma_x,\ d_{1k2} = d_{k2} + \sigma_y\,, \\ \sigma_a &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,, \\ \rho_1 &= \frac{\sigma_2 - \rho\sigma_1}{\sigma_a},\ \rho_2 = \frac{\sigma_1 - \rho\sigma_2}{\sigma_a}\,, \end{aligned}$$

and d_m is given in (14.5). Using the risk-neutral valuation relationship, we can obtain the price of an option paying the best of two assets and cash (PBC) by substituting $\mu_1 = \mu_2 = r$ and discounting the expected payoff in (14.17) by the risk-free rate r:

$$\begin{aligned} PBC = I_1 e^{-g_1\tau} N(d_{1k1},\ d_m,\ \rho_1) + I_2 e^{-g_2\tau} N_2(d_{1k2},\ -d_m + \sigma_a\sqrt{\tau},\ \rho_2) \\ + K e^{-r\tau} N_2(-d_{k1},\ -d_{k2},\ \rho)\,, \end{aligned} \tag{14.18}$$

where

$$\begin{aligned} d_{1k1} &= d_{k1} + \sigma_1\sqrt{\tau}, \\ d_{1k2} &= d_{k2} + \sigma_2\sqrt{\tau}\,, \\ d_{k1} &= \left[\ln\left(\frac{I_1}{K}\right) + \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] \Big/ (\sigma_1\sqrt{\tau})\,, \\ d_{k2} &= \left[\ln\left(\frac{I_2}{K}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\right] \Big/ (\sigma_2\sqrt{\tau})\,, \end{aligned}$$

and σ_a, ρ_1, and ρ_2 are the same as in (14.17).

We can find that the pricing formula of an option paying the best of two assets given in (14.7) is a special case of that of an option paying the best of two assets and cash in (14.18). Since the two bivariate normal cumulative functions $N_2(d_{1k1}, d_m, \rho_1)$ and $N_2(d_{1k2}, -d_m + \sigma_a\sqrt{\tau}, \rho_2)$ approach $N(d_m)$ and $N(-d_m + \sigma_a\sqrt{\tau})$, respectively,* (14.18) approaches (14.7) when K approaches zero.

Example 14.3. Find the price of the option paying the best of two assets in Example 14.1 and a cash payment of \$18.00, given the interest rate $r =$ 8%.

Substituting $I_1 = \$20$, $I_2 = \$15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.15$, $g_1 = 0.05$, $g_2 = 0.04$, $\tau = 0.25$, $\rho = 0.85$, $r = 0.08$, $K = \$18$. $\sigma_a = 0.0949$, $d_m = 6.0359$ into (14.18) yields:

$$
\begin{aligned}
\rho_1 &= \frac{\sigma_2 - \rho\sigma_1}{\sigma_a} = -0.0316\,,\\
\rho_2 &= \frac{\sigma_1 - \rho\sigma_2}{\sigma_a} = 0.5532\,,\\
d_{k1} &= \left[\ln\left(\frac{I_1}{K}\right) + \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] \Big/ (\sigma_1\sqrt{\tau}) = 1.209\,,\\
d_{k2} &= \left[\ln\left(\frac{I_2}{K}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\right] \Big/ (\sigma_2\sqrt{\tau}) = -2.335\,,\\
d_{1k1} &= d_{k1} + \sigma_1\sqrt{\tau} = 1.299\,,\\
d_{1k2} &= d_{k2} + \sigma_2\sqrt{\tau} = -2.26\,,
\end{aligned}
$$

$$
\begin{aligned}
PBC &= I_1 e^{-g_1\tau} N_2(d_{1k1},\ d_m,\ \rho_1) + I_2 e^{-g_2\tau} N_2(d_{1k2},\ -d_m + \sigma_a\sqrt{\tau},\ \rho_2)\\
&\quad + K e^{-r\tau} N_2(-d_{k1},\ -d_{k2},\ \rho)\\
&= 20e^{-0.05\times0.25} N_2(1.299,\ 6.0359,\ -0.0316)\\
&\quad + 15e^{-0.04\times0.25} N_2(-2.26,\ -5.9984,\ 0.5532)\\
&\quad + 18e^{-0.08\times0.25} N_2(-1.209,\ 2.335,\ 0.85)\\
&= \$19.8358\,.
\end{aligned}
$$

*Using the definition of the standard cumulative function of the bivariate normal distribution given in (A11.1), we can obtain:

$$
\begin{aligned}
N_2(\infty, b, \rho) &= \int_{-\infty}^{b} f(v) N\left[(\infty - \rho v)/\sqrt{1-\rho^2}\right] dv\\
&= \int_{-\infty}^{b} f(v) N(\infty) dv = \int_{-\infty}^{b} f(v) dv = N(b)\,.
\end{aligned}
$$

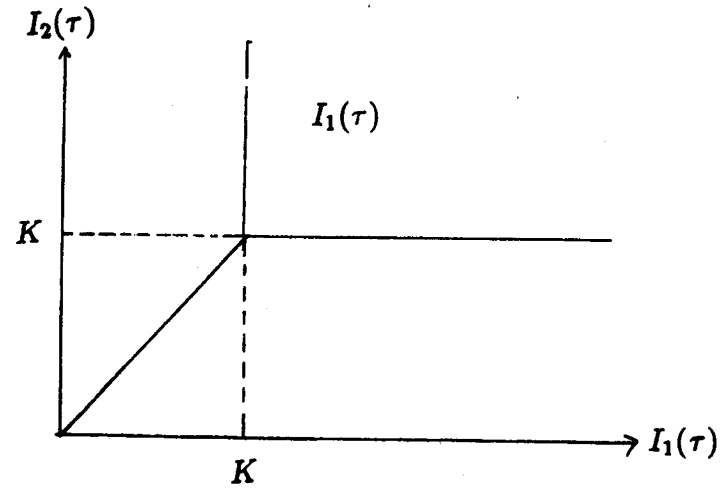

Fig. 14.3. The integration domain for an option paying the worst of two assets and nonzero cash.

The pricing formula in (14.18) is for options paying the best of two assets and a prespecified amount of cash. Our last task in this section is to provide a corresponding formula for options paying the worst of two assets and a prespecified amount of cash. Figure 14.3 depicts the integration domain for such an option. Following a similar procedure, we can obtain the price of an option paying the worst of two assets and cash (PWC) using the integration domain shown in Figure 14.3 and the joint density function given in (IV4) and (IV5):

$$\begin{aligned} PWC = {} & I_1 e^{-g_1\tau} N_2(-d_{1k1}, -d_m - \rho_2\sigma_1\sqrt{\tau}, \rho_2) \\ & + I_2 e^{-g_2\tau} N_2(-d_{1k2}, d_m - \rho_1\sigma_2\sqrt{\tau},\ \rho_1) \\ & + K e^{-r\tau} N_2(d_{k1},\ d_{k2},\ \rho) \end{aligned} \tag{14.19}$$

where all parameters are the same as in (14.18).

Example 14.4. Find the prices of the corresponding options paying the worst of the two assets and a cash payment in Example 14.3.

Substituting $I_1 = \$20$, $I_2 = \$15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.15$, $g_1 = 0.05$, $g_2 = 0.04$, $\tau = 0.25$, $\rho = 0.85$, $r = 0.08$, $K = \$18$, $\sigma_a = 0.0949$, $d_m = 6.0359$, and values of some related intermediate parameters into formula (14.19) yields:

$$
\begin{aligned}
PWC &= I_1 e^{-g_1\tau} N_2(-d_{1k1}, -d_m, \rho_2) + I_2 e^{-g_2\tau} N_2(-d_{1k2}, d_m - \sigma_a\sqrt{\tau}, \rho_1) \\
&\quad + K e^{-r\tau} N_2(d_{k1}, d_{k2}, \rho) \\
&= 20e^{-0.05\times 0.25} N_2(-1.299, -6.0359, 0.5532) \\
&\quad + 15e^{-0.04\times 0.25} N_2(2.26, 5.9984, -0.0316) \\
&\quad + 18 \times e^{-0.08\times 0.25} N_2(1.209, -2.335, 0.85) \\
&= 20 \times 0.9876 \times 0.0000 + 15 \times 0.9901 \times 0.9881 + 18 \times 0.982 \times 0.0098 \\
&= 0.0000 + 14.6748 + 0.1729 = \$14.8477\,.
\end{aligned}
$$

We leave it as an exercise to check that the pricing formula given in (14.19) degenerates to the pricing formula given in (14.8) when the cash payment approaches infinity. This is consistent with the intuition that the cash payment becomes irrelevant when it becomes extremely large. This degenerated case is symmetric to the case that formula (14.18) degenerates to 14.7 when the corresponding cash payment approaches zero.

14.6. SUMMARY AND CONCLUSIONS

We have analyzed options paying the best or worst of two risky assets and cash in this chapter. We first examined options paying the best or worst of two risky assets and provided closed-form solutions for their prices. As no cash is involved, there is no distinction between a call and a put for these options. As their integration domain is similar to that of exchange options, options paying the best or worst of two risky assets are also simple correlation options. Since the integration domain is simple, their corresponding pricing formula can be expressed in terms of univariate normal cumulative functions.

Options paying the best or worst of two risky assets and cash are more complicated than the corresponding options without cash as the cash payment makes the payoff patterns of these options complicated. Their prices can be expressed in terms of bivariate normal cumulative functions instead of univariate normal cumulative functions. Options paying the best of two risky assets are special cases of options paying the best of two risky assets and cash when the cash payment is zero, and options paying the worst of two risky assets are special cases of options paying the worst of two risky assets and cash when the cash payment approaches infinity.

QUESTION AND EXERCISES

Questions

14.1. What is an option paying the best of two assets?

14.2. What is an option paying the worst of two assets?

14.3. What is an option paying the best of two assets and cash?

14.4. What is an option paying the worst of two assets and cash?

14.5. How does the price of an option paying the best of two assets change with the correlation coefficient between these two assets? Why?

14.6. How does the price of an option paying the worst of two assets change with the correlation coefficient between these two assets? Why?

14.7. Why are options paying the best or worst of two assets considered simple correlation options?

14.8. Are there call or put options paying the best or worst of two assets and cash?

Exercises

14.1. Find the prices of the options paying the best and worst of two assets to expire in one year with the spot prices $I_1 = \$25$, $I_2 = \$20$, the volatilities $\sigma_1 = 15\%$ and $\sigma_2 = 25\%$, the yields on the two stocks are $g_1 = 3.5\%$ and $g_2 = 2.5\%$, and the two stock returns are correlated with the correlation coefficient $\rho = 65\%$.

14.2. Find the prices of the options in Exercise 14.1 if the correlation coefficient is changed to 25% and other parameters remain the same.

14.3. Find the maximum and minimum prices of the options in Exercise 14.1 for all possible correlation coefficients.

14.4. Find the prices of the options paying the best of two assets and cash \$22 to expire in one year with other parameters the same as in Exericise 14.1.

14.5. Find the prices of the options paying the worst of two assets and cash \$22 to expire in one year with other parameters the same as in Exericise 14.1.

14.6.* Show that the identity in (14.13) is correct.

14.7.* Derive the sensitivity in (14.14a).

14.8.* Show that the pricing formula for options paying the worst of two assets and cash in (14.19) includes the pricing formula for options paying the worst of two assets in (14.8) as a special case.

14.9.* Show that the thetas of the options paying the best and worst of two assets are opposite in sign and the same in magnitude when the yields on the two underlying assets are both zero.

14.10.* Show the expected payoff of an option paying the best of two assets in (14.3).

14.11.* Show that the pricing formula in (14.18) for options paying the best of two assets and cash degenerates to that in (14.7) for options paying the best of two assets when K approaches zero.

14.12.* Find the corresponding pricing formulas for options paying either $\max[\omega_1 I_1, (\tau), \omega_2 I_2(\tau), K]$ or $\min[\omega_1 I_1(\tau), \omega_2 I_2(\tau), K]$, where ω_1 and ω_2 are nonnegative weights.

14.13.* Show that the pricing formula given in (14.19) for options paying the worst of two assets and cash degenerates to the pricing formula given in (14.8) for options paying the best of two assets when K approaches infinity.

Chapter 15

STANDARD DIGITAL OPTIONS AND CORRELATION DIGITAL OPTIONS

15.1. INTRODUCTION

Digital options are also known as binary or bet options. Because of their simple payoff patterns and other unique characteristics, they attract many participants in the over-the-counter (OTC) marketplace. Generally speaking, the payoff of a digital option can be either a fixed amount of cash, an asset, or the difference between an asset price and a prespecified level which is often different from the strike price. These digital options are known as cash-or-nothing (CON), asset-or-nothing (AON), and gap options, respectively. See Rubinstein (1991) for a good discussion of these options. Digital options have been familiar most recently in accrual note structures, which essentially include a series of digital options, and in mini-premium foreign exchange trades.

As in outside barrier options analyzed in Chapter 11, we may call the asset that is involved in the payoff of a digital option the payment asset, which is almost always the same as the underlying asset or the measurement instrument. However, it is not necessarily the case. If the payment asset is different from the underlying asset, there will be two assets involved in the digital option. We may call these digital options involving two assets correlation digital options, and those involving only one asset ordinary digital options. We will show in this chapter that correlation digital options have more flexibility than ordinary digital options, and hence, may have greater potential for practical use.

A correlation digital option is very similar in nature to an outside barrier option discussed in Chapter 11 because both have the measurement asset separated from the payment asset. As a matter of fact, correlation digital options can be regarded as European-style outside barrier options because their payoffs are determined according to whether the measurement asset prices surpass a certain level at maturity. As a matter of fact, the so called "pure vega" digital options are actually correlation digital options with the measurement asset specified as the implied volume of another option. Actually, many existing exotic products have similar properties to those of correlation digital options. For example, most interest-rate swaps are based on either one-month or three-month LIBOR. There are many exotic swaps in the market which possess knock-in or knockout properties, depending on whether LIBOR exceeds a certain prespecified trigger rate on some prespecified date. LIBOR in these swaps is very similar to the measurement instrument in a correlation digital option and the floating leg can be considered as its payoff asset. In general, it is difficult to hedge swaps with knockout properties. To some extent, correlation digital options provide a good method to hedge these swaps. They can also be used to hedge and/or speculate in financial assets which are highly sensitive to inflation, using gold forward prices as the measurement instrument. Or they can be used as managerial compensation packages in which the measurement instrument can be sales or production quantities and the payoff can be either cash, stocks, or a combination of the two. Therefore, a systematic treatment of these correlation digital options will prove to be very useful.

Some digital options such as CON or AON options are more simple than vanilla options in their payoffs. Due to their simple payoff patterns, some people argue that they should be considered as basic building blocks of vanilla options. Most recently, Pechtl (1995) showed that the payoff of a vanilla option can be duplicated with the sum of an infinite number of digital options. Whereas this argument is interesting theoretically, it may not be of much practical use.

The purpose of this chapter is to introduce ordinary digital options, both European- and American-style, and then extend these options to correlation digital options. We will largely follow Zhang (1995d) in this Chapter. We will study how to price correlation digital options and analyze their sensitivities. We will show that correlation digital options include all three types of ordinary digital options as special cases, and, as their name implies, the correlation coefficient play an important role in determining the prices, sensitivities, and other valuation aspects of correlation digital options. Again, we confine our analysis to a Black-Scholes environment for the purpose of transparency as well as easy comparisons.

15.2. STANDARD DIGITAL OPTIONS

The simplest digital option is a cash-or-nothing (CON) option. A CON option is very much like a bet. If the underlying asset surpasses (resp. falls

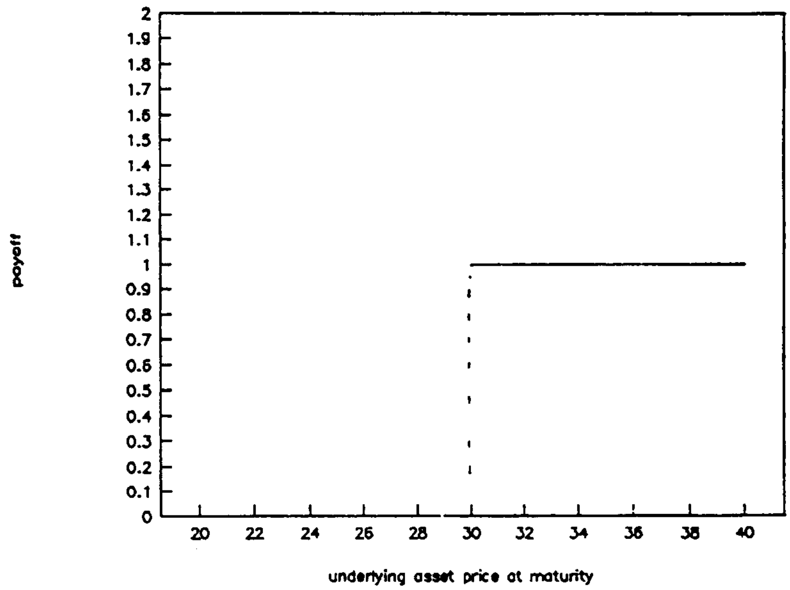

Fig. 15.1. The payoff of a CON option with strike price $K = \$30$.

below) a prespecified level, then a prespecified amount of cash is paid to the buyer of a CON call (resp. put) option. Otherwise the option expires worthless. Let us study the simplest CON option with a payment of \$1.00 because the prices of any other CON options with different prespecified payments are simply the products of these payments and the price of the one-dollar CON option. The payoff pattern of such a simple CON call option is given in Figure 15.1, which indicates that the payoff jumps from zero to \$1.00 at the strike price. The simple CON option with one-dollar payoff is also called a one-dollar CON option.

In a Black-Scholes environment, the underlying asset return is assumed to follow a lognormal random walk given in (IV1) and the simple CON option can be priced very conveniently because it is the present value of the possible one-dollar future payoff. Since the probability that a CON option will be in-the-money or the probability that $\omega S(\tau) > \omega K$ is $N(\omega d)$ in a Black-Scholes environment,[1] the expected payoff of the one-dollar CON option is simply $N(\omega d)$ and its price ($1DCON$) is therefore

$$1DCON = \mathrm{e}^{-r\tau} N(\omega d)\,, \tag{15.1}$$

[1]This can be readily shown using the change of variable: $P[\omega S(\tau) > \omega K] = P[\omega x > \omega \ln(K/S)] = N\{\omega[u > \ln(K/S) - \mu_x]\sigma_x] = P[\omega u > -\omega d] = N(\omega d)$, where $x = \ln[S(\tau)/S]$ is the log-return of the underlying asset which is lognormally distributed using the solution in (2.11) and $u = (x - \mu_x)/\sigma_x$ is the standardized normal variable for x.

which is the expected payoff $N(\omega d)$ discounted at the risk-free rate r continuously.

Using (15.1), we know that the price of a one-dollar CON call option is $e^{-r\tau}N(d)$ and that of a one-dollar CON put option is $e^{-r\tau}N(-d)$.

Example 15.1. Find the prices of the one-dollar CON call and put options to expire in one year if the spot and strike prices are \$100, interest rate 8%, the payout rate of the underlying asset 3%, and the volatility of the underlying asset 20%.

Substituting $S = K = \$100$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, and $\tau = 1$ into (15.1) yields

$$\begin{aligned} d &= [\ln(S/K) + (r - g - \sigma^2)\tau]/(\sigma\sqrt{\tau}) \\ &= 0.15\,, \end{aligned}$$

and the prices of the CON call and put are

$$1DCONCall = e^{-0.08\times 1}N(0.15) = \$0.5166$$

and

$$1DCONPut = e^{-0.08\times 1}N(-0.15) = \$0.4065\,.$$

Another kind of digital options is called gap options. Gap options are direct extensions of vanilla options. Figures 2.1 and 2.2 indicate that the payoffs of vanilla call and put options are linear lines kinked at the strike price K. Although their payoffs are kinked, the payoff functions are continuous because the payoff of any vanilla call (put) option starts above (below) the strike price K from zero. Compared to these continuous payoff patterns, the payoff of a gap option starts at a prespecified level X which is often different from the strike price of the vanilla option K. We may simply call X the gap parameter. Figure 15.2 illustrates the possible payoffs of gap options with various gap parameters. Figure 15.2 indicates that the payoff of a gap call option is precisely the same as that of a vanilla call option when the gap parameter X is the same as the strike price of the option and that the payoff of a gap call option is above (resp. below) that of a vanilla call option when the gap parameter X is smaller (resp. greater) than the strike price of the option. This is easily understood because a gap option has a greater (resp. smaller) payoff than the corresponding vanilla call option. Thus, the payoff of a gap option either jumps up or falls down at the strike price K depending whether the gap parameter X is smaller or greater than the strike price K.

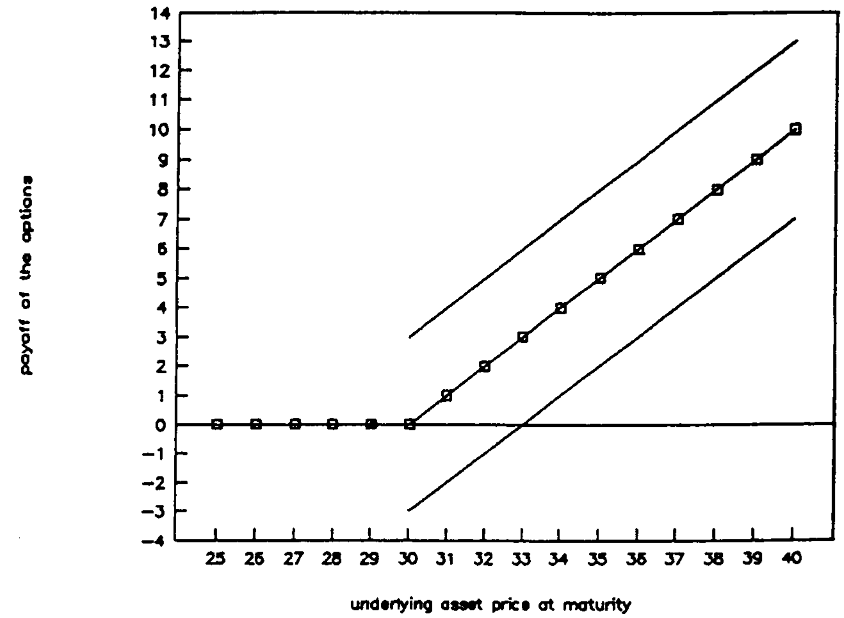

Fig. 15.2. Payoffs of Gap options strike price K = \$30, X = \$27, 30, & 33.

Using Figure 15.2, the payoff of a gap option (*POFGAP*) can be alternatively expressed:

$$POFGAP(\tau) = \omega[S(\tau) - X] \text{ if } \omega S(\tau) > \omega K$$
$$= 0 \text{ if otherwise}, \qquad (15.2)$$

where K is the strike price of the option; X is a prespecified gap parameter which determines the level of gap around the payoff asset price; max(. , .) is a function that gives the larger of two numbers; and ω is a binary operator (1 for a call option and −1 for a put option).

Following a similar method as in Chapter 2 to derive the standard Black-Scholes formula, we can obtain the price of a gap option (*PGAP*) with payoff in (15.2):

$$PGAP = \omega S \, e^{-g\tau} N(\omega d + \omega\sigma\sqrt{\tau}) - \omega X e^{-r\tau} N(\omega d)\,, \qquad (15.3)$$

where $d = \left[\ln(\frac{S}{K} + (r - g - \sigma^2/2)\tau\right]/(\sigma\sqrt{\tau})$ is the same argument as in the extended Black-Scholes formula in (3.2) and g is the payout rate of the underlying asset.

The pricing formula in (15.3) of gap options is almost the same as the extended Black-Scholes formula in (10.31) with the only exception that the strike price K is replaced by the gap parameter X in the second term. Substituting $X = K$ into (15.3) yields exactly the extended Black-Scholes formula in (10.31). Therefore a vanilla options is a special case of gap options when the gap parameter is the same as the strike price.

Example 15.2 Find the prices of the gap call options to expire in half a year if the spot and strike prices are \$100 and \$105, respectively, interest rate 7%, the payout rate of the underlying asset is 3%, the volatility of the underlying asset 20%, and the gap parameter $X = \$102$ and \$107.

Substituting $S = \$100$, $K = \$105$, $X = \$102$, $r = 0.07$, $g = 0.03$, $\sigma = 0.20$, and $\tau = 0.50$ into (15.3) yields

$$d = [\ln(100/105) + (0.07 - 0.03 - 0.20^2/2) \times 0.5]/(0.20\sqrt{0.50}) = 0.071\,,$$

and

$$\begin{aligned} PGAP(\omega = 1\&X = 102) &= S\ \mathrm{e}^{-g\tau}N(d + \sigma\sqrt{\tau}) - X\mathrm{e}^{-r\tau}N(d) \\ &= 100\mathrm{e}^{-0.03\times 0.5}N(0.07 + 0.141) - 102\mathrm{e}^{-0.07\times 0.5}N(0.071) \\ &= \$5.527\,; \end{aligned}$$

and the price of the gap call option with $X = \$107$ can be similarly obtained

$$\begin{aligned} PGAP(\omega = 1\&X = 107) &= 100\mathrm{e}^{-0.03\times 0.5}N(0.07 + 0.141) - 107\mathrm{e}^{-0.07\times 0.5} \\ &\quad N(0.071) \\ &= \$2.977\,. \end{aligned}$$

Asset-or-nothing (AON) options are also very popular. As their name implies, AON options provide their holders the right to own the underlying assets if the options expire in-the-money. Obviously, an AON option is a special gap option when the gap parameter is zero. Therefore, the price of an AON option (AON) can be obtained by substituting $X = 0$ into (15.3):

$$AON = S\mathrm{e}^{-g\tau}N(\omega d + \omega\sigma\sqrt{\tau})\,, \tag{15.4}$$

where all parameters are the same as in (15.3).

We will discuss the properties of CON, AON, and gap options in more details when we study correlation digital options in the following sections.

Example 15.3. Find the prices of the AON call and put options to expire in half a year with the gap parameter $X = \$102$ and other parameters the same as in Example 15.2.

Substituting $S = \$100$, $K = \$105$, $X = \$102$, $r = 0.07$, $g = 0.03$, $\sigma = 0.20$, and $\tau = 0.50$ into (15.2) yields

$$AON(\omega = 1) = S\,\mathrm{e}^{-g\tau}N(d + \sigma\sqrt{\tau}) = 100^{-0.03\times 0.5}N(0.071 + 0.141)$$
$$= \$57.53\,,$$

and

$$AON(\omega = -1) = S\mathrm{e}^{-g\tau}N(-d - \sigma\sqrt{\tau})$$
$$= 100\mathrm{e}^{-0.03\times 0.5}N(-0.071 - 0.141) = \$40.98\,.$$

The AON options studied above are AON options with one strike price, whether an asset is paid or not depending on whether the underlying asset price is below or above this strike price. There is another kind of AON options which are also very popular in practice. This kind of AON options are called supershares which were first proposed by Hakansson (1976) and priced by Garman (1978). Supershares can be considered as special AON options. A supershare is an AON option because the asset is paid if the underlying asset price at the option's maturity ends up within a certain range and nothing is paid otherwise. Suppose that the range for a supershare is specified as from K_1 to K_2. A supershare can be considered as a portfolio of two regular AON options: a long AON call at K_1 and short AON call at K_2. Using the pricing formula for regular AON options given in (15.4), we can readily find the price of a supershare (PSPS) as follows

$$PSPS = S\mathrm{e}^{-g\tau}\{N[d(S, K_1) + \sigma\sqrt{\tau}] - N[d(S, K_2) + \sigma\sqrt{\tau}]\}$$

where $d(S, K)$ is the same argument as in the extended Black-Scholes formula with spot and strike prices S and K, respectively.

15.3. AMERICAN DIGITAL OPTIONS

An American digital option pays off one dollar immediately if the strike price is touched at any time during the life of the option. American digital options can be considered as digital options with nondeferrable payment. As in the analysis of rebates for "out" options in Chapter 10, the one-dollar payment can also be deferred to maturity. Such digital options with the one-dollar payment deferred to maturity are called one-touch digitals, implying that one dollar is paid at maturity if the strike price or the breakpoint is touched any time during the life of the option. In other words, one-touch digitals are American digitals when the one dollar is deferred to maturity. We try to provide closed-form solutions for these American digital options in this section.

15.3.1. Nondeferrable American Digitals

American digitals are not new products, they are special cases of the knockout barrier options analyzed in Chapters 10 and 11. The payoff of a nondeferred American digital is a special case of the nondeferred rebate when the rebate is fixed at one dollar. Since the price of an American digital option is the present value of one dollar without growth for a knockout barrier option, we can readily obtain the pricing formula of an American digital option by using the closed-form solution for the present value of the rebate of a window out barrier options in Chapter 11. Substituting $R = 1$ and $\eta = 0$ into the formula of the present value of the rebate of a window out barrier option given in (11.51) yields the price for a one-dollar American digital option (P1AD):

$$\begin{aligned} P1AD = &\left(\frac{H}{S}\right)^{q_1} e^{-(r+vq_1-\sigma^2 q_1^2/2)\tau_1}\Big[N_2(D_1, -DD_1, c) + N_2(-D_1, DD_1, c)\Big] \\ &+ \left(\frac{H}{S}\right)^{q_{-1}} e^{-(r+vq_{-1}-\sigma^2 q_{-1}^2/2)\tau_1} \\ &\times \Big[N_2(D_{-1}, -DD_{-1}, c) + N_2(-D_{-1}, DD_{-1}, c)\Big], \end{aligned} \tag{15.5}$$

where

$$\begin{aligned} D_\nu &= d_{bs}(S, H, \tau_1) - \sigma q_\nu \sqrt{\tau_1}, \\ DD_\nu &= d_{bs}(S, H, \tau_1 + \tau_e) - \sigma q_\nu \sqrt{\tau_1 + \tau_e}, \\ c &= -\sqrt{\frac{\tau_1}{\tau_1 + \tau_e}}, \\ \psi(r) &= \sqrt{v^2 + 2r\sigma^2}, \\ q_v(r) &= \frac{v + \nu\psi(r)}{\sigma^2}, \nu = 1 \text{ or } -1, \end{aligned}$$

$d_{bs}(S, H, s)$ is the same as in (10.31) with spot and strike prices S and H, and time to maturity s, respectively; τ_1, τ_e, and τ are the time when the barrier starts to be effective, the time when the barrier ends to be effective, and the time to maturity of the option, respectively; $N_2(a, b, \rho)$ is the cumulative function of a standard bivariate normal distribution with upper bounds a and b and correlation coefficient ρ, and all other parameters are the same as in (11.51).

The pricing formula for a one-dollar American digital option given in (15.5) is a very general one because it permits the barrier to be effective within a subset during the life of the option as in one-window barrier options. It is obvious that the pricing formula given in (15.5) degenerates to the present value of a one-dollar American option when the barrier is effective throughout the life of the option when the forward starting time $\tau_1 = 0$ and $\tau_e = \tau$ as in vanilla options. Substituting $\tau_1 = 0$ an $\tau_e = \tau$ (15.5) yields the price of a one-dollar American digital option when the barrier is effective throughout the whole life of the option as in vanilla barrier options:

$$P1AD(\tau_1 = 0 \ \& \ \tau_e = \tau) = \left(\frac{H}{S}\right)^{q_1(r)} N[\theta Q_1(r)] + \left(\frac{H}{S}\right)^{q_{-1}(r)} N[\theta Q_{-1}(r)]\,, \tag{15.6}$$

where

$$\psi(r) = \sqrt{v^2 + 2r\sigma^2}\,,$$
$$Q_\nu(r) = \frac{\ln(H/S) + \nu\tau\psi(r)}{\sigma\sqrt{\tau}}, \nu = 1 \text{ or } -1\,,$$
$$q_\nu(r) = \frac{v + \nu\psi(r)}{\sigma^2}\,,$$

where all parameters are the same as in (15.5).

Example 15.4. Find the price of the one-dollar American digital option to expire in one year if the spot and strike prices are \$100 and \$95, interest rate 8%, the payout rate of the underlying asset is 3%, the volatility of the underlying asset 20%, the forward starting time is the present, and the earlier ending time is also one year.

Substituting $S = K = \$100$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau = 1$ and $v = r - g - \sigma^2/2 = 0.03$ into (15.6) yields

$$d_{bs} = [\ln(100/95) + 0.03 \times 1]/(0.20\sqrt{1}) = 0.4065\,,$$
$$\psi(r) = \sqrt{0.03^3 + 2 \times 0.08 \times 0.20^2} = 0.0854\,,$$
$$q_1(r) = (0.03 + 0.0854)/0.20^2 = 2.886\,,$$
$$q_{-1}(r) = (0.03 - 0.0854)/0.20^2 = -1.385\,,$$
$$Q_1 = [\ln(95/100) + 1 \times 0.0854]/(0.20 \times \sqrt{1}) = 0.1705\,,$$
$$Q_{-1} = [\ln(95/100) - 1 \times 0.0854]/(0.20 \times \sqrt{1}) = -0.6835\,,$$
$$P1AD(\tau_1 = 0\tau_e = \tau) = 0.95^{2.886}N(0.1705) + 0.95^{-1.835}N(-0.6835)$$
$$= \$0.7459\,.$$

Example 15.5. Find the price of the one-dollar American digital option in Example 15.4 if the forward starting time is three months, and the ending time is the same as the maturity time of the option.

Substituting $S = K = \$100$, $r = 0.08$, $g = 0.03$, $\sigma = 0.20$, $\tau_1 = 0.25$, $\tau = \tau_e = 1$, and $v = r - g - \sigma^2/2 = 0.03$ into (15.5) and using some results in Example 15.4 yields

$$(r + vq_1 - q_1^2\sigma^2/2)\tau_1 = -0.0433\,,$$

$$(r + vq_{-1} - q_{-1}^2/2)\tau_1 = 0.0208\,,$$

$$d_{bs}(S, H, \tau_1) = 0.5879\,,$$

$$d_{bs}(S, H, \tau_1 + \tau_e) = 0.3971\,,$$

$$D_1 = 0.5879 - 0.20 \times 2.886\sqrt{0.25} = 0.2993\,,$$

$$D_{-1} = 0.5879 - 0.20 \times (-1.385)\sqrt{0.25} = 0.7264\,,$$

$$DD_1 = 0.3971 - 0.20 \times 2.886\sqrt{1.25} = 0.2482\,,$$

$$DD_{-1} = 0.3971 - 0.20 \times (-1.385)\sqrt{1.25} = 0.7068\,,$$

$$\rho = -\sqrt{0.25/1.25} = -0.4472\,,$$

and therefore the American digital option price is

$$\begin{aligned} P1AD = {} & 0.95^{2.886}\mathrm{e}^{-0.0433}\Big[N_2(0.2993, 0.2468, -0.4472) \\ & + N_2(-0.2993, -0.2482, -0.4472)\Big] \\ & + 0.95^{-1.385}\mathrm{e}^{-0.0208}\Big[N_2(0.7264, -0.7068, -0.4472) \\ & + N_2(-0.7264, 0.7068, -0.4472)\Big] \\ = {} & \$0.5998\,. \end{aligned}$$

Comparing the results in Examples 15.4 and 15.5, we can readily find that the price of the American digital option with forward start feature \$0.5998 is much cheaper than the price of the corresponding option without the forward start feature \$0.7549. The cheaper price results from the fact that the barrier is effective in a shorter period of time and the probability that the barrier is touched is less.

15.3.2. One-Touch Digitals

A one-touch digital option is actually an American digital option with payment deferred to the maturity of the optoin if the breakpoint is touched any time during the life of the option. Using the arguments made in Chapter 10 for the present values of deferrable rebates for knockout options, we can obtain the price of a deferrable American digital option corresponding to (15.6)

$$OTD = e^{-r\tau}\left[N\left(\frac{v-a}{\sigma\sqrt{\tau}}\right) + \left(\frac{H}{S}\right)^{2v/\sigma^2} N\left(\frac{v+a}{\sigma\sqrt{\tau}}\right)\right], \tag{15.7}$$

where all parameters are the same as in (15.6).

With the above arguments, we can regard a knockout barrier option with nondeferrable rebate as a portfolio of a vanilla barrier option without any rebate and an American digital option, and a standard barrier option with deferrable rebate as a portfolio of a vanilla barrier option without any rebate and a one-touch digital option.

15.4. DOUBLE-DIGITAL OPTIONS

Double-digital options are also called double digitals, or range binaries. As in the case of double-barrier options with two barriers, there are two boundaries for a double digital, one normally above and the other below the spot price. A European double-digital option pays one dollar if the underlying asset price at the option maturity ends within the range defined by the two boundaries, and nothing if otherwise. Thus, the price of a double digital is simply the probability that the underlying asset price ends up within the two boundaries discounted by the risk-free interest rate. Using the density function given in (11.63) with double barriers, we can obtain the price of a double digital (PDD):

$$\begin{aligned} PDD = e^{-r\tau} \sum_{n=-\infty}^{+\infty} \Bigg[\left(\frac{U}{L}\right)^{2nv/\sigma^2} & \left\{ N\left[\omega d_{bs}(S,K,v) + \left(\frac{\omega X_n'}{\sigma\sqrt{\tau}}\right)\right] \right. \\ & \left. - N\left[\omega d_{bs}(S,W_\omega,v) + \left(\frac{\omega X_n'}{\sigma\sqrt{\tau}}\right)\right]\right\} \\ & - \left(\frac{U}{S}\right)^{2v/\sigma^2} \left(\frac{L}{U}\right)^{2nv/\sigma^2} \left\{ N\left[\omega d_{bs}(S,K,v) + \left(\frac{\omega X_n''}{\sigma\sqrt{\tau}}\right)\right] \right. \\ & \left. - N\left[\omega d_{bs}(S,W_\omega,v) + \left(\frac{\omega X_n''}{\sigma\sqrt{\tau}}\right)\right]\right\}\Bigg], \end{aligned} \tag{15.8}$$

where $W_\omega = U$ if $\omega = 1$ and $W_\omega = L$ if $\omega = -1$, U and L are the upper and lower bounds for the option, and all other parameters are the same as in (11.66).

Alternatively, we can obtain *PDD* if we use the density function given in (11.72):

$$FCRDP = e^{-r\tau}\sigma^4(a-b)\left(\frac{L}{S}\right)^{v/\sigma^2}\sum_{n=1}^{\infty}$$

$$\times\left\{\frac{na_n e^{-\lambda_n\tau}}{n^2\pi^2\sigma^4+v^2(a-b)^2}\left[(-1)^{n-1}\left(\frac{U}{L}\right)^{v/\sigma^2}\right]^{(1+\omega)/2}+\omega\right\}, \tag{15.9}$$

where all parameters are the same as in (11.73).

The above analysis is for a European double-digital option. Their corresponding American digital option pays off one dollar as soon as either the upper bound or the lower bound is touched. As the prices of American digital options with one barrier are special cases of the present values of rebates of the knockout options with one barrier, the prices of American digital options with double barriers are special cases of the present values of the rebates double-barrier knockout options. We can obtain the price of an American double digital option (P1ADD) by substituting $R = 1$ and $\eta = 0$ into the present value of the rebate of a knockout option with double barriers given in (11.72)

$$P1ADD = V_u + V_l \tag{15.10}$$

where V_u and V_l are the same as in (11.72).

15.5. CORRELATION DIGITAL OPTIONS

Suppose that there are two assets or indices, one of which is called the measurement instrument or asset, and the other the payment asset. Let $S(t)$ and $M(t)$ represent the prices of the payment asset and the measurement asset, respectively, as in outside barrier options in Chapter 11. Suppose further that both the underlying and the payoff asset prices $S(t)$ and $M(t)$ follow the standard stochastic process given in (IV1), the returns of the two assets are correlated with the correlation coefficient ρ, and σ and σ_2 are the instantaneous standard deviations of the two prices, respectively.

The payoff of a European-style correlation digital option (POCD) can now be expressed as follows:

$$\begin{aligned} POCD &= \omega[S(\tau)-X] \text{ if } \omega M(\tau) \geq \omega K \\ &= 0 \text{ if otherwise}, \end{aligned} \tag{15.11}$$

where K is the strike price of the option; X is a prespecified price to determine the level of gap around the payoff asset price; max (. , .) is a function that gives the larger of two numbers, and ω is a binary operator (1 for a call option and -1 for a put option).

It is obvious that (15.11) becomes precisely the same as the payoff of an ordinary gap option if the payoff asset is exactly the same as the underlying asset, and it becomes the same as an ordinary asset-or-nothing binary option if X is also set to zero.

15.6. PRICING CORRELATION DIGITAL OPTIONS

In order to price all kinds of options illustrated so far in this book, we need the distribution of the underlying asset price. Let x and y represent the log-returns of the payment and the measurement assets, respectively. With the joint density function between x and y given in (IV4) and (IV5), we can find the conditional density function of x under the condition of $\omega M(\tau) > \omega K$, or $\omega y > \omega \ln(M/S)$, $\omega v > \omega d(M, K, \sigma^2)$:

$$f[u|\omega v > \omega d(M,K,\sigma_2,g_2)] = f(u)N\left[\frac{d(M,K,\sigma_2,g_2)+\rho u}{\sqrt{1-\rho^2}}\right], \quad (15.12)$$

where

$$d(A,B,C,D) = \frac{\ln(A/B) + (r - D - C^2/2)\tau}{C\sqrt{\tau}},$$

is the argument in the Black-Scholes formula with spot and strike prices A and B, volatility C and payout rate D, respectively; M, σ_2, g_2 are the spot price, volatility, and payout rate of the measurement asset, respectively;

$$u = \frac{x - (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \text{ and } v = \frac{y - (r - g_2 - \sigma_2^2)\tau}{\sigma_2\sqrt{\tau}},$$

are the two standardized variables corresponding to the log-returns of the payment and measurement assets, respectively; and ω is the same option operator in (15.11).

Using the conditional density function given in (15.9), we can obtain the expected payoff of a European correlation digital call option given in (15.9):

$$E(POCD) = \omega S e^{(r-g)\tau} N_2[\omega d_1(S,K,\sigma,g), \omega d(M,K,\sigma_2,g_2) + \omega\rho\sigma\sqrt{\tau}, \rho]$$
$$- \omega X N_2[\omega d(S,K,\sigma,g),\ \omega d(M,K,\sigma_2,g_2), \rho], \quad (15.13)$$

where $d_1(S,K,\sigma,g) = d(S,K,\sigma,g) + \sigma\sqrt{\tau}$, and $N_2[a,b,\rho]$ is the value of the cumulative function of the standard bivariate normal distribution with upper limits a and b for the first and the second arguments and correlation coefficient ρ, and other parameters are the same as in (15.12).

The no-arbitrage argument permits us to use the risk-neutral valuation approach by discounting the expected payoff of an option at expiration by the risk-free rate of return. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate, we can obtain the correlation digital option price ($CDOP$) by discounting the expected payoff given in (15.13) the risk-free rate r:

$$\begin{aligned} CDOP = {} & \omega S e^{-g\tau} N_2[\omega d_1(S,K,\sigma,g), \theta d(M,K,\sigma_2,g_2) + \theta\rho\sigma_2\sqrt{\tau}, \omega\theta\rho] \\ & - \omega X e^{-r\tau} N_2[\omega d(S,K,\sigma,g), \theta d(M,K,\sigma_2,g_2), \omega\theta\rho]\,, \qquad (15.14) \end{aligned}$$

where $CDOP$ is the correlation digital option price and all parameters are the same as in (15.13).

The pricing formula in (15.14) has two interesting characteristics. One is that it is similar to the plain vanilla option pricing formula as there are only two terms and the univariate normal cumulative functions $N(.)$ are replaced by the bivariate normal cumulative functions $N_2(.,.,.)$. The other is that the correlation coefficient plays an important role in the pricing formula.

Example 15.6. Find the prices of the correlation digital call and put options to expire in half a year, given the spot prices of the payment and measurement assets \$100 and \$50, the payout rates of the two assets 3% and 5%, the volatilities of the two assets 20% and 10%, respectively, the interest rate is 8%, the strike price is \$45, the correlation coefficient between the two assets 80%, and the gap parameter $X = \$105$.

Substituting $S = \$100$, $M = \$80$, $K = \$85$, $X = \$105$, $\sigma = 0.20$, $\sigma_2 = 0.15$, $r = 0.08$, $g = 0.03$, $g_2 = 0.05$, $\sigma = 0.20$, $\sigma_2 = 0.15$, and $\tau = 0.50$ into (15.14) yields

$$d(S,K,\sigma,g) = \frac{\ln(100/85)(0.08 - 0.03 - 0.20^2/2)0.50}{0.20\sqrt{0.50}} = 1.2552\,,$$

$$d(M,K,\sigma_2,g_2) = \frac{\ln(80/85) + (0.08 - 0.05 - 0.10'2)0.50}{0.10\sqrt{0.50}} = -0.6806\,,$$

the call option price is

$$\begin{aligned} CDOP(\omega = 1) &= 100e^{-0.03\times 0.50}N_2(1.25520.20\sqrt{0.50}, -0.6806 \\ &\quad + 0.80 \times 0.10\sqrt{0.50}, 0.80) \\ &\quad - 105e^{0.08\times 0.50}N_2(1.2552, -0.6806, 0.80) \\ &= \$1.214\,, \end{aligned}$$

and the corresponding put option price is

$$\begin{aligned} CDOP(\omega = -1) &= -100e^{-0.03\times 0.50}N_2(-1.2552 - 0.20\sqrt{0.50}, 0.6806 \\ &\quad - 0.80 \times 0.10\sqrt{0.50}, 0.80) \\ &\quad + 105e^{-0.08\times 0.50}N_2(-1.2552, 0.6806, 0.80) \\ &= \$2.554\,. \end{aligned}$$

Table 15.1 lists the correlation digital call option values for various correlation coefficient ρ ranging from -0.9 to 0.90, given the underlying asset spot price S = \$100, the payment spot price S = \$100, the strike price K = \$100, the gap parameter X = \$100, time to maturity τ = 1 year, the volatility of the underlying asset $\acute{\sigma}$ = 10%, the volatility of the payoff asset σ = 10%, the interest rate r = 10%, and other combinations of these given parameters specified in Table 15.1. From Table 15.1 we can observe that the correlation digital call option value increases strictly with the correlation coefficient given other parameters the same and that the value varies significantly with various correlation coefficients. For instance, the price of the at-the-money option with the correlation coeffiwent $\rho = 0.90$ is \$4.7976, ten times as large as the price 0.9419 with $\rho = -0.90$. In fact, the correlation option value increases nearly linearly with the correlation coefficient given other parameters the same.

Figure 15.3 depicts the prices of correlation digital options for various correlation coefficients from -90 to 90%, given the spot prices of the underlying asset and the measurement asset \$100, strike price \$90, time to maturity half a year, interest rate 8%, the payouts of the two assets 3%, the volatilities of the two assets 20%, and the gap parameters \$95 and \$98, respectively. The dotted curve represents the prices of the correlation digital options with gap parameter \$98 and the undotted curve the prices of the corelation digital options with gap parameter \$95. It is obvious from Figure 15.3 that the option prices are lower with higher gap parameters.

Table 15.1. Values of correlation digital call options and their sensitivities for various correlation coefficients

$S = S_p$ $K = X = \$100$, $\tau = 0.5$, $\sigma = \sigma_p = 0.10$, $r = 0.10$				
ρ	CDOP	Delta	Gamma	Chi
−0.9	0.4914	0.2112	0.0562	16.0719
−0.8	0.7793	0.2411	0.0588	8.3617
−0.7	1.0381	0.2657	0.0557	5.7691
−0.6	1.2828	0.2875	0.0566	4.4650
−0.5	1.5192	0.3076	0.0555	3.6805
−0.4	1.7502	0.3265	0.0544	3.1583
−0.3	1.9776	0.3466	0.0544	2.7876
−0.2	2.2026	0.3622	0.0553	2.5131
−0.1	2.4261	0.3796	0.0553	2.3041
0.0	2.6489	0.3967	0.0553	2.1424
0.1	2.8714	0.4139	0.0552	2.0169
0.2	3.0946	0.4312	0.0552	1.9206
0.3	3.3191	0.4489	0.0552	1.8497
0.4	3.5459	0.4695	0.0552	1.8037
0.5	3.7761	0.4886	0.0552	1.7824
0.6	4.0115	0.5090	0.0551	1.7927
0.7	4.2550	0.5313	0.0551	1.8500
0.8	4.5121	0.5569	0.0551	1.9977
0.9	4.7976	0.5890	0.0550	2.4134

$S = S_p$ $K = X = \$100$, $\tau = 1$, $\sigma = \sigma_p = 0.10$, $r = 0.10$				
ρ	CDOP	Delta	Gamma	Chi
−0.9	0.8923	0.2612	0.0391	23.7379
−0.8	1.3779	0.2919	0.0387	12.5052
−0.7	1.8070	0.3173	0.0386	8.7165
−0.6	2.2088	0.3398	0.0385	6.8069
−0.5	2.5943	0.3605	0.0384	5.6572
−0.4	2.9694	0.3801	0.0383	4.8922
−0.3	3.3375	0.3990	0.0383	4.3503
−0.2	3.7010	0.4173	0.0382	3.9508
−0.1	4.0616	0.4353	0.0382	3.6487
0.0	4.4207	0.4532	0.0382	3.4178
0.1	4.7795	0.4711	0.0382	3.2421
0.2	5.1395	0.4891	0.0382	3.1119
0.3	5.5021	0.5075	0.0381	3.0223
0.4	5.8689	0.5265	0.0381	2.9727
0.5	6.2424	0.5462	0.0381	2.9674
0.6	6.6258	0.5672	0.0380	3.0190
0.7	7.0249	0.5901	0.0380	3.1586
0.8	7.4504	0.6161	0.0380	3.4711
0.9	7.9312	0.6485	0.0379	4.3033

Table 15.1 (*Continued*)

$S = S_p = K = X = \$100$, $\tau = 1$, $\sigma = 0.20$, $\sigma_p = 0.10$, $r = 0.10$				
ρ	CDOP	Delta	Gamma	Chi
−0.9	0.9092	0.2649	0.0396	23.9318
−0.8	1.3966	0.2950	0.0391	12.5699
−0.7	1.8269	0.3201	0.0389	8.7514
−0.6	2.2295	0.3424	0.0387	6.8300
−0.5	2.6157	0.3631	0.0386	5.6746
−0.4	2.9913	0.3826	0.0386	4.9065
−0.3	3.3598	0.4014	0.0385	4.3629
−0.2	3.7235	0.4197	0.0385	3.9623
−0.1	4.0843	0.4377	0.0384	3.6597
0.0	4.4435	0.4556	0.0384	3.4288
0.1	4.8022	0.4735	0.0384	3.2533
0.2	5.1621	0.4915	0.0383	3.1236
0.3	5.5244	0.5100	0.0383	3.0350
0.4	5.8909	0.5289	0.0383	2.9868
0.5	6.2638	0.5488	0.0383	2.9836
0.6	6.6448	0.5698	0.0383	3.0385
0.7	7.0430	0.5929	0.0383	3.1836
0.8	7.4373	0.6191	0.0383	3.5071
0.9	7.9461	0.6521	0.0384	4.3717

$S = S_p = X = \$100$, $K = \$90$, $\tau = 1$, $\sigma = \sigma_p = 0.10$, $r = 0.10$				
ρ	CDOP	Delta	Gamma	Chi
−0.9	5.2707	0.7323	0.0751	21.8966
−0.8	5.5420	0.7276	0.0729	11.5480
−0.7	5.8201	0.7256	0.0707	8.1772
−0.6	6.0932	0.7269	0.0689	6.5444
−0.5	6.3570	0.7307	0.0675	5.6068
−0.4	6.6101	0.7364	0.0665	5.0209
−0.3	6.8521	0.7434	0.0658	4.6419
−0.2	7.0831	0.7514	0.0653	4.4000
−0.1	7.3032	0.7604	0.0650	4.2594
0.0	7.5125	0.7702	0.0649	4.2021
0.1	7.7105	0.7807	0.0649	4.2220
0.2	7.8985	0.7920	0.0652	4.3227
0.3	8.0746	0.8041	0.0657	4.5186
0.4	8.2384	0.8172	0.0664	4.8408
0.5	8.3886	0.8313	0.0674	5.3510
0.6	8.5229	0.8467	0.0687	6.1773
0.7	8.6372	0.8634	0.0705	7.6260
0.8	8.7239	0.8811	0.0727	10.6272
0.9	8.7699	0.8963	0.0750	19.8725

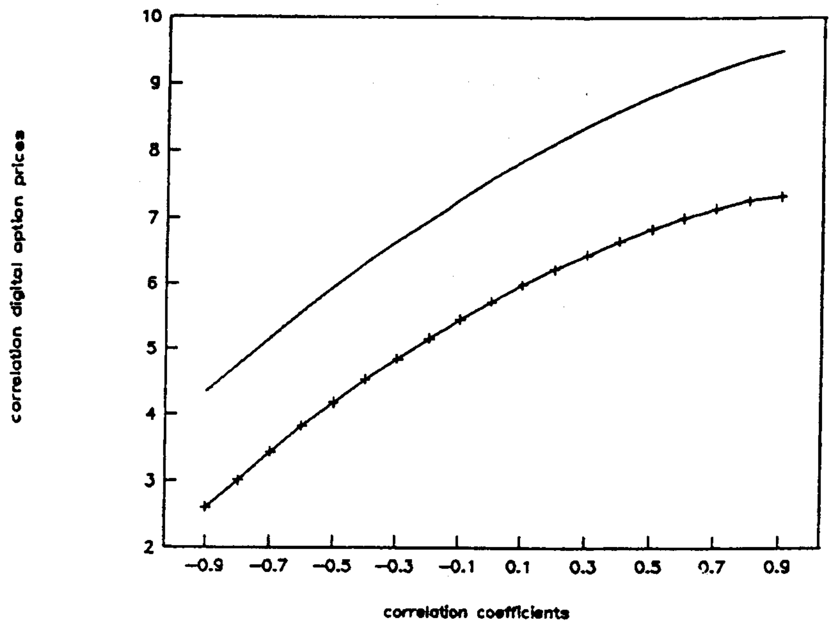

Fig. 15.3. Correlation digital prices for various correlation coefficients.

We can also observe that the chis of the correlation digital options are all positive with chosen paramters.

15.7. SPECIAL CASES OF CORRELATION DIGITAL OPTIONS

The pricing formula of a correlation digital option in (15.14) is very general as it includes all three ordinary digital options as special cases. We show these cases in this section.

15.7.1. Ordinary Gap Options

If the payoff asset is exactly the same as the underlying asset, then $S = M$, $\sigma_2 = \sigma$ $g_2 = g$ and $\rho = 1$. Substituting $g_2 = g$, $\sigma_2 = \sigma$, and $\rho = 1$ into (15.14) yields

$$OGP = \omega S e^{-g\tau} N_2\Big(\omega d_1, \omega d_1, 1\Big) - \omega X e^{-r\tau} N_2(\omega d, \omega d, 1) . \qquad (15.15)$$

Formula (15.15) cannot be used directly because a bivariate normal cumulative function with a perfect correlation coefficient $\rho = 1$ does not exist.

However, this can be easily overcome by the following identity[2]

$$N_2(z, z, 1) = N(z),$$

for and $z \in (-\infty, \infty)$.

Substituting (15.16) into (15.15) yields

$$OGP = \omega e^{-g\tau} SN(\omega d + \omega\sigma\sqrt{\tau}) - \omega X e^{-r\tau} N(\omega d), \qquad (15.16)$$

which is precisely the pricing formula of an ordinary gap option given in (10.31).

Vanilla options are a special case of ordinary gap options by substituting $X = K$ into (15.16).

15.7.2. Ordinary Asset-Or-Nothing Options

The pricing formula of an ordinary asset-or-nothing option (AON) given in (15.3) can be obtained by substituting $X = 0$ into (15.17).

15.7.3. Ordinary Cash-Or-Nothing Options

Although an AON option is a special case of an ordinary gap option, the ordinary cash-or-nothing (CON) option a not special case of the ordinary gap option in a strict sense, as its price cannot be obtained from the pricing formula of an ordinary gap options directly. However, the ordinary CON option is a special case of the correlation digital option when $M = Cash$, and $X = g_2 = \sigma_2 = \rho = 0$. Substituting these values into (15.14) and discounting it at the risk-free rate r yields[3]

$$CON = e^{-r\tau} Cash\ N(\omega d). \qquad (15.17)$$

15.7.4. "Another Asset-Or-Nothing Options"

The payoff of an ordinary asset-or-nothing option is the same as the underlying asset. The payoff of a correlated AON, or "another asset-or-nothing" (AAON) option is the other asset. The pricing formula of an AAON can be obtained by substituting $X = 0$ into (15.14)

$$AAON = \omega S e^{-g\tau} N_2[\omega d_1(S, K, \sigma, g), \omega d(M, K, \sigma_2, g_2) + \omega\rho\sigma_2\sqrt{\tau}, \rho], \qquad (15.18)$$

[2]This is a special case of the identity given in (11.38) and is proved in Appendix of Chapter 11, thus $N_2(z, z, 1) = N[\min(z, z)] = N(z)$.

[3]$d(M, K, \sigma_2)$ approaches infinity as $M = Cash$ and $X = \mu = \sigma_2 = \rho = 0$. When the correlation coefficient is zero, the two variables are stochastically independent, it can be readily shown $N_2(\theta_1, \theta_2, 0) = N(\theta_1)N(\theta_2)$. Therefore $N_2[d_p + \rho\sigma_p\sqrt{\tau}, 0] = N(\infty)N(d) = N(d)$.

where *CDOP* is the correlation digital option price and all parameters are the same as in (15.14).

15.7.5. The Trivial Case of Independence

The trivial case of independence between the two payment and measurement assets can help us understand the pricing formula given in (15.14) better. Substituting $\rho = 0$ into (15.14) and after simplifications yields the following

$$CDOP(\rho = 0) = N[\omega d(M, K, \sigma_2, g_2)]C_{\mathrm{bs}}(S, K, \sigma, g, \omega)\,, \qquad (15.19)$$

where $C_{\mathrm{bs}}(S, K, \sigma, g, \omega)$ is the vanilla option price with the spot price S, strike price K, volatility σ, payout rate g and option operator ω as given in (10.31).

Since the first factor in (15.19) is actually the probability that the measurement asset price ends up above (resp. below) the strike price for a call (resp. put) option, the pricing formula given in (15.19) can be interpreted as the pricing formula of a vanilla option multiplied by the probability that the option is "knocked" in at maturity. This result is consistent with our intuition because the two assets are stochastically independent and the pricing formula of the option on the payment asset is actually the same as that of the vanilla option with probability adjustment.

15.8. SENSITIVITIES

Due to their unique payoff patterns, digital options have sensitivities very different from those of most other options. In this section we will compare the sensitivities of the traditional and correlation digital options.

15.8.1. Deltas

The delta of an ordinary gap option can be obtained by taking the first-order derivative of (15.3) with respect to S:

$$\delta_{og} = we^{-g\tau}N(\omega d + \omega\sigma\sqrt{\tau}) + \frac{K - X}{S\sigma\sqrt{\tau}}e^{-r\tau}(d)\,, \qquad (15.20)$$

where d is the same as in (15.3).

It is obvious that the delta of the ordinary gap call option in (15.20) becomes precisely the same as the delta of a plain vanilla option when $K = X$. Substituting $X = 0$ into (15.20) yields the delta of an ordinary AON option:

$$\delta_{oaon} = we^{-g\tau}N(\omega d + \omega\sigma\sqrt{\tau}) + e^{-g\tau}\frac{f(d_1)}{\sigma\sqrt{\tau}}. \tag{15.21}$$

The delta of an ordinary CON call option can be obtained by taking the first-order derivative of (15.14) with respect to S:

$$\delta_{con} = \omega\frac{Cash}{S\sigma\sqrt{\tau}}e^{-r\tau}f(d). \tag{15.22}$$

From (15.21), we know that the delta of an ordinary AON call option jumps from zero to

$$\delta_{oaon}(S = K) = N\left[\frac{(r - g + \sigma^2/2)\sqrt{\tau}}{\sigma}\right] + \frac{e^{-g\tau}}{\sigma\sqrt{\tau}}f\left[\frac{(r - g + \sigma^2/2)\sqrt{\tau}}{\sigma}\right],$$

as the underlying asset price increases from $S < K$ to $S = K$. The delta of an ordinary asset-or-nothing call option at $S = K$, $\delta_{oaon}(S = K)$, is clearly always greater than that of a plain vanilla option by an extra positive amount. This implies that an ordinary AON option is more sensitive to the underlying asset price change than its corresponding plain vanilla option.

From (15.22), we know that the delta of a CON call option jumps from zero to

$$\delta_{con}(S = K) = \frac{Cash}{S\sigma\sqrt{\tau}}e^{-r\tau}f\left[\frac{(r - g - \sigma^2/2)\sqrt{\tau}}{\sigma}\right],$$

as the underlying asset price increases from $S < K$ to $S = K$.

The delta of a correlation digital option with respect to the current payoff asset price can be obtained as follows (see Appendix of this chapter for the proof):

$$\begin{aligned}\delta cd = {} & \omega e^{-g\tau}N_2[\omega d_1(S, K, \sigma, g), \omega d(M, K, \sigma_2, g_2) + \omega\rho\sigma_2\sqrt{\tau}, \rho] \\ & - \rho\frac{f[d_1(S, K, \sigma, g)]}{\sigma\sqrt{\tau}}\left\{N\left[\frac{\omega d_1(S, K, \sigma, g)}{\sqrt{1 + \rho}}\right] - N\left[\frac{\omega d(S, K, \sigma, g)}{\sqrt{1 + \rho}}\right]\right\},\end{aligned} \tag{15.23}$$

where all parameters are the same as in (15.14) and $f(.)$ is the density function of a standard normal distribution.

We calculated the deltas of various correlation call options, the results of which are listed in Table 15.1, for various correlation coefficients, given

the same combination of other parameters. From Table 15.1 we can observe that the deltas vary significantly with various correlation coefficients. This is consistent with the fact that correlation digital option values change significantly with various correlation coefficients.

15.8.2. Gammas

The gamma of an ordinary gap option can be readily obtained by taking the first-order derivative of (15.23) with respect to S:

$$\gamma_{og} = \frac{\gamma_{pv}}{K}\left\{X + \frac{(X-K)d}{\sigma\sqrt{\tau}}\right\}, \tag{15.24}$$

where $\gamma_{pv} = f(d_1)/[S\sigma\sqrt{\tau}] > 0$ is the gamma of a vanilla option.

It is obvious that the gamma of an ordinary gap call option in (15.24) becomes precisely the same as that of a vanilla option when $K = X$. Substituting $X = 0$ into (15.24) yields the gamma of an ordinary AON option,

$$\gamma_{oaon} = -\frac{\gamma_{pv}}{\sigma\sqrt{\tau}}d. \tag{15.25}$$

The gamma of an ordinary CON call option can be obtained by taking the first-order derivative of (15.22) with respect to S:

$$\gamma_{con} = \frac{-\omega\, Cash}{S^2\sigma^2\tau}\mathrm{e}^{-r\tau}d_1 f(d). \tag{15.26}$$

From (15.25), we know that the gamma of an AON option jumps or drops from zero to

$$\gamma_{oaon}(S=K) = -\frac{\gamma_{pv}}{\sigma}(r - g\frac{\sigma^2}{2})\sqrt{\tau},$$

as the underlying asset price increases from $S < K$ to $S = K$. The amount of change at $S = K$ depends on the volatility of the underlying asset, interest rate, and time to maturity of the option. The gamma of an ordinary AON option at $S = K$, $\gamma_{oaon}(S = K)$, is clearly very different from that of a plain vanilla option. The gamma of an ordinary AON option can be of the same (opposite) sign as the corresponding vanilla option if d is negative (positive). As d is positive (negative) for a deep-in(out-of)-the-money vanilla call option, the gamma of an ordinary AON option changes in the same (opposite) direction with the corresponding vanilla option if it is deep-out-of(in)-the-money.

the same combination of other parameters. From Table 15.1 we can observe that the deltas vary significantly with various correlation coefficients. This is consistent with the fact that correlation digital option values change significantly with various correlation coefficients.

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It is obvious that the gamma of an ordinary gap call option in (15.24) becomes precisely the same as that of a vanilla option when $K = X$. Substituting $X = 0$ into (15.24) yields the gamma of an ordinary AON option,

$$\gamma_{oaon} = -\frac{\gamma_{pv}}{\sigma\sqrt{\tau}}d. \tag{15.25}$$

The gamma of an ordinary CON call option can be obtained by taking the first-order derivative of (15.22) with respect to S:

$$\gamma_{con} = \frac{-\omega\, Cash}{S^2\sigma^2\tau}e^{-r\tau}d_1 f(d). \tag{15.26}$$

From (15.25), we know that the gamma of an AON option jumps or drops from zero to

$$\gamma_{oaon}(S = K) = -\frac{\gamma_{pv}}{\sigma}(r - g\frac{\sigma^2}{2})\sqrt{\tau},$$

as the underlying asset price increases from $S < K$ to $S = K$. The amount of change at $S = K$ depends on the volatility of the underlying asset, interest rate, and time to maturity of the option. The gamma of an ordinary AON option at $S = K$, $\gamma_{oaon}(S = K)$, is clearly very different from that of a plain vanilla option. The gamma of an ordinary AON option can be of the same (opposite) sign as the corresponding vanilla option if d is negative (positive). As d is positive (negative) for a deep-in(out-of)-the-money vanilla call option, the gamma of an ordinary AON option changes in the same (opposite) direction with the corresponding vanilla option if it is deep-out-of(in)-the-money.

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$$\gamma_{og} = \frac{\gamma_{pv}}{K}\left\{X + \frac{(X-K)d}{\sigma\sqrt{\tau}}\right\}, \tag{15.24}$$

where $\gamma_{pv} = f(d_1)/[S\sigma\sqrt{\tau}] > 0$ is the gamma of a vanilla option.

It is obvious that the gamma of an ordinary gap call option in (15.24) becomes precisely the same as that of a vanilla option when $K = X$. Substituting $X = 0$ into (15.24) yields the gamma of an ordinary AON option,

$$\gamma_{oaon} = -\frac{\gamma_{pv}}{\sigma\sqrt{\tau}}d. \tag{15.25}$$

The gamma of an ordinary CON call option can be obtained by taking the first-order derivative of (15.22) with respect to S:

$$\gamma_{con} = \frac{-\omega\, Cash}{S^2\sigma^2\tau}\mathrm{e}^{-r\tau}d_1 f(d). \tag{15.26}$$

From (15.25), we know that the gamma of an AON option jumps or drops from zero to

$$\gamma_{oaon}(S = K) = -\frac{\gamma_{pv}}{\sigma}(r - g\frac{\sigma^2}{2})\sqrt{\tau},$$

as the underlying asset price increases from $S < K$ to $S = K$. The amount of change at $S = K$ depends on the volatility of the underlying asset, interest rate, and time to maturity of the option. The gamma of an ordinary AON option at $S = K$, $\gamma_{oaon}(S = K)$, is clearly very different from that of a plain vanilla option. The gamma of an ordinary AON option can be of the same (opposite) sign as the corresponding vanilla option if d is negative (positive). As d is positive (negative) for a deep-in(out-of)-the-money vanilla call option, the gamma of an ordinary AON option changes in the same (opposite) direction with the corresponding vanilla option if it is deep-out-of(in)-the-money.

The gamma of a correlation digital option can be obtained as follows (see Appendix of this chapter for an outline of the proof):

$$\gamma_{cb} = \frac{-\rho f(d_1)}{S\sigma\sqrt{\tau}} N[(h_1) + N(h_2)] + \frac{\rho f(d_1)}{S^2\sigma\sqrt{\tau}(1-\rho^2)}[d_1 N(h_1) - S_p d_p N(h_2)]$$
$$- \frac{\rho}{S^2\sigma\sqrt{\tau}(\sqrt{1-\rho^2})^3}\left[G(d_1, d' + \rho\sigma\sqrt{\tau}, \rho) - \frac{Xe^{-r\tau}}{S}G(d, d', \rho)\right], \tag{15.27}$$

where

$$G(a, b, \rho) = -(1-\rho^2)f(a)f(a_1) + b\sqrt{1-\rho^2}f(b)N(a_2),$$
$$h_1 = d_1/\sqrt{1+\rho}, h_2 = d/\sqrt{1+\rho},$$
$$a_1 = (b - \rho a)/\sqrt{1-\rho^2}, a_2 = (a - \rho b)/\sqrt{1-\rho^2},$$
$$d_1 = d_1(S, K, \sigma, g), d' = d(M, K, \sigma_2, g_2),$$

and all other parameters are the same as in (15.14).

The gamma expression is rather complicated as it involves six terms, two of them being in terms of double integrations. Although the gamma expression is in closed-form, we cannot see clearly how the gamma changes with various parameters. We have calculated the gamma values for various sets of given parameters and the results are listed in Table 15.1. From Table 15.1 we can observe that gammas vary moderately with various correlation coefficients.

15.8.3. Chi

Sensitivities of vanilla option values with respect to various option parameters have been named in Greek letters and these names have become very popular. As we argued in Chapter 13 that the Greek letter χ is pronounced as "chi" in English and has the same first letter "c" as correlation coefficient, we may simply use chi to stand for the sensitivity of a correlation option value with respect to its correlation coefficient. Taking the partial derivative of (15.14) with respect to ρ yields the chi of a correlation digital option:

$$\chi_{cb} = \frac{-S}{(1-\rho^2)^{3/2}}\left[\rho(\rho d' + \sigma\sqrt{\tau})\sqrt{1-\rho^2}f(d_1)N(h_1) + G(d_1, d' + \rho\sigma\sqrt{\tau}, \rho)\right]$$
$$+ \frac{Xe^{-r\tau}}{(1-\rho^2)^{3/2}}\left[\rho^2 d\sqrt{1-\rho^2}f(d)N(h_2) + G(d, d', \rho)\right], \tag{15.28}$$

where $G(a, b, \rho)$, $F(a, b, \rho)$ are the same as in (15.27) and all parameters are the same as in (15.14).

Expression (15.28) indicates that the sensitivity of a correlation digital option value with respect to its correlation coefficient, the chi, is expressed in similar functions as its delta and gamma. Table 15.1 lists the chis for various correlation coefficients given the same combination of other parameters. It indicates that the chi changes tremendously with various correlation coefficients, especially those closer to -1. Other calculations show that the chi can become extremely large when ρ approaches -1.

15.9. SUMMARY AND CONCLUSIONS

We introduced and priced both European and American digital options as well as double digital options in this chapter. We have introduced the concept of correlation digital options with one measurement asset and one payoff asset. We have provided closed-form solutions for correlation digital option prices and showed that these options include all existing ordinary European digital options as special cases. Our analysis shows that digital options have sensitivities very different from those of vanilla options. Correlation digital options should have great potential because of their unique characteristics and simplicity in pricing. Indeed, many existing exotic products possess properties similar to those of correlation digital options.

Digital options can be combined with many other kinds of exotic options to form more complicated exotic options. For example, Rubinstein and Reiner (1991) studied binary barrier options and provided a detailed "family tree" of 28 possible binary barrier options. Since the objective of this book is to introduce the basic kinds of exotic options, we do not want to include these combination products because the number of combination can be too large for a single book to cover.

QUESTIONS AND EXERCISES

Questions

15.1. What is a CON option?
15.2. What is an AON option?
15.3. What is a gap option?
15.4. Why are digital options considered as the simplest options?
15.5. Under what condition can a gap option become a vanilla option?
15.6. Should an American digital option be cheaper or more expensive than its corresponding European digital option? Why?

15.7. What is the most important characteristic of all digital options?

15.8. What is a correlation digital option? Why can it be very popular?

15.9. Are American digital options with forward start barriers cheaper or more expensive than the corresponding American digital options with barriers effective throughout the lives of the options?

15.10. What are double digital options?

15.11. What is the price of a correlation digital option if the payment asset and the measurement asset are stochastically independent?

Exercises

15.1. Find the prices of the one-dollar CON call and put options to expire in half a year if the spot and strike prices are \$90, interest rate 10%, the payout rate of the underlying asset is 5%, and the volatility of the underlying asset 15%.

15.2. Find the deltas, gammas, and vegas of the CON options in Exercise 15.1.

15.3. Find the prices of the gap call and put options to expire in eight months if the spot and strike prices are \$100 and \$105, respectively, interest rate 7%, the payout rate of the underlying asset is 5%, the volatility of the underlying asset 10%, and the gap parameter $X =$ \$105.

15.4. Find the deltas, gammas, and vegas of the gap options in Exercise 15.3.

15.5. Find the prices of the corresponding AON call and put options in Exercise 15.3 and other parameters are the same as in Exercise 15.3.

15.6. Find the deltas, gammas, and vegas of the AON options in Exercise 15.5.

15.7. Find the price of the one-dollar American digital option to expire in five months if the spot and strike prices are \$80, \$85, interest rate 6%, the payout rate of the underlying asset is 2%, the volatility of the underlying asset 15%, the forward starting time is the present, and the earlier ending time is five months.

15.8. Find the price of the one-dollar American digital option if the forward starting time is two months and other parameters remain the same as in Exercise 15.7.

15.9. Find the price of the one-dollar American digital option if the earlier ending time is four months and other parameters remain the same as in Exercise 15.7.

15.10. Find the price of the one-dollar American digital option if the forward starting time is two months and the earlier ending time is four months and other parameters remain the same as in Exercise 15.7.

15.11. Find the price of the one-touch option given all the information the same as in Exercise 15.7.

15.12. Find the price of the European double-digital option if the upper barrier is \$90 and other parameters are the same as in Exercise 15.7.

15.13. Find the price of the corresponding American double-digital option in Exercise 15.12.

15.14. Find the prices of the correlation digital call and put options to expire in eight months, given the spot prices of the payment and measurement assets \$80 and \$90, the payout rate of the two assets 2% and 4%, the volatilities of the two assets 25% and 15%, respectively, interest rate is 9%, strike price is \$85, the correlation coefficient between the two assets 75%, and the gap parameter $X = \$75$.

15.15. Find the deltas, gammas, and chis of the two correlation options in Exercise 15.14.

15.16. Find the prices of the corresponding AAON options in Exercise 15.14.

15.17. Find the deltas, gamma, and chi of the AAON options in Exercise 15.16.

15.18. Find the correlation option prices in Exercise 15.14 if the correlation coefficient is 74% (hint: use the chis in Exercise 15.17).

15.19. Find the correlation option prices in Exercise 15.14 if the correlation coefficient is -1 [hint: use the two identities given in (11.42) and (11.43) in Chapter 11].

15.20. Show that the price of an American digital option given in (15.6) is a special case of the price of an American digital option given in (15.5) when the forward starting time is zero and the earlier ending time is the same as the maturity of the option.

15.21.* Find the following integration in terms of standard univariate normal density and cummulative functions: $I_2 = \int_{-\infty}^{a} f(u)f(A+Bu)du$ [Hint: follow the procedure to derive (A15.5) in Appendix.]

15.22.* Find the following integration in terms of standard univariate normal density and cummulative functions: $\Pi_2 = \int_{-\infty}^{a} uf(u)f(A+Bu)du$. [Hint: follow the procedure and derive Π in (A15.7) and see the results in Exercise 15.21.]

15.23.* Find the following integration using Exercise 15.21: $\int_a^\infty f(u)f(A + Bu)du$.

APPENDIX

The partial derivative of $N_2(a, b, \rho)$ with respect to the upper bound b can be obtained using the two alternative forms of the joint density function of a standard bivariate normal distribution given in (IV4) and (IV5) at the beginning of Part IV:

$$\frac{\partial}{\partial b}N_2(a,b,\rho) = \int_{-\infty}^{a} f(u)f\left(\frac{b-\rho u}{\sqrt{1-\rho^2}}\right)\frac{-\rho du}{\sqrt{1-\rho^2}}$$

$$= \frac{-\rho}{\sqrt{1-\rho^2}}\int_{-\infty}^{a} f(u)f\left(\frac{b-\rho u}{\sqrt{1-\rho^2}}\right)du. \qquad \text{(A15.1)}$$

We need to find an explicit expression for the univariate integration in (A15.1). The univariate integration of the product of two standard density functions $f(z)$ and $f\left[(A + Bz)/\sqrt{1-\rho^2}\right]$ can be generally calculated using the following method to find three unknown variables Σ, M, and, H such that

$$f(u)f\left(\frac{A+Bu}{\sqrt{1-\rho^2}}\right) = \frac{1}{2\pi}\exp\left[-\frac{(1+B^2-\rho^2)u^2 + ABu + A^2}{2(1-\rho^2)}\right]$$

$$= \frac{1}{2\pi}\exp\left[\frac{u^2 - 2Mu + H}{2\sum^2}\right], \qquad \text{(A15.2a)}$$

$$-\frac{1}{2\sum^2} = -\frac{1+B^2-\rho^2}{2(1-\rho^2)}, -\frac{-2M}{2\sum^2} = -\frac{2AB}{2(1-\rho^2)},$$

and

$$-\frac{H}{2\sum^2} = -\frac{A^2}{2(1-\rho^2)}. \qquad \text{(A15.2b)}$$

Solving the three unknown variables Σ, M, and H from the three equations in (A15.2b) yields

$$\sum = \sqrt{(1-\rho^2)/(1+B^2-\rho^2)},$$

$$M = -AB/(1+B^2-\rho^2),$$

and

$$H = A^2/(1 + B^2 - \rho^2). \tag{A15.3}$$

Substituting the solutions of the three variables σ, M, and H into (A15.1), we can readily find

$$\begin{aligned} I &= \int_{-\infty}^{a} f\left(\frac{A + Bu}{\sqrt{1-\rho^2}}\right) \exp\left[-\frac{(u-M)^2 + H - M^2}{2\Sigma^2}\right] du \\ &= \frac{\Sigma}{\sqrt{2\pi}} \exp\left(-\frac{H - M^2}{2\Sigma^2}\right) N\left(\frac{a - M}{\Sigma}\right). \end{aligned} \tag{A15.4}$$

Substituting $A = b$ and $B = -\rho$ into (A15.3) and then substituting Σ, M, and H into (A15.2a) yields the integration of the product of two standard normal density functions:

$$\begin{aligned} I &= \int_{-\infty}^{a} f(u) f\left(\frac{A + Bu}{\sqrt{1-\rho^2}}\right) du \\ &= \sqrt{1-\rho^2} F(A; 0, 1 + B^2 - \rho^2) N\left(\frac{a\sqrt{1 + B^2 - \rho^2}}{\sqrt{1-\rho^2}}\right). \end{aligned} \tag{A15.5}$$

The partial derivative in (A15.1) is thus known. This method can be used to find closed-form expressions for first-order sensitivities such as deltas and vegas for most correlation options in a Black-Scholes environment.

A related univariate integration most often used to find closed-form expressions for second-order sensitivities such as gammas is of the following form:

$$II = \int_{-\infty}^{a} u f(u) f\left(\frac{A + Bu}{\sqrt{1-\rho^2}}\right) du. \tag{A15.6}$$

Using the method of integrating by parts, we can obtain the following

$$\begin{aligned} II &= -f(a) f\left(\frac{A + Ba}{\sqrt{1-\rho^2}}\right) - \int_{-\infty}^{a} f\left(\frac{A + Bu}{\sqrt{1-\rho^2}}\right) \frac{A + Bu}{\sqrt{1-\rho^2}} \frac{B}{\sqrt{1-\rho^2}} du \\ &= -f(a) f\left(\frac{A + Ba}{\sqrt{1-\rho^2}}\right) - \frac{AB}{1-\rho^2} I - \frac{B^2}{1-\rho^2} II, \end{aligned}$$

which is an equation with only one unknown II because I is given in (A15.4).

The solution of II is given as follows:

$$II = \int_{-\infty}^{a} uf(u)f\left(\frac{A+Bu}{\sqrt{1-\rho^2}}\right)du$$

$$= -\frac{1-\rho^2}{1+B^2-\rho^2}f(a)f\left(\frac{A+Ba}{\sqrt{1-\rho^2}}\right)$$

$$- AB\frac{\sqrt{1-\rho^2}}{1+B^2-\rho^2}F(A;0,1+B^2-\rho^2)$$

$$\times N\left(\frac{a\sqrt{1+B^2-\rho^2}+AB/\sqrt{1+B^2-\rho^2}}{\sqrt{1-\rho^2}}\right). \tag{A15.7}$$

Using the result of II in terms of I in (A15.4), we can express the gamma and chi in closed-form in terms of the univariate cumulative functions.

Chapter 16

QUOTIENT OPTIONS

16.1. INTRODUCTION

Quotient options are also called ratio options. As the name implies, a quotient option is an option written on the ratio of two underlying asset prices, indices, or other quantities. Due to their unique nature, quotient options can be used to take advantage of the relative performance of two assets, markets, or portfolios. Although there are other options such as spread options that can perform similar functions, quotient options possess some advantages over spread options because their prices can be expressed conveniently in closed-form in a Black-Scholes environment.

Quotient options have a characteristic which all other options so far covered in this book do not have — notional value or face value. As quotient options are normally written on the ratios of asset prices or market indices, and these ratios, and hence the prices of the options written on them, may be rather small, there may exist some prespecified notional amount or face amount for some quotient options. The total value of such an option is simply the product of the option price and the prespecified notional amount. Since the notional amount is always prespecified, we can simply concentrate on the option price. The purpose of this chapter is to price quotient options in a Black-Scholes environment and to find applications for these options.

16.2. QUOTIENT OPTIONS

The payoff of a European-style option written on the ratio of two instruments can be expressed as follows:

$$PPQT_{1/2} = \max\left[\omega\frac{I_1(\tau)}{I_2(\tau)} - \omega K,\ 0\right], \tag{16.1}$$

or

$$PPQT_{2/1} = \max\left[\omega\frac{I_2(\tau)}{I_1(\tau)} - \omega K,\ 0\right], \tag{16.2}$$

where K is the strike price of the option and ω is a binary operator (1 for a call option and -1 for a put option.)

Assume that the two underlying asset prices follow the stochastic process given in (IV1) as in previous chapters. As the individual asset price is lognormally distributed in a Black-Scholes environment, the ratio of the price of the first asset over that of the second asset, or vice versa, is also lognormally distributed. Thus, the quotient option price can be expressed in closed-form. Since there are two ratios for every two underlying instruments and two types of options for each ratio, there are four kinds of quotient options for every two underlying instruments.

16.3. PRICING QUOTIENT OPTIONS

Figure 16.1 depicts the integration domain or the area in which a quotient call option written on the ratio of two asset prices will be in-the-money. It is obvious that the integration domain is a little more complicated than that of an exchange option because the strike price K is in general not equal to 1. The integration domain is always the area below the straight line starting from the origin with the slope K. It becomes the same as that of an exchange option only when the strike price $K = 1$.

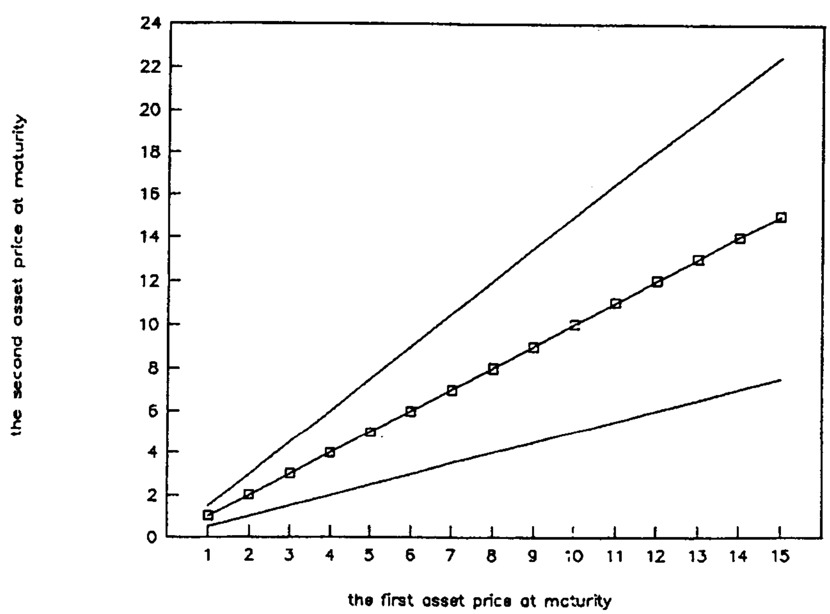

Fig. 16.1. Payoffs of quotient options with strike prices $K = 0.50$, 1.00, and 1.50.

Using the bivariate normal density function given in (IV4) and (IV5), we can obtain the expected payoff of a quotient option given in (16.1):

$$PPQTT_{1/2} = \omega \left[\frac{I_1}{I_2} e^{(\mu_1 - \mu_2)\tau + \sigma_y(\sigma_y - \rho\sigma_x)} N(\omega d_{1ra12}) - KN(\omega d_{ra12}) \right], \tag{16.3}$$

where

$$d_{ra12} = \frac{1}{\sigma_a\sqrt{\tau}} \left[\ln\left(\frac{I_1}{KI_2}\right) + \left(\mu_1 - \mu_2 - \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right)\tau \right],$$

and

$$d_{1ra12} = d_{ra12} + \sigma_a\sqrt{\tau}, \quad \sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$

Arbitrage-free arguments permit us to use the risk-neutral valuation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate. We can obtain the price of quotient option (PQT) by substituting $\mu_i = r - g_i$ (g_1 and g_2 are the pay out rates of the two underlying assets, respectively) and discounting the expected payoff in (16.3) by the risk-free rate:

$$PQT_{1/2} = \omega \left[\frac{I_1}{I_2} e^{(g_2 - g_1 - r)\tau + \sigma_2(\sigma_2 - \rho\sigma_1)\tau} N(\omega d_{1ra12}) - Ke^{-r\tau} N(\omega d_{ra12}) \right], \tag{16.4}$$

where

$$d_{ra12} = \frac{1}{\sigma_a\sqrt{\tau}} \left[\ln\left(\frac{I_1}{KI_2}\right) + \left(g_2 - g_1 - \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right)\tau \right],$$

and

$$d_{1ra12} = d_{ra12} + \sigma_a\sqrt{\tau},$$

σ_a is the same as in (16.3).

Formula (16.4) is obviously of the Black-Scholes type as it is in terms of the univariate cumulative function of the standard normal distribution and the first argument d_{1ra12} is always $\sigma_a\sqrt{\tau}$ greater than the second argument d_{ra12} in the cumulative function as in the Black-Scholes formula. It can be verified that the formula of the expected payoff in (16.3) degenerates into that for a vanilla option when $I_2 = 1$, $\mu_2 = \sigma_2 = g_2 = 0$ because the aggregate volatility σ_a becomes the same as the volatility of the first asset. One obvious difference between the quotient option pricing formula in (16.4) and the Black-Scholes formula is that the volatilities and the correlation coefficient parameters appear in the coefficient of the first term. We will explain this term in the following section.

Example 16.1. Find the prices of the quotient call and put options to expire in three months, given the two spot prices \$100, the strike price \$0.98, the volatilities of the two assets 15%, the payout rates for the two assets 3% and 6%, the interest rate 10%, and the correlation coefficient between the returns of the two assets 90%.

Substituting $I_1 = I_2 = \$100$, $K = \$0.98$, $\sigma_1 = \sigma_2 = 0.15$, $\tau = 3/12 = 0.25$, $g_1 = 0.03$, $g_2 = 0.06$, $r = 0.10$, and $\rho = 0.90$ into (16.4) yields

$$\sigma_a = \sqrt{0.15^2 - 2 \times 0.90 \times 0.15 \times 0.15 + 0.15^2} = 0.0671\,,$$

$$d_{ra12} = \frac{1}{\sigma_a\sqrt{0.25}}\left[\ln\left(\frac{100}{0.98 \times 100}\right) + \left(0.06 - 0.03 - \frac{1}{2} \times 0.15^2 + \frac{1}{2} \times 0.15^2\right) \times 0.15\right] = 0.8089\,,$$

$$d_{1ra12} = 0.8089 + 0.067\sqrt{0.25} = 0.8425\,,$$

and the call option price

$$\begin{aligned} &= e^{(0.06-0.03-0.10)\times 0.25+0.15(0.15-0.9\times 0.15)\times 0.25} N(0.8425) \\ &\quad - 0.98e^{-0.10\times 0.25} N(0.8089) \\ &= \$0.0311\,, \end{aligned}$$

and the put option price

$$\begin{aligned} &= -e^{(0.06-0.03-0.10)\times 0.25+0.15(0.15-0.9\times 0.15)\times 0.25} N(-0.8425) \\ &\quad + 0.98e^{-0.10\times 0.25} N(-0.8089) \\ &= \$0.0036\,. \end{aligned}$$

Options can also be written on the ratio of the second instrument over the first. The payouts of such options are given in (16.2). The price of a European option on the ratio of the second instrument over the first can be obtained following a similar procedure as above or simply from (16.4) using the symmetric property between I_1 and I_2,

$$PQT_{2/1} = \omega\left[\frac{I_2}{I_1} e^{(g_1-g_2-r)\tau+\sigma_1(\sigma_1-\rho\sigma_2)\tau} N(\omega d_{1ra21}) - Ke^{-r\tau} N(\omega d_{ra21})\right]\,, \tag{16.5}$$

where

$$d_{ra21} = \frac{1}{\sigma_a\sqrt{\tau}}\left[\ln\left(\frac{I_2}{KI_1}\right) + \left(g_1 - g_2 + \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right)\tau\right] = -d_{ra12} - \frac{2\ln K}{\sigma_a\sqrt{\tau}},$$

$$d_{1ra21} = d_{ra21} + \sigma_a\sqrt{\tau},$$

and

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

where d_{ra12} is the same as in (16.4).

Formula (16.5) is also of the Black-Scholes type as it is in terms of the univariate cumulative function of the standard normal distribution and the first argument d_{1ra21} is always $\sigma_a\sqrt{\tau}$ greater than the second argument d_{ra21} in the cumulative function.

Example 16.2. Find the prices of the call and put options written on the ratio of the second asset price over the first in Example 16.1.

Substituting $I_1 = I_2 = \$100$, $K = \$0.98$, $\sigma_1 = \sigma_2 = 0.15$, $\tau = 3/12 = 0.25$, $g_1 = 0.30$, $g_2 = 0.06$, $r = 0.10$, and $\rho = 0.90$ into (16.4) and using the results in Example 16.1 yields

$$\sigma_a = \sqrt{0.15^2 - 2\times 0.90\times 0.15\times 0.15\times + 0.15^2} = 0.0671,$$

$$d_{ra21} = -d_{ra12} - \frac{2\ln K}{\sigma_a\sqrt{\tau}} = 0.3954,$$

$$d_{1ra21} = 0.4290,$$

and the call option price

$$= e^{(0.30-0.06-0.10)\times 0.25+0.15(0.15-0.9\times 0.15)\times 0.25}N(0.4290)$$
$$- 0.98e^{-0.10\times 0.25}N(0.3954)$$
$$= \$0.0203,$$

and the put option price

$$= -e^{0.03-0.06-0.10)\times 0.25+0.15(0.15-0.9\times 0.15)\times 0.25}N(-0.4290)$$
$$+ 0.98e^{-0.10\times 0.25}N(-0.3954)$$
$$= \$0.0075.$$

16.4. SENSITIVITIES

Using the two arguments d_{1ra12} and d_{ra12} given in (16.4), we can obtain the following identity:

$$I_1 e^{[g_2 - g_1 + \sigma_2(\sigma_2 - \rho\sigma_1]\tau} f(d_{1ra12}) = KI_2 f(d_{ra12}), \qquad (16.6)$$

which can be used to simplify most sensitivity expressions of quotient options.

Like exchange options and options paying the maximum or the minimum of two underlying assets, there are two deltas for a quotient option because there are two underlying assets. Taking partial derivative of the pricing formula in (16.4) with respect to the two spot prices and simplifying the results using (16.6) yields the following two deltas:

$$\begin{aligned}\text{Delta1} &= \frac{\partial}{\partial I_1} PQT_{1/2} \\ &= \frac{\omega}{I_2} e^{(g_2-g_1-r)\tau+\sigma_2(\sigma_2-\rho\sigma_1)\tau} N(\omega d_{1ra12}) > 0\,,\end{aligned} \tag{16.7a}$$

$$\begin{aligned}\text{Delta2} &= \frac{\partial}{\partial I_2} PQT_{1/2} \\ &= -\frac{\omega I_1}{I_2^2} e^{(g_2-g_1-r)\tau+\sigma_2(\sigma_2-\rho\sigma_1)\tau} N(\omega d_{1ra12}) \\ &= -\frac{I_1}{I_2}\text{Delta1} < 0\,.\end{aligned} \tag{16.7b}$$

Formula (16.7b) indicates that the delta of the quotient option price with respect to the second asset price is always of the opposite sign to that with respect to the first asset price. The opposite sign is consistent with the intuition that the two spot prices have opposite effects on the ratio which affects the option price.

Example 16.3. Find the deltas of the quotient call option written on the ratio of the first asset price over that of the second asset in Example 16.1.

Substituting $I_1 = I_2 = \$100$, $K = \$0.98$, $\sigma_1 = \sigma_2 = 0.15$, $\tau\ 3/12 = 0.25$, $g_1 = 0.30$, $g_2 = 0.06$, and $r = 0.10$ into (16.7a) yields

$$\begin{aligned}\text{Delta1}(\omega{=}1) &= \frac{1}{100} e^{(0.06-0.03-0.10)\times0.25+0.15(0.15-0.9\times0.15)\times0.25} N(0.8425) \\ &= 0.0079 = 0.79\%\,, \\ \text{Delta1}(\omega{=}-1) &= \frac{-1}{100} e^{(0.06-0.03-0.10)\times0.25+0.15(0.15-0.9\times0.15)\times0.25} N(-0.8425) \\ &= -0.002 = -0.20\%\,, \\ \text{Delta2}(\omega{=}1) &= -\text{Delta1}(\omega = 1) = -0.79\%\,, \\ \text{Delta2}(\omega{=}-1) &= -\text{Delta2}(\omega = -1) = 0.20\%\,.\end{aligned}$$

Next, we will find the chi, or the sensitivity of the quotient option price with respect to the correlation coefficient. Taking partial derivative of (16.4) with respect to the correlation coefficient ρ and simplifying the result yields the chi of the quotient option using (16.6):

$$\begin{aligned}\frac{\partial}{\partial\rho}PQT_{1/2} &= -\frac{\omega\sigma_1\sigma_2}{I_2}e^{(g_2-g_1-r)\tau+\sigma_2(\sigma_2-\rho\sigma_1)\tau}N(\omega d_{1ra12}) \\ &= -(\sigma_1\sigma_2)\text{Delta1} < 0\,, \qquad (16.8)\end{aligned}$$

which indicates that the quotient option price decreases monotonically with the correlation coefficient. The negative sign of the chi of a quotient option is very intuitive because the more positively (resp. negatively) the two assets are correlated, the more likely the two asset prices move together (resp. in opposite directions) and thus the more likely the ratio is to be smaller (resp. greater), and therefore, the less (resp. more) valuable the call option written on the ratio will be.

Example 16.4. Find the chi of the quotient call and put options written on the ratio of the first asset price over that of the second asset in Example 16.1.

Using the results in Example 16.3 and (16.8), we can find the chi of the quotient options:

$$-(\sigma_1\sigma_2)\text{Delta1}(\omega = 1) = -0.15 \times 0.15 \times 0.0079 = -0.0002 = -0.02\%\,,$$

and

$$-(\sigma_1\sigma_2)\text{Delta1}(\omega = -1) = -0.15 \times 0.15 \times (-0.0020) = 0.00005 = 0.005\%\,.$$

16.5. APPLICATIONS

Quotient options can be used to take advantage of the relative performance of two underlying instruments or markets. We will take an example to see how they can be used.

Example 16.5. Suppose that there are two stocks with the spot prices $I_1 =$ \$100, $I_2 =$ \$100, the volatilities $\sigma_1 = 18\%$ and $\sigma_2 = 15\%$, their dividend rates $g_1 = 4\%$, $g_2 = 3\%$, the two stock returns are correlated with the correlation coefficient $\rho = 75\%$, the interest rate $r = 5\%$, and the strike price of the option $K = 1$, then what are the prices of the quotient call and put options on the ratio of the first asset price over that of the second asset to expire in one year?

Substituting the given parameters into (16.4) yields:

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.12\,,$$

$$d_{ra12} = \left[\ln\left(\frac{100}{1 \times 100}\right) + \left(0.03 - 0.04 - \frac{1}{2} \times 0.12^2\right) \times 1\right] \Big/ (0.12\sqrt{1})$$

$$= -0.2033\,,$$

$$d_{1ra12} = -0.2033 + 0.12\sqrt{1} = -0.0833\,.$$

Therefore the call option price

$$= 1e^{(0.03-0.04-0.05)\times 1 + 0.15\times(0.15-0.75\times 0.2)\times 1} N(-0.0833)$$

$$- e^{-0.05\times 1} N(-0.2033)$$

$$= \$0.0453\,,$$

and the price of the put option

$$= -1e^{(0.03-0.04-0.05)\times 1 + 0.15\times(0.15-0.75\times 0.2)\times 1} N(0.0833) + e^{-0.05\times 1} N(0.2033)$$

$$= \$0.0557\,.$$

16.6. SUMMARY AND CONCLUSIONS

To some degree, quotient options can be used to achieve similar results as spread options because they are both written on the relative price changes of two underlying assets. As it is to be shown in Chapter 21, although there exist closed-form solutions for spread options in the Black-Scholes environment, the parameters are more complicated than those in the formulas given in (16.4) and (16.5). Due to the simplicity of their pricing formulas, it is more convenient to use quotient options.

QUESTIONS AND EXERCISES

16.1. What are quotient options?

16.2. How many kinds of quotient options are there for each pair of underlying instruments?

16.3. Why are ratio options similar to spread options?

16.4. Find the quotient option price in Example 16.1 if $K = \$1.1$, $\rho = 0.95$, and other parameters remain unchanged.

16.5. Find the deltas of the two quotient options in Exercise 16.4.

16.6. Find the chi of the two quotient options in Exercise 16.4.

16.7. Find the prices of quotient options written on the ratio of the second asset price over the first asset price in Exercise 16.4.

16.8. Find the deltas of the two quotient options in Exercise 16.7.

16.9. Find the chi of the two quotient options in Exercise 16.7.

16.10. Find the prices of the quotient options in Exercise 16.4 if $K = \$0.95$ and other parameters remain unchanged.

Chapter 17

PRODUCT OPTIONS AND FOREIGN DOMESTIC OPTIONS

17.1. INTRODUCTION

A product option is an option written on the product of two underlying asset prices or indices. A direct application of product options is foreign domestic options or foreign equity options in domestic currency. A foreign domestic option can be either a foreign equity or commodity option with the strike price in domestic currency. Product options could also be used to hedge the revenue of one company because the revenue is the product of the commodity sales and the product price. The purpose of this chapter is to price product options in a Black-Scholes environment and to apply the theory to foreign domestic options and other situations.

17.2. PRODUCT OPTIONS

The payoff of a European option on the product of two underlying instruments can be expressed as follows:

$$PDUCT = \max\left[\omega I_1(\tau)I_2(\tau) - \omega K,\ 0\right], \qquad (17.1)$$

where K is the strike price of the option and ω is a binary operator (1 for a call option and -1 for a put option).

In a Black-Scholes environment, the two underlying instruments are assumed to follow the stochastic process given in (IV1). The two instruments do not both have to be asset prices. If one underlying instrument is a foreign asset price and the other is the foreign exchange rate measured in domestic currency per unit of foreign currency, the payoff in (17.1) becomes the same as that of a domestic option with the strike price K in domestic currency. If one underlying instrument is a domestic commodity asset price and the other is the commodity sales of a company, the payoff in (17.1) becomes that of an option written on the revenue of that company. Since the product of two lognormally distributed variables are also lognormally distributed, closed-form pricing formulas of product options can be readily obtained.

17.3. PRICING PRODUCT OPTIONS

Figure 17.1 depicts the integration domain or the area in which a product call option can be in-the-money. The curve is obviously a hyperbolic curve rather than a straight line, which is the case all the option payoff patterns so far covered in this book. For any point above (resp. below) the curve in Figure 17.1, the product of the two prices is above (resp. below) the strike price. Thus, the area above (resp. below) the curve is the integration domain for a product call (resp. put) option.

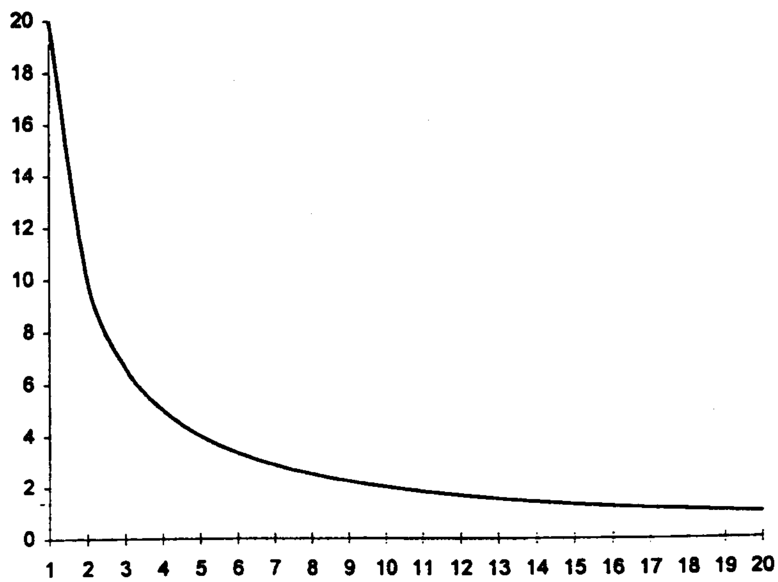

Fig. 17.1. Integration domain for a product option with $K = 20$.

Using the bivariate normal density function given in (IV3) and (IV4) and the integration domain shown in Figure 17.1, we can obtain the expected payoff of a product option in (17.1) as follows:

$$E(PDUCT) = \omega I_1 I_2 e^{(\mu_1+\mu_2+\rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}) - \omega K N(\omega d_{pu}), \quad (17.2)$$

where

$$d_{pu} = \frac{1}{\sigma pu\sqrt{\tau}}\left\{\ln\left(\frac{I_1 I_2}{K}\right) + \left[\mu_1 + \mu_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\tau\right\},$$

$$d_{1pu} = d_{pu} + \sigma_{pu}\sqrt{\tau},$$

and

$$\sigma_{pu} = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,.$$

We can obtain the price of a product option (PPDC) by substituting $\mu_i = r - g_i$ (g_1 and g_2 are the payout rates of the two underlying assets) into (17.2) and discounting the expected payoff in (17.2) by the risk-free rate r:

$$PPDC = \omega I_1 I_2 e^{(r-g_1-g_2+\rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}) - \omega K e^{-r\tau} N(\omega d_{pu}), \tag{17.3}$$

where

$$d_{pu} = \frac{1}{\sigma_{pu}\sqrt{\tau}}\left\{\ln\left(\frac{I_1 I_2}{K}\right) + \left[2r - g_1 - g_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\tau\right]\right\},$$

$$d_{1pu} = d_{pu} + \sigma_{pu}\sqrt{\tau},$$

and

$$\sigma_{pu} = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,.$$

We can observe that (17.3) is clearly of the Black-Scholes type as it is in terms of the univarite cumulative function of the standard normal distribution and the first argument d_{1pu} is always d_{pu} greater than the second argument in the cumulative functions as in the Black-Scholes formula. The volatility function σ_{pu} can be understood as an aggregate volatility of the two assets as it is the effective volatility used in the pricing formula. Obviously it is similar to the aggregate volatility σ_a in the pricing formulas of exchange options, options paying the maximum or minimum of two assets, and ratio options in previous chapters. However, the aggregate volatility σ_{pu} is different from the aggregate volatility σ_a because it is always greater than the latter for positive correlation coefficients resulting from the positive sign in the product term of the expression. It can be verified that the pricing formula in (17.3) degenerates into the Black-Scholes formula when $I_2 = 1$, $\mu_2 = \sigma_2 = g_2 = 0$ because the aggregate volatility σ_{pu} becomes the same as the volatility of the first asset.

Example 17.1. Find the prices of product options to expire in one year if the spot prices of the two stocks are $I_1 = \$4$, $I_2 = \$15$, the volatilities of the two stock returns $\sigma_1 = 10\%$ and $\sigma_2 = 15\%$, the dividend rates of the two stocks $g_1 = g_2 = 0$, the two stock returns are correlated with the correlation coefficient $\rho = 50\%$, the interest rate $r = 5\%$, and the strike price of the product option $K = \$60$.

Substituting the given parameters into (17.3), we obtain:

$$\sigma_{pu} = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.2179\,,$$

$$d_{pu} = \left[\ln\left(\frac{I_1}{I_2}\right) + \left(2r - g_1 - g_2 - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right)\tau\right] \Big/ (\sigma_{pu}\sqrt{\tau})$$
$$= 3.8435\,,$$

$$d_{1pu} = d_{pu} + \sigma_{pu}\sqrt{\tau} = 4.0614\,.$$

Therefore, using (17.3), the price of the product call option becomes

$$\begin{aligned} PPDC(\omega = 1) &= I_1 I_2 e^{(r-g_1-g_2+\rho\sigma_1\sigma_2)\tau} N(d_{1pu}) - K e^{-r\tau} N(d_{pu}) \\ &= 4 \times 15 e^{(0.05-0-0+0.50\times0.1\times0.15)\times1} N(4.0614) \\ &\quad - 60 e^{-0.05\times1} N(3.8435) \\ &= \$6.4804\,; \end{aligned}$$

and the price of the corresponding put option becomes

$$\begin{aligned} PPDC(\omega = -1) &= -I_1 I_2 e^{(r-g_1-g_2+\rho\sigma_1\sigma_2)\tau} N(-d_{1pu}) + K e^{-r\tau} N(-d_{pu}) \\ &= -4 \times 15 e^{(0.05-0-0+0.50\times0.1\times0.15)\times1} N(-4.0614) \\ &\quad + 60 e^{-0.05\times1} N(-3.8435) \\ &= \$0.0019\,. \end{aligned}$$

17.4. FOREIGN DOMESTIC OPTIONS

An immediate application of the product option pricing formula is for foreign domestic options or foreign equity options with domestic strike prices because the product of a foreign equity price and the exchange rate per unit of foreign currency at the option maturity is the foreign equity price in terms of domestic currency. The foreign exchange rate per unit of foreign currency can be modeled as a stochastic process given in (IV1) with the underlying payout rate the same as the foreign interest rate. Assume that the first asset is the foreign equity and the second asset the foreign exchange rate, the payout rate of the second asset g_2 is simply the foreign interest rate r_f. Substituting $g_2 = r_f$ into (17.3) yields the price of a foreign domestic option (PFD):

$$PFD = \omega I_1 I_2 e^{(r-r_f-g_1+\rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}) - \omega K e^{-r\tau} N(\omega d_{pu})\,, \quad (17.4)$$

where

$$d_{pu} = \frac{1}{\sigma_{pu}\sqrt{\tau}} \left\{ \ln\left(\frac{I_1 I_2}{K}\right) + \left[2r - r_f - g_1 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\tau \right\},$$

$$d_{1pu} = d_{pu} + \sigma_{pu}\sqrt{\tau},$$

and σ_{pu} is the same as in (17.3).

Example 17.2. Find the prices of the call and put options written on a Sony stock with the strike price $K = \$60$ to expire in one year, given that the spot price of the Sony stock $I_1 = ¥5350$, the volatility of this stock $\sigma_1 = 10\%$, the Japanese interest rate $r_f = 4\%$, the US interest rate $r = 5\%$, the current US dollar/Japanese yen exchange rate $I_2 = \$0.0111$/yen, the volatility of the exchange rate $\sigma_2 = 15\%$, and the stock return and the US dollar/yen exchange rate are correlated with the correlation coefficient $\rho = -25\%$.

Since the options in this example are foreign domestic options, we can use (17.4) directly. Substituting the given parameters into (17.4), we get:

$$\sigma_{pu} = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.1581,$$

$$d_{pu} = \frac{1}{\sigma_{pu}\sqrt{\tau}} \left\{ \ln\left(\frac{I_1 I_2}{K}\right) + \left[2r - g_1 - r_f - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\tau \right\}$$
$$= 0.2178,$$

$$d_{1pu} = 0.3759.$$

Therefore, using (17.4), the price of the Japanese stock call option becomes

$$I_1 I_2 e^{(r-g_1-r_f+\rho\sigma_1\sigma_2)\tau} N(d_{1pu}) - K e^{-r\tau} N(d_{pu})$$
$$= 5350 \times 0.0111 e^{(0.05-0-0.04-0.25\times0.1\times0.15)\times1} N(0.3759)$$
$$- 60e^{-0.05\times1} N(0.2178)$$
$$= \$5.212;$$

and the Japanese stock put option becomes

$$- I_1 I_2 e^{(r-g_1-r_f+\rho\sigma_1\sigma_2)\tau} N(-d_{1pu}) + K e^{-r\tau} N(-d_{pu})$$
$$= -5350 \times 0.111 e^{(0.05-0-0.04-0.25\times0.1\times0.15)\times1} N(-0.3759)$$
$$+ 60e^{-0.05\times1} N(-0.2178)$$
$$= \$2.472.$$

17.5. REVENUE OPTIONS

As the revenue of any company is the product of their production sold and the selling price of their product, options can be written on the product to hedge the risks on the revenue. Assume the price of a company's product to be the first asset price I_1 which follows the stochastic process in (IV1) and the company production sold I_2 also follows the same stochastic process. The only difference between the stochastic process of the asset price and that of the production sold is that the drift of the asset price process has to be $r - g_1$ in the risk-neutral world and the drift of the quantity process does not need to be constrained by this condition.

We can obtain the price of a revenue option (PREVU) by substituting $\mu_1 = r - g_1$ into (17.2) and discounting the expected payoff given in (17.2) by the risk-free rate r:

$$PREVU = \omega I_1 I_2 e^{(\mu_2 - g_1 + \rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}) - \omega K e^{-r\tau} N(\omega d_{pu}), \quad (17.5)$$

where

$$d_{pu} = \frac{1}{\sigma_{pu}\sqrt{\tau}}\left[\ln\left(\frac{I_1 I_2}{K}\right) + \left(r - g_1 + \mu_2 - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right)\tau\right],$$

$$d_{1pu} = d_{pu} + \sigma_{pu}\sqrt{\tau},$$

$$\sigma_{pu} = \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

and μ_2 is the instantaneous mean of the quantity of product is sold.

Example 17.3. Suppose the current price of the product of XYZ company I_1 = \$20, the volatility of the product price σ_1 = 12%, the current sales I_2 = 1 million, the volatility of sales is 20%, the product price and the sales are correlated with the negative correlation coefficient 20%, the instantaneous drift of the sales is 15%, the payout rate of the first asset $g_1 = 0$, the interest rate r = 6%. Then what are the revenue option prices with the strike price K = \$20 million?

Substituting the given parameters into (17.5) yields

$$\begin{aligned}
\sigma_{pu} &= \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2} = \sqrt{0.10^2 + 2 \times (-0.20) \times 0.10 \times 0.15 + 0.15^2} \\
&= 0.1628, \\
d_{pu} &= \frac{1}{\sigma_{pu}\sqrt{\tau}}\left\{\ln\left(\frac{I_1 I_2}{K}\right) + \left[r - g_1 + \mu_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\tau\right\} \\
&= \frac{1}{0.1628\sqrt{\tau}}\left\{\ln\left(\frac{20 \times 1}{20}\right) + \left[0.06 - 0 + 0.15 - \frac{1}{2}(0.10^2 + 0.15^2)\right] \times 1\right\} \\
&= 1.1901, \\
d_{pu} &= d_{pu} + \sigma_{pu}\sqrt{\tau} = 1.1901 + 0.1628 = 1.3529,
\end{aligned}$$

and the revenue call option price

$$\begin{aligned}
&I_1 I_2 e^{(\mu_2 - g_1 + \rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}) - \omega K e^{-r\tau} N(\omega d_{pu}) \\
&= 20 \times 1e^{(0.15-0-0.2\times0.1\times0.15)\times1} N(1.3529) - 20e^{-0.06\times1} N(1.1901) \\
&= \$4.4959 \text{ million};
\end{aligned}$$

and the revenue put option price

$$\begin{aligned}
&- I_1 I_2 e^{(\mu_2 - g_1 + \rho\sigma_1\sigma_2)\tau} N(-d_{1pu}) + K e^{-r\tau} N(-d_{pu}) \\
&= -20 \times 1e^{(0.15-0-0.2\times0.1\times0.15)\times1} N(-1.3529) + 20e^{-0.06\times1} N(-1.1901) \\
&= \$0.1641 \text{ million}.
\end{aligned}$$

17.6. SENSITIVITIES

Using the two arguments d_{1pu} and d_{pu} in (17.3), we can obtain the following identity:

$$I_1 I_2 e^{(r - g_2 - g_2 + \rho\sigma_1\sigma_2)\tau} f(d_{1pu}) = K f(d_{pu}), \tag{17.6}$$

which can be used to simplify most sensitivity expressions for product options.

Product options have two deltas as ratio options because there are two underlying assets. Taking partial derivative of the pricing formula in (17.3) with respect to the two spot prices and simplifying the results using (17.6) yields the following two deltas:

$$\begin{aligned}
\text{Delta}_i &= \frac{\partial}{\partial I_i} PPDC \\
&= \omega I_j e^{(r - g_2 - g_1 + \rho\sigma_1\sigma_2)\tau} N(\omega d_{1pu}), \; i, \; j = 1 \text{ or } 2, \text{ and } i \neq j,
\end{aligned} \tag{17.7}$$

which indicates that the delta of a product option with respect to any one of the two asset prices equals the other asset spot price multiplied by the product of a normal cumulative function value and a positive coefficient. This symmetric results from the symmetric property of the product because the order of the two asset prices does not affect the product value.

The sensitivity of the product option price with respect to the correlation coefficient is obtained by taking partial derivative of (17.3) with respect to the correlation coefficient ρ and simplifying the result using (17.6):

$$\frac{\partial}{\partial \rho} PPDC = \frac{\omega\sqrt{\tau}\sigma_1\sigma_2}{\sigma_{pu}} e^{(r - g_2 - g_1 + \rho\sigma_1\sigma_2)\tau} f(d_{1pu}), \tag{17.8}$$

which indicates that the product call (resp. put) option price increases (resp. decreases) monotonically with the correlation coefficient. The positive sign of the chi of a product option is intuitive because the more closely the two assets are correlated, the larger the product, and the greater the call option price will be.

Example 17.4. Find the deltas of the product call option in Example 17.1.

Substituting the given parameters into (17.7) using the results in Example 17.1 yields:

$$\text{Delta}_1 = I_2 e^{(0.05-0-0+0.50\times 0.1\times 0.15)\times 1} N(4.0614) = 15.888\,,$$

and

$$\text{Delta}_2 = I_1 e^{(0.05-0-0+0.50\times 0.1\times 0.15)\times 1} N(4.0614) = 4.237\,.$$

17.7. SUMMARY AND CONCLUSIONS

We have found closed-form solutions for product options or options written on the product of two underlying instruments. An immediate application of the product option pricing formula is to price foreign domestic options — foreign equity or commodity options with strike prices in domestic currency. The pricing formula of product options can also be used to price options written on the revenue of a company when one underlying asset is the selling price of a product and the other is the quantity of that product sold. The compact closed-form solution for product option prices in a Black-Scholes environment is very convenient to use and can be applied to many other problems.

QUESTIONS AND EXERCISES

17.1. What are product options?

17.2. What are foreign domestic options?

17.3. What is the connection between product options and foreign domestic options?

17.4. Give two examples to show how product options could be used.

17.5. How are the integration domains of product options different from those of other options?

17.6. Find the prices of product options to expire in half a year if the spot prices of the two stocks are $I_1 = \$5$, $I_2 = \$20$, the volatilities of these two stock returns $\sigma_1 = 20\%$ and $\sigma_2 = 25\%$, the dividend rates $g_1 = g_2 = 4\%$, the two stock returns are correlated with the correlation coefficient $\rho = 75\%$, the interest rate $r = 8\%$, and the strike price of the product option $K = \$100$.

17.7. Find the deltas of the product options in Exercise 17.6.

17.8. Find the chi of the product options in Exercise 17.6.

17.9. Find the prices of Sony domestic options in Example 17.2 if the correlation coefficient between the Sony stock and the dollar/yen exchange rate is -40% and other parameters remain unchanged as in Example 17.2.

17.10. Find the deltas of the two options in Exercise 17.9.

17.11. Find the prices of the call and put options written on a Toyota Motor stock with the strike price $K = \$20$ to expire in one year, given that the spot price of the Toyota Motor stock $I_1 = ¥183$, the volatility of this stock is 13%, the Japanese interest rate $r_f = 4\%$, the US interest rate $r = 6\%$, the current US dollar/Japanese yen exchange rate $I_2 = \$0.0111$/yen, the volatility of the exchange rate $\sigma_2 = 16\%$, and the stock return and the dollar/yen exchange rate are correlated with the correlation coefficient -35%.

17.12. Find the deltas of the two Toyota Motor stock options in Exercise 17.11.

17.13. Find the chi of the two Toyota Motor options in Exercise 17.11.

17.14. Find the prices of the two Toyota Motor options in Exercise 17.11 if the correlation coefficient between the Toyota Motor stock return and the dollar/yen exchange rate is -50% and other parameters are the same as in Exercise 17.11.

17.15. Find the prices of the call and put options written on a Volkswagen stock with the strike price $K = \$325$ to expire in half a year, given that the spot price of the Volkswagen stock $I_1 = 454$ German marks, the volatility of this stock is 10%, the German interest rate 6%, the US interest rate $r = 8\%$, the current US dollar/mark exchange rate is $I_2 = \$0.70$/mark, the volatility of the exchange rate $\sigma_2 = 15\%$, and the stock return and the US dollar/mark exchange rate are correlated with the correlation coefficient -15%.

17.16.* Show the identity given in (17.6).

Chapter 18

FOREIGN EQUITY OPTIONS

18.1. INTRODUCTION

Foreign equity options, as their name implies, are options written on foreign equity with strike prices in foreign currency. These options could be priced for foreign investors using the Black-Scholes formula directly in foreign currency. However, domestic investors may also be interested in foreign equity options. For instance, American investors may be interested in Japanese stock or German stock options to either speculate in the Japanese or German equity markets or hedge their exposure in these markets. It is obvious that exchange risks are involved for domestic investors because the payoffs of foreign equity options are in foreign currency and they have to be converted into domestic currency. In other words, foreign equity returns are correlated with exchange rates. With increasing development in financial market globalization, the demand for foreign equity options has grown significantly in the past decade and will grow at an even higher speed in the coming years.

This is the second chapter which involves foreign currency. We studied foreign domestic options as an application of product options in Chapter 17. Reiner (1992) analyzed four basic types of currency-related options and called them currency-translated options. The purpose of this chapter is to analyze foreign equity options in a Black-Scholes environment and introduce some of their applications.

18.2. FOREIGN EQUITY OPTIONS

The payoff of a foreign equity option in foreign currency can be given the same as in (2.1):

$$PFE = \max\left[\omega I_1(\tau) - \omega K_f,\ 0\right], \tag{18.1}$$

where $I_1(\tau)$ and K_f stand for the foreign equity price at the option maturity and the strike price in foreign currency, respectively; ω is the same binary operator (1 for a call option and -1 for a put option).

In a Black-Scholes environment the underlying instrument is assumed to follow a standard geometric Brownian process. We assume that the foreign equity price follows the stochastic process given in (IV1). Since the payoff in (18.1) is in foreign currency, it has to be converted into domestic currency for domestic investors. Assume that the exchange rate $I_2(\tau)$ is in domestic currency per unit of foreign currency and it follows the stochastic process given in (IV1) with the payout rate $g_2 = r_f$, the foreign interest rate as in Chapter 17. The payoff of a foreign equity option in domestic currency (PFEDC) is simply the product of the payoff given in (18.1) and the exchange rate $I_2(\tau)$:

$$PFEDC = I_2(\tau) \max\left[\omega I_1(\tau) - \omega K_f,\ 0\right], \tag{18.2}$$

which can be expressed alternatively as:

$$PFEDC = \max\left[\omega I_1(\tau) I_2(\tau) - \omega K_f I_2(\tau)\right], \tag{18.3}$$

resulting from multiplying the exchange rate $I_2(\tau)$ into both terms of the max (. , .) function in (18.2).

The payoff in (18.3) indicates that a foreign equity option can be understood as a product option with a floating strike price $K_f I_2(\tau)$. This is somewhat similar to the Asian options with floating strike prices in Chapters 5 to 7 and to the foreign domestic options in Chapter 16.

18.3. PRICING FOREIGN EQUITY OPTIONS

The price of a foreign equity option in foreign currency can be given directly using the extended Black-Scholes formula in (10.31) because the payoff in (18.1) is the same as that of a call option in (2.1) when $\omega = 1$ and as that of a put option in (2.2) when $\omega = -1$, with the exception that the payoff in (18.1) is in foreign currency and those in (2.1) and (2.2) are in domestic currency. In order to compare the difference between the price of a foreign equity option in foreign currency with that in domestic currency, we first express the foreign equity option price in foreign currency (FEOPFC) by using the extended Black-Scholes formula given in (10.31):

$$FEOPFC = \omega I_1 e^{-g_1\tau} N(\omega d_{1f}) - \omega K_f e^{-r_f\tau} N(\omega d_f), \tag{18.4}$$

where

$$d_f = \frac{\ln(I_1/K_f) + (r_f - g_1 - \sigma_1^2/2)\tau}{\sigma_1\sqrt{\tau}},$$
$$d_{1f} = d_f + \sigma_1\sqrt{\tau},$$

ω is the binary operator (1 for a call option and -1 for a put option), r_f is the foreign interest rate, and other parameters are the same as in (18.2).

The pricing formula of a foreign equity option in foreign currency is exactly the same as the Black-Scholes formula in (10.31) if we substitute the domestic interest rate and the domestic strike price with the foreign interest rate and the foreign strike price, respectively.

Since the exchange rate is normally correlated to the stock price, we cannot price foreign equity options in domestic currency directly using the extended Black-Scholes formula given in (18.4). However, the problem can be solved using the joint distribution function between the stock return and the exchange rate given in (IV4) and (IV5). As illustrated in (18.2), the exchange rate merely converts or modifies the payoff of a foreign equity option, and the integration domain for a foreign equity option in domestic currency is the same as that in foreign currency, or the payoff in (18.2) is nonzero for foreign equity prices above the strike price for a call option and below it for a put option. Using the joint density function given in (IV4) and (IV5) and the integration domain discussed above, we can obtain the expected payoff of a foreign equity option in domestic currency in (18.2):

$$\begin{aligned} E(PFEDC) = {} & \omega I_1 I_2 \mathrm{e}^{(\mu_1+\mu_2+\rho\sigma_1\sigma_2)\tau} N[\omega(d_{1f} + \rho\sigma_2\sqrt{\tau})] \\ & - \omega K_f I_2 \mathrm{e}^{u_2\tau} N[\omega(d_f + \rho\sigma_2\sqrt{\tau})], \end{aligned} \tag{18.5}$$

where d_f and d_{1f} are the same as in (18.4), σ_2 and ρ represent the volatility of the exchange rate and the correlation coefficient between the exchange rate and the foreign equity return, respectively, and μ_1 and μ_2 represent the drifts of the foreign equity return and the exchange rate, respectively.

The foreign equity option price in domestic currency (FEPDC) is obtained by substituting $\mu_1 = r_f - g_1$ and $\mu_2 = r - r_f$ into (18.5) and discounting (18.5) at the risk-free domestic interest rate r:

$$\begin{aligned} FEPDC = {} & I_2\{\omega I_1 \mathrm{e}^{(\rho\sigma_1\sigma_2 - g_1)\tau} N[\omega(d_{1f} + \rho\sigma_2\sqrt{\tau})] \\ & - \omega K_f \mathrm{e}^{-r_f\tau} N[\omega(d_f + \rho\sigma_2\sqrt{\tau})]\}, \end{aligned} \tag{18.6}$$

where all parameters are the same as in (18.5).

The pricing formula of a foreign equity option in (18.6) is obviously of the Black-Scholes type because it is expressed in terms of the cumulative functions of the standard normal distribution and the argument in the first cumulative normal function is always $\sigma_1\sqrt{\tau}$ greater than that in the second cumulative function. It is different from all other correlation option pricing formulas in the sense that the arguments in both the cumulative functions have a common term $\rho\sigma_2\sqrt{\tau}$ resulting from the exchange rate which is multiplied in front of the regular foreign equity option payoff given in (18.2).

We can check that the term in the brace of the pricing formula in (18.6) becomes precisely the same as the pricing formula of a foreign equity option in foreign currency given in (18.4) when the correlation coefficient is zero. This is intuitive because the foreign equity option is priced independently of the correlation coefficient when the correlation coefficient is zero and the value of the foreign equity option in domestic currency is simply obtained by multiplying the foreign equity option price in foreign currency by the exchange rate. When the correlation coefficient is not zero, we can obtain (18.6) by substituting the spot price I_1 with $I_1 e^{\rho\sigma_1\sigma_2}$ in (18.4) and convert it into domestic currency with the spot exchange rate I_2.

Example 18.1. Find the prices of the BMW stock options to expire in one year with the strike price 800 marks in both German marks and US dollars, given the spot BMW stock price 810 marks, the German interest rate 7%, the US interest rate 8%, the payout rate of the stock g_1= 4%, the volatility of the stock σ_1= 12%, the spot German mark/US dollar exchange rate 1.39 marks/dollar, the exchange rate volatility 15%, and the correlation coefficient between the BMW stock return and the exchange rate is 25%.

The prices of the BMW options in German marks can be obtained by substituting $I_1 = 810$ marks, $K_f = 800$ marks, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, and $g_1 = 0.04$ into (18.4):

$$\begin{aligned}
d_f &= [\ln(810/800) + (0.07 - 0.04 - 0.12^2/2)]/(0.12\sqrt{1}) \\
&= 0.2935\,, \\
d_{1f} &= d_f + \sigma_1\sqrt{\tau} \\
&= 0.2935 + 0.12 \times \sqrt{1} = 0.4135\,, \\
FEOPFC(\omega = 1) &= 810e^{-0.04\times 1}N(0.4135) - 800e^{-0.07\times 1}N(0.2935) \\
&= 54.875 \text{ marks}\,,
\end{aligned}$$

and

$$\begin{aligned}
FEOPFC(\omega = -1) &= -810e^{-0.04\times 1}N(-4135) + 800e^{-0.07\times 1}N(-0.2935) \\
&= 22.551 \text{ marks}\,.
\end{aligned}$$

The prices of the BMW options in US dollars can be obtained by substituting $I_1 = 810$ marks, $I_2 = 1/1.39$, $K_f = 800$ marks, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, $g_1 = 0.04$, $\rho = 0.25$, and $\sigma_2 = 0.15$ into (18.5):

$$d_f + \rho\sigma_2\sqrt{\tau} = 0.2935 + 0.25 \times \sqrt{1} = 0.5435\,,$$

$$d_{1f} + \rho\sigma_2\sqrt{\tau} = 0.4135 + 0.25 \times \sqrt{1} = 0.6635\,,$$

$$\begin{aligned} FEPDC(\omega = 1) &= \frac{1}{1.39}\Big[810\mathrm{e}^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(0.6635) \\ &\qquad - 800\mathrm{e}^{-0.07\times 1}N(0.5435)\Big] \\ &= \$40.644\,, \end{aligned}$$

and

$$\begin{aligned} FEPDC(\omega = -1) &= \frac{1}{1.39}\Big[-810\mathrm{e}^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(-0.6635) \\ &\qquad + 800\mathrm{e}^{-0.07\times 1}N(-0.5435)\Big] \\ &= \$14.881\,. \end{aligned}$$

Comparing the results in Example 18.1, we can find that the BMW stock option prices in US dollars cannot be obtained directly from converting the corresponding prices in German marks using the current exchange rate. If we convert the BMW stock call and put option prices 54.875 and 22.551 marks into US dollars at the current exchange rate 1.39 marks/dollar, we would obtain \$39.478 and \$16.224, respectively. Yet the prices of these options are \$40.644 and \$14.881 from Example 18.1. Thus, the direct conversion method used above undervalues the call option by \$1.166 and overvalues the put option by \$1.343, respectively.

Example 18.2. Find the prices of the Honda Motor stock options to expire in nine months with the strike price ¥1600 in both Japanese yen and US dollars, given the spot Honda Motor stock price ¥1540, the Japanese interest rate 3%, the US interest rate 8%, the payout rate of the stock $g_1 = 2\%$, the volatility of the Honda Motor stock $\sigma_1 = 14\%$, the spot Japanese yen/US dollar exchange rate ¥90/dollar, the exchange rate volatility 18%, and the correlation coefficient between the BMW stock return and the exchange rate is -30%.

The prices of the Honda Motor options in Japanese yen can be obtained by substituting $I_1 = ¥1540$, $K_f = ¥1600$, $r_f = 3\%$, $\tau = 9/12$, $\sigma_1 = 0.14$, and $g_1 = 0.02$ into (18.4):

$$\begin{aligned} d_f &= [\ln(1540/1600) + (0.03 - 0.02 - 0.14^2/2) \times 1]/(0.14\sqrt{1}) \\ &= -0.3140, \\ d_{1f} &= d_f + \sigma_1\sqrt{\tau} \\ &= -0.3140 + 0.14\sqrt{9/12} = -0.1923, \end{aligned}$$

$$\begin{aligned} FEOPFC(\omega = 1) &= 1540\mathrm{e}^{-0.02\times 1}N(-0.1923) - 1600\mathrm{e}^{-0.03\times 1}N(-0.3140) \\ &= ¥53189, \\ FEOPFC(\omega = -1) &= -1540\mathrm{e}^{-0.02\times 1}N(0.1923) + 1600\mathrm{e}^{-0.03\times 1}N(0.3140) \\ &= ¥100.518. \end{aligned}$$

The prices of the Honda Motor options in US dollars can be obtained by substituting $I_1 = ¥1540$, $I_2 = 1/90$, $K_f = ¥1600$, $r_f = 3\%$, $\tau = 9/12$, $\sigma_1 = 0.14$, $g_1 = 0.02$, $\rho = -0.30$, and $\sigma_2 = 0.18$ into (18.5):

$$\begin{aligned} d_f + \rho\sigma_2\sqrt{\tau} &= -0.3140 - 0.30 \times \sqrt{9/12} = -0.5738, \\ d_{1f} + \rho\sigma_2\sqrt{\tau} &= -0.1923 - 0.30 \times \sqrt{9/12} = -0.4521, \end{aligned}$$

$$\begin{aligned} FEPDC(\omega = 1) &= \frac{1}{90}\Big[1540\mathrm{e}^{(-0.30\times 0.14\times 0.18-0.02)\times 1}N(-0.4521) \\ &\quad - 800\mathrm{e}^{-0.03}N(-0.5738)\Big] = \$0.536, \\ FEPDC(\omega = -1) &= \frac{1}{90}\Big[- 1540\mathrm{e}^{(-0.30\times 0.14\times 0.18-0.02)\times 1}N(0.4521) \\ &\quad + 800\mathrm{e}^{-0.03}N(0.5738) = \$1.142. \end{aligned}$$

Comparing the results prices in Example 18.2, we can find that the Honda Motor stock option prices in US dollars cannot be obtained directly from converting the corresponding prices in Japanese yens using the current exchange rate. We can convert the Honda Motor stock call and put option prices ¥53.189 and ¥100.518 into US dollars at the current exchange rate 90 yen/dollar and obtain \$0.591 and \$1.117, respectively. Thus, the direct conversion method overvalues the call option by \$0.055 and undervalues the put option by \$0.025, respectively. The reason that the call option is undervalued in Example 18.1 and overvalued in Example 18.2 is that the correlation coefficient is negative in Example 18.2 while it is positive in Example 18.1.

18.4. SENSITIVITIES

Using the expressions of d_f and d_{1f} in (18.4) and (18.6), we can have the following identity:

$$I_1\mathrm{e}^{(\rho\sigma_1\sigma_2-g_1)\tau}f(d_{1f} + \rho\sigma_2\sqrt{\tau}) = \mathrm{e}^{-r_f\tau}K_f f(d_f + \rho\sigma_2\sqrt{\tau}). \qquad (18.7)$$

The delta of a foreign equity option with respect to the spot price of the foreign equity can be obtained by taking partial derivative of (18.6) and simplifying the result using (18.7):

$$Delta_1 = \frac{\partial FEPDC}{\partial I_1} = \omega I_2 e^{(\rho\sigma_1\sigma_2 - g_1)\tau} N[\omega(d_{1f} + \rho\sigma_2\sqrt{\tau})] \,. \quad (18.8)$$

It is obvious that the delta formula in (18.8) degenerates to that of a vanilla option multiplied by the spot exchange rate when the correlation coefficient is zero.

The delta of a foreign equity option with respect to the spot exchange rate I_2 can be obtained directly by taking partial derivative of (18.6) with respect to I_2:

$$\begin{aligned} Delta_2 = \frac{\partial FEPDC}{\partial I_2} &= \omega I_1 e^{(\rho\sigma_1\sigma_2 - g_1)\tau} N[\omega(d_{1f} + \rho\sigma_2\sqrt{\tau})] \\ &\quad - \omega K_f e^{-r_f \tau} N[\omega(d_f + \rho\sigma_2\sqrt{\tau})] \,, \end{aligned} \quad (18.9)$$

which is very similar to the foreign equity option pricing formula in foreign currency given in (18.4). It is interesting to observe that the sensitivity given in (18.9) becomes exactly the foreign equity option pricing formula in foreign currency given in (18.4) when $\rho = 0$, implying that the pricing formula (18.4) is a special case of the sensitivity in (18.9).

Example 18.3. Find the deltas of the options in Example 18.1.

Substituting $I_1 = 810$ DM, $I_2 = 1/1.39$, $K_f = 800$ DM, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, $g_1 = 0.04$, $\rho = 0.25$, and $\sigma_2 = 0.15$ into (18.8) yields

$$\begin{aligned} Delta_1(\omega = 1) &= \frac{1}{1.39} e^{(0.25\times0.12\times0.15-0.04)\times1} N(0.6635) \\ &= \$0.518 = 51.8\% \,, \\ Delta_1(\omega = -1) &= \frac{1}{1.39} e^{(0.25\times0.12\times0.15-0.04)\times1} N(-0.6635) \\ &= -\$0.176 = 17.6\% \,, \\ Delta_2(\omega = 1) &= 810 e^{(0.25\times0.12\times0.15-0.04)\times1} N(0.6635) \\ &\quad - 800 e^{-0.07\times1} N(0.5435) = \$56.495 \,, \\ Delta_2(\omega = -1) &= -810 e^{(0.25\times0.12\times0.15-0.04)\times1} N(-0.6635) \\ &\quad + 800 e^{-0.07\times1} N(-0.5435) = \$20.684 \,. \end{aligned}$$

The chi of a foreign equity option can be readily obtained by taking partial derivative of (18.6) with respect to ρ and simplifying the result using (18.7):

$$\frac{\partial FEPDC}{\partial \rho} = \omega\sigma_1\sigma_2\tau I_1 I_2 e^{(\rho\sigma_1\sigma_2 - g_1)\tau} N[\omega(d_{1f} + \rho\sigma_2\sqrt{\tau})]\,, \quad (18.10)$$

which is always positive for call options and negative for put options.

Example 18.4. Find the chi of the options in Example 18.2.

Substituting $I_1 = ¥1540$, $I_2 = 1/90$, $K_f = ¥1600$, $r_f = 3\%$, $\tau = 9/12$, $\sigma_1 = 0.14$, $g_1 = 0.02$, $\rho = -0.30$, and $\sigma_2 = 0.18$ into (18.10) yields:

$$\text{chi } (\omega = 1) = 0.14 \times 0.18 \times \frac{9}{12} \times \frac{1}{90} \times 1540e^{(-0.30\times0.14\times0.18-0.02)\times1}$$
$$N(-0.4521) = 0.1024 = 10.24\%\,,$$
$$\text{chi } (\omega = -1) = -0.14 \times 0.18 \times \frac{9}{12} \times \frac{1}{90} \times 1540e^{(-0.30\times0.14\times0.18-0.02)\times1}$$
$$N(0.4521) = -0.2122 = -21.22\%\,.$$

As argued above, a foreign equity option can be priced independently using (18.4) and converted into US dollars by multiplying the spot exchange rate when the correlation coefficient is zero. In general, however, it cannot be priced using the Black-Scholes formula and then converting the option price at the current exchange rate as our examples have shown in this chapter.

18.5. SUMMARY AND CONCLUSIONS

In this chapter, we have obtained closed-form solutions for foreign equity option prices in both foreign and domestic currencies and found analytical expressions for their delta and chi in domestic currency. Our examples showed that it is appropriate to use the Black-Scholes formula directly to price foreign equity options in foreign currency and then convert the option prices to domestic currency at the spot exchange rate only when the correlation coefficient between the stock return and the exchange rate is zero. In general, when the correlation coefficient is not zero, the direct conversion method is not appropriate to price foreign equity options. Therefore, it is always more appropriate to use the general pricing formula (18.8), or using the Black-Scholes formula in (18.4) by substituting the spot price I_1 with $I_1 e^{\rho\sigma_1\sigma_2\tau}$ and convert the price in domestic price at the spot exchange rate I_2.

As we will argue later in the following chapters, exchange risks are not restricted only in foreign equity options although they are related to them.

QUESTIONS AND EXERCISES

18.1. What are foreign equity options?

18.2. What is the difference between the price of a foreign equity option in foreign currency and that in domestic currency?

18.3. Can foreign equity option prices in domestic currency be simply obtained using the extended Black-Scholes formula in (18.4) and the spot exchange rate? Why?

18.4. Why does the direct conversion method undervalue the BMW call option and overvalue the corresponding put option?

18.5. What is the correct way to price foreign equity options?

18.6. Why is the integration domain for a foreign equity option in domestic currency the same as that in foreign currency?

18.7. Does the direct conversion method using the spot exchange rate overvalue or undervalue the Honda Motor stock put option in Example 18.2?

18.8. Why does the direct conversion method has opposite effects on the options in Examples 18.1 and 18.2 (it undervalues the call in Example 18.1 and overvalues it in Example 18.2, and it overvalues the put in Example 18.1 and undervalues it in Example 18.2)?

18.9. Why do the two arguments in the two cumulative functions in (18.6) both contain the same term $\rho\sigma_2\sqrt{\tau}$?

18.10. Find the prices of the BMW options if the correlation coefficient between the BMW stock return and the mark/dollar exchange rate is changed to -25% and other parameters remain the same as in Example 18.1.

18.11. Find the deltas of the two options in Exercise 18.10.

18.12. Find the chi of the two options in Exercise 18.10.

18.13. Find the Honda Motor option prices in Example 18.2 if the spot yen/dollar exchange rate is changed to ¥85/dollar and other parameters remain the same as in Example 18.2.

18.14. Find the deltas of the two options in Exercise 18.13.

18.15. Find the prices of the Honda Motor options in Example 18.2 if the volatility of the dollar/yen exchange rate increases to 25% and other parameters remain unchanged as in Example 18.2.

18.16. Find the chi of the two options in Exercise 18.15.

Chapter 19

EQUITY-LINKED FOREIGN EXCHANGE OPTIONS

19.1. INTRODUCTION

We studied foreign equity options in Chapter 18. Foreign equity options for domestic investors are actually instruments which put a floor (resp. a ceiling) on the foreign equity price for a call option (resp. put option) without any restriction on the exchange rate. Some domestic investors might prefer to put a floor or a ceiling on the exchange rate and confine the equity forward with a currency option. They can be best satisfied by equity-linked foreign exchange options. Equity-linked foreign exchange options provide protection against foreign equity exposure. Thus, they are symmetric to foreign equity options for domestic investors, because they put a floor (resp. a ceiling) on the foreign exchange rate for a call option (resp. put option) without any restriction on the forward equity price. They can also be considered as foreign currency options with uncertain notional quantity which is correlated to the exchange rate.

Bankers Trust offered such products under the name "Elf-X", the abbreviation for "equity-linked foreign exchange options". Marcus and Modest (1986) analyzed such options and found applications for them in agricultural price supports. More recently, Reiner (1992) provided a good analysis of equity-linked foreign exchange options and priced these options in both domestic and foreign currencies. This chapter will introduce these options, price them in a Black-Scholes environment, and find applications for them.

19.2. EQUITY-LINKED FOREIGN EXCHANGE OPTIONS

The payoff of a standard foreign exchange option (PFFXOP) can be given using the parameters in Chapter 18:

$$PFFXOP = \max\left[\omega I_2(\tau) - \omega K_e,\ 0\right], \tag{19.1}$$

where $I_2(\tau)$ and K_e stand for the exchange rate at the option maturity and the strike rate of the option, respectively, and ω is the binary operator (1 for a call option and -1 for a put option.)

The exchange rate $I_2(\tau)$ is in domestic currency per unit of foreign currency, the same as in Chapter 18 for foreign equity options. Assume that the foreign exchange rate $I_2(\tau)$ follows the stochastic process given in (IV1) with the payout rate of the foreign currency $g_2 = r_f$, the foreign interest rate. Using this process, we can easily obtain the foreign exchange option price using the extended Black-Scholes formula in (2.14). The payoff given in (19.1) can be considered as a foreign exchange option with unitary notional quantity. An equity-linked foreign exchange option is a foreign exchange option with the forward foreign equity price as the notional quantity $I_1(\tau)$. The payoff of an equity-linked foreign exchange option (PEQFXOP) can be given formally

$$PEQFXOP = I_1(\tau)\max[\omega I_2(\tau) - \omega K_e,\ 0]\,, \tag{19.2}$$

where $I_1(\tau)$ is the forward foreign asset price at the option maturity.

The payoff in (19.2) is in domestic currency because the exchange rate is expressed in domestic currency per unit of foreign currency and $I_1(\tau)$ is in foreign currency. Comparing the payoff function in (18.2) with (19.2), we can easily find that either can be obtained by changing the positions of $I_1(\tau)$ and $I_2(\tau)$ in the other expression if we neglect the constant strike prices K_f and K_e. Thus, the payoff in (18.2) can be considered as symmetric to that given in (19.2). Since the payoff of an equity-linked foreign exchange option is symmetric to that of a foreign equity option in domestic currency, we say that these two kinds of options are analytically symmetric.

The payoff in (19.2) indicates that a call option puts a floor and a put option puts a ceiling on the exchange rate, and the foreign equity exposure is not protected at all. In other words, no restriction is imposed in the foreign equity price, as in the exchange rate in a foreign equity option in a domestic currency in Chapter 18.

19.3. PRICING EQUITY-LINKED FOREIGN EXCHANGE OPTIONS

For convenient comparison, we express the price of a standard foreign exchange option (FXOP) with the payoff given in (19.1) as:

$$FXOP = \omega I_2 \mathrm{e}^{-r_f\tau} N(\omega d_{1x}) - \omega K_e \mathrm{e}^{-r\tau} N(\omega d_x)\,, \tag{19.3}$$

where

$$d_x = \frac{\ln(I_2/K_e) + (r - r_f - \sigma_2^2/2)\tau}{\sigma_2\sqrt{\tau}},$$

$$d_{1x} = d_x + \sigma_2\sqrt{\tau},$$

ω is the binary operator (1 for a call option and -1 for a put option), r and r_f represent the domestic and foreign interest rates, respectively, and other parameters are the same as in (18.2).

Since the exchange rate is normally correlated to the foreign stock price as in foreign equity options studied in Chapter 18, we cannot price equity-linked foreign exchange options in domestic currency directly using the extended Black-Scholes formula in (19.3). The problem of pricing an equity-linked foreign exchange option is symmetric to pricing a foreign equity option in domestic currency. Using the joint density functions given in (IV4) and (IV5) and the integration domain from K_e to infinity for a call option and from negative infinity to K_e for a put option, we can obtain the expected payoff of a equity-linked foreign exchange option in domestic currency given in (19.2):

$$\begin{aligned} E(PEQFXOP) &= \omega I_1 I_2 e^{(\mu_1+\mu_2+\rho\sigma_1\sigma_2)\tau} N[\omega(d_{1x} + \rho\sigma_1\sqrt{\tau})] \\ &\quad - \omega K_e I_1 e^{u_2\tau} N[\omega(d_x + \rho\sigma_1\sqrt{\tau})], \end{aligned} \tag{19.4}$$

where d_x and d_{1x} are the same as in (19.3), σ_2 and ρ represent the volatility of the exchange rate and the correlation coefficient between the exchange and the foreign equity returns, respectively, and μ_1 and μ_2 represent the drifts of the foreign equity return and the exchange rate, respectively.

The price of an equity-linked foreign exchange option (EQFX) is obtained by substituting $\mu_1 = r_f - g_1$ and $\mu_2 = r - r_f$ into (19.4) and discounting (19.4) at the risk-free domestic interest rate r:

$$\begin{aligned} EQFX = I_1\{&\omega I_2 e^{(\rho\sigma_1\sigma_2 - g_1)\tau} N[\omega(d_{1x} + \rho\sigma_1\sqrt{\tau})] \\ &-\omega K_e e^{-(r-r_f+g_1)\tau} N[\omega(d_x + \rho\sigma_1\sqrt{\tau})]\}, \end{aligned} \tag{19.5}$$

where all parameters are the same as in (19.4).

The pricing formula of a foreign equity option in (19.5) is obviously of the Black-Scholes type because it is expressed in terms of the cumulative functions of the standard normal distribution and the argument in the first cumulative normal function is always $\sigma_2\sqrt{\tau}$ greater than that in the second cumulative function. It is different from other correlation option pricing formulas in the sense that the arguments in both the cumulative functions have a common term $\rho\sigma_1\sqrt{\tau}$ resulting from the exchange rate which is multiplied in front of the regular foreign equity option payoff given in (19.2).

We can check that the term in the brace of the pricing formula in (19.5) becomes precisely the same as that of a foreign exchange option in foreign currency given in (19.3) when the correlation coefficient is zero. This is intuitive because the equity-linked foreign exchange option is priced independently of the correlation coefficient when it is zero and the value of the equity-linked foreign exchange option in domestic currency is simply obtained by multiplying the foreign exchange option price in domestic currency per unit of foreign currency by the spot exchange rate.

Example 19.1: Find the prices of the BMW stock-linked German mark options to expire in one year with the strike price 1.4 marks/US dollar, given the spot BMW stock price 810 marks, the German interest rate 7%, the US interest rate 8%, the payout rate of the stock $g_1 = 4\%$, the volatility of the foreign stock $\sigma_1 = 12\%$, the spot German mark/US dollar exchange rate 1.39 marks/dollar, the exchange rate volatility 15%, and the correlation coefficient between the BMW stock return and the exchange rate is 25%.

The prices of the BMW options can be obtained by substituting $K_e = 1/1.40 = \$0.714$, $I_2 = 1/1.39 = \$0.719$, $r = 8\%$, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, and $\sigma_2 = 0.15$ into (19.3):

$$\begin{aligned}
d_x &= [\ln(0.719/0.714) + (0.07 - 0.08 - 0.12^2/2)]/(0.12\sqrt{1}) \\
&= -0.0852\,, \\
d_{1x} &= d_x + \sigma_1\sqrt{\tau} \\
&= -0.0852 + 0.12 \times \sqrt{1} = 0.0348\,, \\
FEOPFC(\omega = 1) &= 0.719\mathrm{e}^{-0.08\times 1}N(0.0348) - 0.714\mathrm{e}^{-0.07\times 1}N(-0.0852) \\
&= \$0.031\,, \\
FEOPFC(\omega = -1) &= -0.719\mathrm{e}^{-0.08\times 1}N(-0.0348) \\
&\quad + 0.714\mathrm{e}^{-0.07\times 1}N(0.0852) = \$0.032\,.
\end{aligned}$$

The prices of the BMW stock-linked mark-dollar options in US dollars can be obtained by substituting $I_1 = 810$ marks, $K_e = 1/1.40 = \$0.714$, $I_2 = 1/1.39 = \$0.719$, $r = 8\%$, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.12$, $g_1 = 0.04$, and $\rho = 0.25$ into (19.5):

$$\begin{aligned}
d_x + \rho\sigma_1\sqrt{\tau} &= -0.0852 + 0.25 \times 0.12 \times \sqrt{1} = 0.055\,, \\
d_{1x} + \rho\sigma_1\sqrt{\tau} &= 0.0348 + 0.25 \times 0.12 \times \sqrt{1} = 0.0648\,,
\end{aligned}$$

$$
\begin{aligned}
FEPDC(\omega = 1) &= 810[0.719\mathrm{e}^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(0.0648) \\
&\quad - 0.714\mathrm{e}^{-(0.08-0.07+0.04)\times 1}N(0.055)] \\
&= \$8.463\,, \\
FEPDC(\omega = -1) &= 810[-0.719\mathrm{e}^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(-0.0648) \\
&\quad + 0.714\mathrm{e}^{-(0.08-0.07+0.04)\times 1}N(-0.055)] \\
&= \$3.560\,.
\end{aligned}
$$

Example 19.2: Find the prices of the Honda Motor stock-linked Japanese yen options to expire in one year with the strike price ¥95/US dollar, given the spot Honda Motor stock price ¥1540, the Japanese interest rate 3%, the US interest rate 8%, the payout rate of the stock $g_1 = 2\%$, the volatility of the stock $\sigma_1 = 14\%$, the spot Japanese yen/US dollar exchange rate 100 yen/dollar, the exchange rate volatility 18%, and the correlation coefficient between the Honda Motor stock return and the exchange rate is -30%.

The prices of the Honda Motor stock options can be obtained by substituting $K_e = 1/95 = \$0.0105$, $I_2 = 1/100 = \$0.0100$, $r = 8 = 0.14$, and $\sigma_2 = 0.18$ into (19.3):

$$
\begin{aligned}
d_x &= [\ln(0.01/0.0105) + (0.02 - 0.08 - 0.14^2/2)]/(0.14\sqrt{1}) \\
&= -1.1949\,, \\
d_{1x} &= d_x + \sigma_1\sqrt{\tau} \\
&= -1.2514 + 0.14 \times \sqrt{1} = -1.0549\,, \\
FEOPFC(\omega = 1) &= 0.0100\mathrm{e}^{-0.08\times 1}N(-1.0549) \\
&\quad - 0.0105\mathrm{e}^{-0.02\times 1}N(-1.1949) = \$0.00011\,, \\
FEOPFC(\omega = -1) &= -0.0100\mathrm{e}^{-0.08\times 1}N(1.0549) \\
&\quad + 0.0105\mathrm{e}^{-0.02\times 1}N(1.1949) = \$0.0012\,.
\end{aligned}
$$

The prices of the Honda Motor stock-linked yen-dollar options in US dollars can be obtained by substituting $I_1 = ¥1540$, $K_e = 1/95 = \$0.0105$, $I_2 = 1/100 = \$0.01$, $r = 8\%$, $r_f = 2\%$, $\tau = 1$, $\sigma_1 = 0.14$, $\sigma_2 = 0.18$, $g_1 = 0.02$, and $\rho = -0.30$ into (19.5):

$$
\begin{aligned}
d_x + \rho\sigma_1\sqrt{\tau} &= -1.1949 - 0.30 \times \sqrt{1} = -1.2369\,, \\
d_{1x} + \rho\sigma_1\sqrt{\tau} &= -1.0549 - 0.30 \times 0.14 \times \sqrt{1} = -1.0969\,, \\
FEPDC(\omega = 1) &= 1540 \times [0.0100\mathrm{e}^{(0.25\times 0.14\times 0.18-0.02)\times 1}N(-1.0969) \\
&\quad - 0.015\mathrm{e}^{-(0.08-0.02+0.02)\times 1}N(-1.2369)] \\
&= \$0.385\,,
\end{aligned}
$$

$$\begin{aligned} FEPDC(\omega = -1) &= 1540 \times [-0.0100\mathrm{e}^{(0.25\times 0.14\times 0.18-0.02)\times 1}N(1.0969) \\ &\quad + 0.0105\mathrm{e}^{-(0.08-0.02+0.02)\times 1}N(1.2369) \\ &= \$0.192\,. \end{aligned}$$

19.4. SENSITIVITIES

Using the expressions of d_x and d_{1x} in (19.3) and (19.5), we can have the following identity:

$$I_1\mathrm{e}^{(\rho\sigma_1\sigma_2-r)\tau}f(d_{1x}+\rho\sigma_1\sqrt{\tau}) = \mathrm{e}^{-r_f\tau}K_e f(d_x+\rho\sigma_1\sqrt{\tau})\,. \tag{19.6}$$

The delta of a equity-linked foreign exchange option with respect to the spot exchange rate can be obtained by taking partial derivative of (19.5) and simplifying the result using (19.6):

$$Delta.Exch = \frac{\partial FEPDC}{\partial I_2} = \omega I_1\mathrm{e}^{(\rho\sigma_1\sigma_2-r)\tau}N[\omega(d_{1x}+\rho\sigma_1\sqrt{\tau})]\,. \tag{19.7}$$

It is obvious that the delta formula in (19.7) degenerates to that of a vanilla option multiplied by the spot foreign equity price when the correlation coefficient is zero.

The delta of equity-linked foreign exchange option with respect to the spot foreign equity price I_1 can be obtained directly by taking partial derivative of (19.5) with respect to I_1:

$$\begin{aligned} Delta.FEP = \frac{\partial FEPDC}{\partial I_1} &= \omega I_2\mathrm{e}^{(\rho\sigma_1\sigma_2-r)\tau}N[\omega(d_{1x}+\rho\sigma_1\sqrt{\tau})] \\ &\quad - \omega K_e\mathrm{e}^{-r_f\tau}N[\omega(d_x+\rho\sigma_1\sqrt{\tau})]\,, \end{aligned} \tag{19.8}$$

which is very similar to the foreign exchange option pricing formula in foreign currency given in (19.3). It is interesting to observe that the sensitivity given in (19.8) becomes exactly the foreign exchange option pricing formula in foreign currency given in (19.3) when $\rho = 0$, implying that the pricing formula (19.3) is a special case of the sensitivity in (19.8).

Example 19.3: Find the deltas of the options in Example 19.1.

Substituting $I_1 = 810$ marks, $K_e = 1/1.40 = \$0.714$, $I_2 = 1/1.39 = \$0.719$, $r = 8\%$, $r_f = 7\%$, $\tau = 1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.12$, $g_1 = 0.04$, and $\rho =$

0.25 into (19.7) and (19.8) yields:

$$\begin{aligned}
Delta.Exch(\omega = 1) &= 0.719e^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(0.0648)\\
&= 36.49\%\,,\\
Delta.Exch(\omega = -1) &= -0.719e^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(-0.0648)\\
&= -32.91\%\,,\\
Delta.FEP(\omega = 1) &= 0.719e^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(0.0648)\\
&\quad - 0.714e^{-(0.08-0.07+0.04)\times 1}N(0.055)\\
&= 1.04\%\,,\\
Delta.FEP(\omega = -1) &= -0.719e^{(0.25\times 0.12\times 0.15-0.04)\times 1}N(-0.0648)\\
&\quad + 0.714e^{-(0.08-0.07+0.04)\times 1}N(-0.055)\\
&= -6.39\%\,.
\end{aligned}$$

The chi of a equity-linked foreign exchange option can be obtained by taking partial derivative of (19.5) with respect to ρ and simplifying the result using (19.6):

$$\frac{\partial EQFX}{\partial \rho} = \omega\sigma_1\sigma_2\tau I_1 I_2 e^{(\rho\sigma_1\sigma_2-r)\tau}N[\omega(d_{1x}+\rho\sigma_1\sqrt{\tau})]\,, \qquad (19.9)$$

which is always positive for call options and negative for put options.

Example 19.4: Find the chi of the options in Example 19.2.

Substituting I_1 = ¥1540 , $K_e = 1/95 = \$0.0105$, $I_2 = 1/100 = \$0.01$, $r = 8\%$, $r_f = 2\%$, $\tau = 1$, $\sigma_1 = 0.14$, $\sigma_2 = 0.18$, $g_1 = 0.02$, and $\rho = -0.30$ into (19.9) yields:

$$\begin{aligned}
\text{chi}(\omega = 1) &= 0.14 \times 0.18 \times 1540 \times 0.01e^{(0.25\times 0.14\times 0.18-0.02)\times 1}N(-1.0969)\\
&= 5.22\%\,,\\
\text{chi}(\omega = -1) &= -0.14 \times 0.18 \times 1540 \times 0.01e^{(0.25\times 0.14\times 0.18-0.02)\times 1}\\
&\quad N(-1.0969) = -33.08\%\,.
\end{aligned}$$

As argued above, the equity-linked foreign exchange option can be priced independently using (19.3) and converted into US dollars by multiplying the spot exchange rate when the correlation coefficient is zero.

19.5. SUMMARY AND CONCLUSIONS

In this chapter, we have obtained closed-form solutions for equity-linked foreign exchange option prices and found analytical expressions for their delta and chi. Equity-linked foreign exchange options possess properties of foreign exchange options with foreign equity characteristics. They can be understood as foreign exchange options with notional values specified as the forward foreign equity prices. Mathematically, an equity-linked foreign exchange option is symmetric to a foreign equity option studied in Chapter 18, because the payoff of the former is the product of the payoff of a foreign equity option and the foreign exchange rate, and that of the latter is the product of the payoff of a foreign exchange option and the forward foreign equity price. Thus the pricing formula of the former can be readily obtained simply by replacing the exchange rate in the pricing formula of the latter with the foreign equity price.

QUESTIONS AND EXERCISES

19.1. What is an equity-linked foreign exchange option?

19.2. Why are equity-linked foreign exchange options symmetric to foreign equity options in domestic currency?

19.3. Can equity-linked foreign exchange option prices in domestic currency be simply obtained using the extended Black-Scholes formula in (19.4) and multiplying the spot exchange rate? Why?

19.4. Why is the integration domain for a equity-linked foreign exchange option in domestic currency the same as that for a equity-linked foreign option in foreign currency?

19.5. Why do the two arguments in the two cumulative functions in (19.6) both contain the same term $\rho\sigma_1\sqrt{\tau}$?

19.6. Find the prices of the BMW options if the correlation between the BMW stock return and the mark/dollar exchange rate is changed to −25% and other parameters remain the same as in Example 19.1.

19.7. Find the deltas of the two options in Exercise 19.6.

19.8. Find the chi of the two options in Exercise 19.6.

19.9. Find the Honda Motor option prices in Example 19.2 if the spot yen/dollar exchange rate is changed to ¥85/dollar and other parameters remain the same as in Example 19.2.

19.10. Find the deltas of the two options in Exercise 19.9.

19.11. Find the prices of the Honda Motor options in Example 19.2 if the volatility of the dollar/yen exchange rate increases to 25% and other parameters remain unchanged as in Example 19.2.

19.12. Find the chi of the two options in Exercise 19.11.

Chapter 20

QUANTO OPTIONS

20.1. INTRODUCTION

The global links through currency and bond markets are well-established and known, equity-related derivative markets have become globalized. For example, many foreign stocks are traded in the New York Stock Exchange (NYSE) either directly or through American Depository Receipts (ADRs), and equities of several US firms are listed in foreign exchanges. Nikkei stock index futures and options have been trading in Singapore, Chicago Board of Options Exchange (CBOE) in the USA, and Toronto Stock Exchange in Canada [see Zhang (1995e) for more detailed information of Nikkei-related products trading in the leading exchanges around the world]. Other products related to foreign stock index have been introduced in the American stock exchange and other US exchanges.

Transactions of foreign equity derivatives always involve foreign exchange risks. Quanto options are designed to cope with such problems. A quanto option, also known as a quanto, is the abbreviation of a "quantity-adjusting option" or "guaranteed exchange rate option". Quantos are mostly used in currency-related markets with the price of one underlying asset converted to another underlying asset at a fixed guaranteed rate. For example, a Japanese oil importer facing uncertainties of the oil price denominated in US dollars and the dollar-yen exchange rate can simply buy a quanto call option to lock the oil price with a fixed dollar-yen exchange rate. Studying the effectiveness of hedging instruments, Ho, Stapleton, and Subrabmanyan (1995) showed that quanto put options provide better down-side protection than vanilla put options since they take into account the effects of the correlation between the exchange rate and the foreign asset. Vanilla options on foreign assets or on exchange rates are relatively inefficient and therefore more expensive.

Quantos are among the few most popular exotic options trading not only in the OTC marketplace but also in organized exchanges. The American stock exchange began to trade quantos in 1992. Reiner (1992) first priced and discussed how to hedge quanto options. Dravid, Richardson, and Sun (1993) applied the quanto option pricing method to price Nikkei warrants and tested the pricing formula using actual market data. Huang, Subrahmanyam, and Yu (1995) priced American-style quanto options. The purpose of this chapter is to introduce and illustrate how to price quanto options in a Black-Scholes environment.

20.2. QUANTO OPTIONS

The payoff of a quanto in foreign currency is the same as the one given in (18.1) for a foreign equity option. As in Chapter 18, we assume that the exchange rate $I_2(\tau)$ is in domestic currency per unit of foreign currency and follows the stochastic process given in (IV1) with the payout rate $g_2 = r_f$, the foreign interest rate. The exchange rate $I_2(\tau)$ at the option maturity can be expressed as follows after solving the stochastic equation (IV1):

$$I_2(\tau) = I_2 e^{(r - r_f + \sigma_2^2/2)\tau + \sigma_2 z_2(\tau)}, \tag{20.1}$$

where $z_2(\tau)$ is a standard Gauss-Wiener process, r, r_f, I_2, and σ_2 represent the domestic and foreign interest rates, the current exchange rate, and the volatility of the exchange rate, respectively.

The exchange rate in (20.1) gives the amount of domestic currency per unit of foreign currency. It can also be expressed in terms of the foreign currency per unit of domestic currency as the reciprocal of the exchange rate given in (20.1). Let $I_2'(\tau)$ represent the reciprocal of $I_2(\tau)$ given in (20.1), we then have the following:

$$I_2'(\tau) = I_2' e^{(r_f - r - \sigma_2^2/2)\tau + \sigma_2 z_{-2}(\tau)}, \tag{20.2}$$

where $z_{-2}(\tau) = -z_2(\tau)$ is the opposite Gauss-Wiener process of $z_2(\tau)$ given in (20.1), $I_2' = 1/I_2$ is the reciprocal of the current exchange rate, and other parameters are the same as in (20.1).

Since the Gauss-Wiener process in the reciprocal exchange rate in (20.2) is the opposite of the one in (20.1), the correlation coefficient between the reciprocal exchange rate and the return of the foreign equity is simply the opposite of that between the exchange rate and the return of the foreign equity, and is equal to $-\rho$.

The payoff of a quanto option in domestic currency (PQTODC) is the same as that of a foreign equity option in domestic currency given in (18.2) but with a fixed exchange rate $\bar{I}_2$:

$$PQTODC = \bar{I}_2 \max[\omega I_1(\tau) - \omega K_f,\ 0]\,, \tag{20.3}$$

where $\bar{I}_2$ is the prespecified exchange rate in domestic currency per unit of foreign currency and other parameters are the same as in (18.2).

20.3. PRICING QUANTOS

In order to price a quanto option with the payoff given in (20.3), we need to convert the payoff in domestic currency to foreign currency. The reason is that a quanto option is hedged in foreign currency. The payoff in foreign currency (PQTFC) is simply the product of the payoff in (20.3) and the reciprocal of the exchange rate given in (20.2):

$$PQTOFC = \bar{I}_2 I_2'(\tau) \max[\omega I_1(\tau) - \omega K_f,\ 0]\,, \tag{20.4}$$

where all parameters are the same as in (20.2) and (20.3).

Since the reciprocal exchange rate is lognormally distributed and correlated to the stock price with the correlation coefficient $-\rho$, we can find the price of a quanto option in foreign currency. As a matter of fact, the expected payoff of a quanto option in foreign currency can be obtained by substituting $\mu_1 = r_f - g_1$, $\mu_2 = r_f - r$ (the drift for the reciprocal exchange rate), and the correlation coefficient $-\rho$ into (18.5):

$$\begin{aligned} E(PQTOFC) = \bar{I}_2 I_2' \{ & \omega I_1 e^{(2r_f - g_1 - r - \rho\sigma_1\sigma_2)\tau} N[\omega(d_{1f} - \rho\sigma_2\sqrt{\tau})] \\ & - \omega K_f e^{(r_f - r)\tau} N[\omega(d_f - \rho\sigma_2\sqrt{\tau})]\}\,, \end{aligned} \tag{20.5}$$

where

$$d_f = \frac{\ln(I_1/K_f) + (r_f - g_1 - \sigma_1^2/2)\tau}{\sigma_1\sqrt{\tau}}\,,$$

$$d_{1f} = d_f + \sigma_1\sqrt{\tau}\,,$$

and other parameters are the same as in (20.4).

The price of a quanto option in foreign currency (QTOF) is obtained by discounting (20.5) at the risk-free foreign interest rate r_f:

$$\begin{aligned} QTOF = \bar{I}_2 I_2' \{ & \omega I_1 e^{(r_f - g_1 - r - \rho\sigma_1\sigma_2)\tau} N[\omega(d_{1f} - \rho\sigma_2\sqrt{\tau})] \\ & - \omega K_f e^{-r\tau} N[\omega(d_f - \rho\sigma_2\sqrt{\tau})]\}\,, \end{aligned} \tag{20.6}$$

where all parameters are the same as in (20.5).

Since the price of the quanto option in (20.6) is in foreign currency, we can obtain the corresponding price in domestic currency (QTOD) by dividing (20.6) by the current reciprocal of the exchange rate I_2':

$$QTOD = \bar{I}_2 e^{-r\tau}[\omega I_F N(\omega d_{1F}) - \omega K_f N(\omega d_{2F})], \qquad (20.7)$$

where $I_F = I_1 e^{-(g_1 - r_f)\tau - \rho\sigma_1\sigma_2\tau}$, $d_{2F} = \frac{\ln\frac{I_F}{K_f} - \frac{1}{2}\sigma_1^2\tau}{\sigma_1\sqrt{2}}$, $d_{1F} = d_{2F} + \sigma\sqrt{\tau}$, and all other parameters are the same as (20.6) and (20.5).

The pricing formula of quanto options in (20.7) is obviously of the Black-Scholes type because it is expressed in terms of the cumulative functions of the standard normal distribution and the argument in the first cumulative normal function is always $\sigma_1\sqrt{\tau}$ greater than that in the second cumulative function. It is similar to the pricing formula of foreign equity options given in (18.6) and that of equity-linked foreign exchange options given in (19.5) in the sense that the arguments in both the cumulative functions have a common term $-\rho\sigma_2\sqrt{\tau}$ resulting from the reciprocal exchange rate which is multiplied in front of the regular foreign equity option payoff given in (20.4).

We can check that the term in the brace of the pricing formula in (20.7) becomes precisely the same as that of a foreign equity option in foreign currency given in (18.4) when the correlation coefficient is zero. This is intuitive because the foreign equity option is priced independently of the exchange rate when the correlation coefficient is zero and the value of the foreign equity option in domestic currency is simply obtained by multiplying the foreign equity option price in foreign currency by the exchange rate.

Example 20.1. Find the prices of the quanto BMW stock options to expire in one year with the strike price 800 marks, given the spot BMW stock price 810 marks, the German interest rate 7%, the US interest rate 8%, the payout rate of the stock $g_1 = 4\%$, the volatility of the stock $\sigma_2 = 12\%$, the fixed German mark/US dollar exchange rate 1.39 marks/dollar, the exchange rate volatility 15%, and the correlation coefficient between the BMW stock return and the exchange rate 25%.

The prices of the BMW quanto options can be obtained by substituting $I_1 = 810$ marks, $K_f = 800$ marks, $r_f = 7\%$, $r = 0.08$, $\bar{I}_2 = 1/1.39 = \$0.719$, $\tau = 1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.15$, $\rho = 0.25$, and $g_1 = 0.04$ into (20.7):

$$d_f = [\ln(810/800) + (0.07 - 0.04 - 0.12^2/2)]/(0.12\sqrt{1}) = 0.2935,$$

$$\begin{aligned} d_{1f} &= d_f + \sigma_1\sqrt{\tau} \\ &= 0.2935 + 0.12 \times \sqrt{1} = 0.4135\,, \end{aligned}$$

$$
\begin{aligned}
\rho\sigma_2\sqrt{\tau} &= 0.25 \times 0.15 \times \sqrt{1} = 0.0375\,, \\
QTOD(\omega = 1) &= 0.719\Big[810\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(0.4135 - 0.0375) \\
&\quad - 800\mathrm{e}^{-0.08\times1}N(0.2935 - 0.0375)\Big] \\
&= \$37.449\,, \\
QTOD(\omega = -1) &= 0.719\Big[-810\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(-0.4135 + 0.0375) \\
&\quad + 800\mathrm{e}^{-0.08\times1}N(-0.2935 + 0.0375)\Big] \\
&= \$16.893\,.
\end{aligned}
$$

Example 20.2. Find the prices of the quanto options in Example 20.1 if the correlation coefficient is changed to -0.25 and other parameters remain unchanged.

The prices of the BMW quanto options can be obtained by substituting $I_1 = 810$ marks, $K_f = 800$ marks, $r_f = 7\%$, $r = 0.08$, $\bar{I}_2 = 1/1.39 = \$0.719$, $\tau = 1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.15$, $\rho = -0.25$, and $g_1 = 0.04$ into (20.7):

$$
\begin{aligned}
QTOD(\omega = 1) &= 0.719\Big[810\mathrm{e}^{(0.07-0.04-0.08+0.25\times0.12\times0.15)\times1} \\
&\quad N(0.4135 + 0.0375) \\
&\quad - 800\mathrm{e}^{-0.08\times1}N(0.2935 + 0.0375)\Big] \\
&= \$40.713\,, \\
QTOD(\omega = -1) &= 0.719\Big[-810\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(-0.4135 - 0.0375) \\
&\quad + 800\mathrm{e}^{-0.08\times1}N(-0.2935 + 0.25 \times 0.15\sqrt{1})\Big] \\
&= \$15.207\,.
\end{aligned}
$$

The results in Examples 20.1 and 20.2 indicate that the quanto call option price is higher and the put option price lower with a smaller correlation coefficient. We will return to the sensitivities of quanto options in the following section.

Example 20.3. Find the prices of the quanto Honda Motor options to expire in nine months with the strike price ¥1600, given the spot Honda Motor stock price ¥1540, and other parameters remain the same as in Example 18.2.

The prices of the Honda Motor quanto call options in Japanese yen can be obtained by substituting $I_2 = 1/90 = \$0.0111$, $I_1 = ¥1540$, $K_f = ¥1600$,

$r_f = 0.03$, $r = 0.08$, $\tau = 9/12 = 0.75$, $\sigma_1 = 0.14$, $\sigma_2 = 0.18$, $\rho = -0.30$, and $g_1 = 0.02$ into (20.7):

$$\begin{aligned}
d_f &= [\ln(1540/1600) + (0.03 - 0.02 - 0.14^2/2) \\
&\qquad \times 0.75]/(0.14\sqrt{0.75}) = -0.3140\,, \\
d_{1f} &= d_f + \sigma_1\sqrt{\tau} \\
&= -0.3140 + 0.14\sqrt{9/12} = -0.1928\,, \\
\rho\sigma_2\sqrt{\tau} &= -0.30 \times 0.18 \times \sqrt{0.75} = -0.0468\,, \\
QTOD(\omega = 1) &= 0.0111\Big[1540\mathrm{e}^{(0.03-0.02-0.08+0.30\times0.14\times0.18)\times0.75} \\
&\qquad N(-0.3140 + 0.0468) \\
&\qquad - 1600\mathrm{e}^{-0.08\times0.75}N(-0.1928 + 0.0468)\Big] \\
&= \$0.608\,, \\
QTOD(\omega = -1) &= 0.0111\Big[-1540\mathrm{e}^{(0.03-0.02-0.08+0.30\times0.14\times0.18)\times0.75} \\
&\qquad N(0.3140 - 0.0468) \\
&\qquad + 1600\mathrm{e}^{-0.08\times0.75}N(0.1928 - 0.0468)\Big] \\
&= \$1.024\,.
\end{aligned}$$

20.4. SENSITIVITIES

Using the expressions of d_f and d_{1f} in (18.4) and (18.6), we can have the following identity for quanto options:

$$I_1\mathrm{e}^{(r_f-g_1-\rho\sigma_1\sigma_2)\tau}f(d_{1f} - \rho\sigma_2\sqrt{\tau}) = K_f f(d_f - \rho\sigma_2\sqrt{\tau})\,. \tag{20.8}$$

The delta of a quanto option with respect to the spot price of the foreign equity can be obtained by taking partial derivative of (20.6) and simplifying the result using (20.7):

$$Delta.PFE = \frac{\partial QTOD}{\partial I_1} = \omega\bar{I}_2\mathrm{e}^{(r_f-g_1\tau-\rho\sigma_1\sigma_2)\tau}N[\omega(d_{1f} - \rho\sigma_2\sqrt{\tau})]\,. \tag{20.9}$$

It is obvious that the delta formula in (20.9) degenerates to that of a vanilla option multiplied by the spot exchange rate when the correlation coefficient is zero.

The delta of a quanto option with respect to the spot exchange rate $\bar{I}_2$ can be obtained directly by taking partial derivative of (20.6) with respect to $\bar{I}_2$:

$$\begin{aligned}
Delta.Exch = \frac{\partial QTOD}{\partial \bar{I}_2} &= \omega I_1\mathrm{e}^{(r_f-g_1-r-\rho\sigma_1\sigma_2)\tau}N[\omega(d_{1f} - \rho\sigma_2\sqrt{\tau})] \\
&\quad - \omega K_f\mathrm{e}^{-r\tau}N[\omega(d_f - \rho\sigma_2\sqrt{\tau})]\,,
\end{aligned} \tag{20.10}$$

which becomes the pricing formula of a foreign equity option in foreign currency given in (18.4) when $\rho = 0$ and $r = r_f$, implying that the pricing formula (18.4) is a special case of the sensitivity in (20.10).

Example 20.4. Find the deltas of the options in Example 20.1.

Substituting $I_1 = 810$ marks, $K_f = 800$ marks, $r_f = 7\%$, $r = 0.08$, $\bar{I}_2 = 1/1.39 = \$0.719$, $\tau = 1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.15$, $\rho = 0.25$, and $g_1 = 0.04$ into (20.8) yields:

$$\begin{aligned}
Delta.PFE(\omega = 1) &= 0.719\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(0.4135 - 0.0375) \\
&= 44\%\,, \\
Delta.PFE(\omega = -1) &= -0.719\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(-0.4135 + 0.0375) \\
&= -24.1\%\,, \\
Delta.Exch(\omega = 1) &= 810\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(0.4135 - 0.0375) \\
&\quad - 800\mathrm{e}^{-0.08\times1}N(0.2935 - 0.0375) \\
&= 52.085\,, \\
Delta.Exch(\omega = -1) &= -810\mathrm{e}^{(0.07-0.04-0.08-0.25\times0.12\times0.15)\times1} \\
&\quad N(-0.4135 + 0.0375) \\
&\quad + 800\mathrm{e}^{-0.08\times1}N(-0.2935 + 0.0375) \\
&= 23.495\,.
\end{aligned}$$

The chi of a quanto option can be obtained by taking partial derivative of (20.6) with respect to ρ and simplifying the result using (20.7):

$$\begin{aligned}
\frac{\partial QTOD}{\partial \rho} &= -\sigma_1\sigma_2\tau I_1 I_2 \mathrm{e}^{(r_f-g_1-r-\rho\sigma_1\sigma_2)\tau} \\
&\quad N[\omega(d_{1f} - \rho\sigma_2\sqrt{\tau})]\,,
\end{aligned} \qquad (20.11)$$

which is always positive for put options and negative for call options.

The sign of the chi expression in (20.11) is consistent with the results given in Examples 20.1 and 20.2 because the quanto call option price is lower and the put option price is higher with a higher correlation coefficient.

Example 20.5. Find the chi of the options in Example 20.3.

Substituting $I_1 = 1/90 = \$0.0111$, $I_1 = ¥1540$, $K_f = ¥1600$, $r_f = 0.03$, $r = 0.08$, $\tau = 9/12 = 0.75$, $\sigma_1 = 0.14$, $\sigma_2 = 0.18$, $\rho = -0.30$, and $g_1 = 0.02$ into (20.10) yields:

$$\begin{aligned}\text{chi}(\omega = 1) &= -0.0111 \times 1540 \times 0.14 \times 0.18 \\ &\quad e^{(0.03-0.02-0.08+0.30\times 0.14\times 0.18)\times 0.75} N(-0.3140 + 0.0468) \\ &= -18.18\%\,, \\ \text{chi}(\omega = -1) &= 0.0111 \times 1540 \times 0.14 \times 0.18 \\ &\quad e^{(0.03-0.02-0.08+0.30\times 0.14\times 0.18)\times 0.75} N(0.3140 - 0.0468) \\ &= -22.96\%\,.\end{aligned}$$

As argued above, a quanto option can be priced independently using (18.4) and converted into US dollar by multiplying the spot exchange rate when the correlation coefficient is zero. In general the two examples above show the call (put) option price using (20.6) in Example 20.3 should be greater (smaller) than the call (put) option price in Example 20.2 [which is correct for the special case of $\rho = 0$ as shown in (18.11)], because the call (put) option price is positively (negatively) related to the correlation coefficient in (18.10). Therefore, the results in Examples 20.3 are consistent to the theoretical analysis in the previous two sections.

20.5. FOREIGN EQUITY OPTIONS AND QUANTO OPTIONS

Quanto options studied in this chapter are very similar to foreign equity options because they are all on foreign equity prices and expressed in domestic currency with all the same parameters. However, there exist significant differences between them because the exchange rate risk is incorporated differently in the two kinds of options. Specifically, the exchange rate is fixed in quanto options, yet it is not fixed in foreign equity options. As a result, the chi of a quanto option is of the opposite sign as that of its corresponding foreign equity option. Since quanto options are very similar to foreign equity options and they are expressed with the same set of parameters, it is highly necessary and useful for us to compare their prices. We will try to achieve some comparative results for these two kinds of options in the remaining part of this section.

In general, we cannot find simple conditions under which the relative magnitudes of the two kinds of options are determined. However, we can obtain some comparative results under special conditions. The results are given in the following corollaries.

Corollary 20.1. The price of a foreign equity option in domestic currency (FEPDC) given in (18.6) and its corresponding quanto option price (QTOD) given in (20.7) satisfy the following relationship

$$FEPDC(\rho = 0) \geq QTOD(\rho = 0) \text{ if and only if } r \geq r_f . \qquad (20.12)$$

Proof. Substituting $\rho = 0$ into (18.6) and (20.8) yields (20.12) after some simple algebraic simplifications. □

Corollary 20.2. The price of a foreign equity option in domestic currency (FEPDC) given in (18.6) and its corresponding quanto option price (QTOD) given in (20.8) satisfy the following relationship

$$\omega\rho[FEPDC(\rho \neq 0) - QTOD(\rho \neq 0)] \geq 0 \text{ if and only if } r \geq r_f , \qquad (20.13a)$$

and

$$\omega\rho[FEPDC(\rho \neq 0) - QTOD(\rho \neq 0)] \leq 0 \text{ if and only if } r \leq r_f , \qquad (20.13b)$$

where ω is the option binary operator (1 for a call and -1 for a put option).

Proof. Immediately from the chi expression for foreign equity options and quanto options given in (18.10) and (20.11), and Corollary 20.1 using the monotonic property of the pricing formulas of the two kinds of options.

As can be seen from Corollaries 20.1 and 20.2 the relative magnitudes of the two interest rates play a crucial role in determining the relative magnitudes of a quanto option price and its corresponding foreign equity option price. This is highly consistent with the fact that a quanto option price is obtained by discounting its expected payoff in foreign currency at foreign interest rate, whereas its corresponding foreign equity option price is obtained by discounting its expected payoff in domestic currency of domestic interest rate. □

20.6 "JOINT" QUANTO OPTIONS

We have introduced and analyzed "fixed" or "true" quanto options. There is an alternative form of quanto options called a "joint" quanto options. For a "joint" quanto options, the value of the option depends on (A) the spot exchange rate at option maturity as a foreign equity option described in Chapter 18 in relation to (B) a guaranteed level of exchange rate

($\bar{I}_2$) as in the previous sections of this chapter. The payoff such an option could be expressed:

$$\max[I_2(\tau), \bar{I}_2] \times \max[\omega I(\tau) - \omega k_f, 0]\,, \tag{20.14}$$

where all parameters are the same as in (20.3).

With method developed to price "true" quanto options in previous sections of this chapter and that to price foreign equity options in Chapter 18, we can readily obtain the pricing formula for a "joint" quanto option (JQTOP):

$$\begin{aligned} JQTOP = \bar{I}_2 \mathrm{e}^{-r\tau}\Big\{&\mathrm{e}^{-(g_1, r_f)\tau\bar{o}\rho\sigma_1\sigma_2\tau} N_2[d_1(\bar{I}_2) + \rho\sigma_1\sqrt{\tau}, d_1(s) - \rho\sigma_2\sqrt{2} - \rho] \\ &- k_f N_2[d(\bar{I}_2), d(s) - \rho\sigma_2\sqrt{2}, -\rho]\Big\} \\ &+ I_2(0)\mathrm{e}^{-g_1\tau + \sigma_2(\sigma_2 - \rho\sigma_1)\tau} N_2[-d(\bar{I}_2) + \sigma_2\sqrt{2}, d_1(s) - \rho\sigma_2\sqrt{2}, -\rho] \\ &- k_f \mathrm{e}^{-r_f\tau} N_2[-d(\bar{I}_2) + \sigma_2\sqrt{2}, d(s) - \rho\sigma_2\sqrt{\tau}, -\rho]\,, \end{aligned} \tag{20.15}$$

where

$$\begin{aligned} d(s) &= \frac{\ln(\frac{s}{k_f}) + (r_f - g_1 - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}\,, \\ d_1(s) &= d(s) + \sigma_1\sqrt{\tau}\,, \\ d(\bar{I}_2) &= \frac{\ln\frac{\bar{I}_2}{I_2(0)} + (r_f - r - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{2}}\,, \\ d_1(\bar{I}_2) &= d(\bar{I}_2) + \sigma_2\sqrt{\tau}\,, \end{aligned}$$

and $I_2(0)$ stands for the spot exchange rate.

It's opivious to observe that the pricing formula given in (20.15) degenerates to the one given in (20.7) when the rate of the guaranteed exchange rate $\bar{I}_2$ for "true" quanto options and spot exchange rate $I_2(0)$ is extremely large, or when the guaranteed exchange rate $\bar{I}_2(0)$ is greatly greater than the exchange rate at maturity. The above observation is intuitive because the payoff given in (20.14) simply degenerates to that for "true" quanto options given in (20.3) under specified conditions.

20.7. SUMMARY AND CONCLUSIONS

Quanto options are among the few most popular types of exotic options. In this chapter, we have obtained closed-form solutions for quanto option prices in domestic currency and found analytical expressions for its deltas and chi in domestic currency. We have compared foreign equity options and quanto options which are both foreign equity options, one with floating exchange rate and one with fixed exchange rate. The most important difference in analyzing them is that quanto option prices are obtained by discounting its expected payoff in foreign currency at foreign interest rate, and foreign equity option prices are obtained by discounting the payoffs in domestic currency at domestic interest rate. Besides this important difference, a quanto option can be regarded with negative correlation adjustment compared to its corresponding foreign equity options.

QUESTIONS AND EXERCISES

20.1. What are quanto options?

20.2. What is the difference between the price of a foreign equity option in foreign currency and that in domestic currency?

20.3. Can foreign equity option prices in domestic currency be simply obtained using the extended Black-Scholes formula in (18.4) and multiplying the spot exchange rate? Why?

20.4. Why is the integration domain for a foreign equity option in domestic currency the same as that for a foreign equity option in foreign currency?

20.5. Why do the two arguments in the two cumulative functions in (20.6) both contain the same term $\rho\sigma_2\sqrt{\tau}$?

20.6. Find the prices of the BMW options if the correlation between the BMW stock return and the mark/dollar exchange rate is changed to -25% and other parameters remain the same as in Example 20.1.

20.7. Find the deltas of the two options in Exercise 20.6.

20.8. Find the chi of the two options in Exercise 20.6.

20.9. Find the Honda Motor option prices in Example 20.2 if the spot yen/dollar exchange rate is changed to ¥85 and other parameters remain the same as in Example 20.2.

20.10. Find the deltas of the two options in Exercise 20.9.

20.11. Find the prices of the Honda Motor options in Example 20.2 if the volatility of the dollar/yen exchange rate increases to 25% and other parameters remain unchanged as in Example 20.2.

20.12. Find the chi of the two options in Exercise 20.11.

20.13. Why a quanto option can be regarded with the negative corelation coefficient adjustment compared to its corresponding foreign equity option?

20.14. What is the most important difference in pricing a quanto option and its corresponding foreign equity option?

20.15.* Prove Corollary 20.1.

20.16.* Prove Corollary 20.2.

Chapter 21

RAINBOW OPTIONS

21.1. INTRODUCTION

In Chapter 14, we studied options paying the best or worst of two assets and cash. Actually options can be written on the maximum or minimum of two or several assets. These options are often called complex options in academic literature. Many problems in finance can be converted to options written on the maximum or minimum of two assets or instruments. Stulz (1982) provided closed-form solutions for such options in a Black-Scholes environment and applied the pricing formulas to several problems. Johnson (1987) generalized the results of options on the maximum or minimum of two assets to options of $n \geq 3$ underlying assets and provided formulas for these options in terms of multinormal cumulative functions. Boyle and Tse (1990) provided a detailed discussion pertaining the pricing of options on the maximum or minimum of n assets. Rubinstein (1991) discussed options written on the maximum or minimum of two instruments and called them two-color rainbow options. These options have been known as rainbow options in the professional world ever since.

The pricing formulas of options on the maximum or minimum of two risky assets can be used to value many financial products such as foreign currency bonds and default-free option bonds. They can also be used to solve problems in corporate finance. Interested readers may go to Stulz (1982) for more specific analysis as to how the pricing formulas can be used to solve specific problems. In this chapter, we will first introduce rainbow options and then discuss how to price these options and apply them in practice.

21.2. TWO-COLOR RAINBOW OPTIONS

Considering the maximum or minimum of two asset prices or indices as the price or index for an imaginary instrument, a two-color rainbow option written on the maximum or minimum of these two assets can be considered as written on this imaginary instrument. The payoff of a two-color rainbow option on the maximum (PRBX) of two assets can be formally given

$$PTCRBMX = \max\{\omega \max[I_1(\tau), I_2(\tau)] - \omega K, 0\}, \tag{21.1}$$

where $I_1(\tau)$ and $I_2(\tau)$ are the prices of the two assets involved at maturity, K is the strike price of the option, and ω is a binary operator (1 for a call option and -1 for a put option).

Similarly, the payoff of a two-color rainbow option on the minimum (PRBN) of two assets can be given

$$PRBN = \max\{\omega \min\ [I_1(\tau), I_2(\tau] - \omega K, 0\}, \tag{21.2}$$

where min(.,.) stands for the function which gives the minimum of two numbers and all the parameters are the same as in (21.1).

It is worth printing out that a put option on the minimum of two can be duplicated with $k - \min[I_1(\tau), I_2(\tau), k] = \max\{k - \min[I_1(\tau), I_2(\tau)]\}$.

21.3. PRICING TWO-COLOR RAINBOW OPTIONS

Assume that the two underlying asset prices $I_1(\tau)$ and $I_2(\tau)$ follow the same stochastic process in (IV1) and the two asset returns are correlated with the correlation coefficient ρ. With these distribution assumptions, we can price a two-color rainbow option using the bivariate normal density function in (IV4) and (IV5). Following a similar procedure as in Stulz (1982), we can obtain the following pricing formula for a call option on the minimum (MNC) of two assets:

$$\begin{aligned} MNC = {} & I_1 e^{-g_1\tau} N_2(d_{11}, d_{12}, \rho_1) + I_2 e^{-g_2\tau} N_2(d_{22}, d_{21}, \rho_2) \\ & - K e^{-r\tau} N_2(d_1, d_2, \rho), \end{aligned} \tag{21.3}$$

where

$$\begin{aligned} d_1 &= \left[\ln\left(\frac{I_1}{K}\right) + \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] / \left(\sigma_1\sqrt{\tau}\right), \\ d_{11} &= d_1 + \sigma_1\sqrt{\tau}, \\ d_2 &= \left[\ln\left(\frac{I_2}{K}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau\right] / (\sigma_2\sqrt{\tau}), \\ d_{22} &= d_2 + \sigma_2\sqrt{\tau}, \\ d_{12} &= \left[\ln\left(\frac{I_2}{I_1}\right) + \left(g_1 - g_2 - \frac{1}{2}\sigma_a^2\right)\tau\right] / (\sigma_a\sqrt{\tau}), \end{aligned}$$

$$d_{21} = \left[\ln\left(\frac{I_1}{I_2}\right) + \left(g_2 - g_1 - \frac{1}{2}\sigma_a^2\right)\tau\right] / (\sigma_a\sqrt{\tau}),$$

$$\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sigma_a}, \rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\sigma - a},$$

$$\sigma_a = \sqrt{\sigma_a^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

g_1 and g_2 represent the payout rates of the two underlying assets, and $N_2(a, b, \rho)$ is the bivariate normal cumulative function with the upper bounds a and b and the correlation coefficient ρ.

We can check that the pricing formula in (21.3) becomes exactly the same as that in Stulz (1982) when the payout rates of the two assets $g_1 = g_2 = 0$.

Example 21.1. Find the price for a call option written on the minimum of two risky assets to expire in eight months with the strike price \$102, given the spot prices of the two assets \$100 and \$95, the payout rates of the two assets zero, the interest rate 8%, the correlation coefficient between the two assets 75%, the volatilities of the two assets 15% and 20%, respectively.

Substituting $I_1 = \$100$, $I_2 = \$95$, $K = \$102$, $\tau = 8/12 = 0.667$, $g_1 = g_2 = 0$, $r = 0.08$, $\rho = 0.75$, $\sigma_1 = 0.15$, $\sigma_2 = 0.20$ into (21.3) yields

$$d_1 = \left[\ln\left(\frac{100}{102}\right) + \left(0.08 - 0 - \frac{1}{2} \times 0.15^2\right)0.667\right] / (0.15\sqrt{0.667}) = 0.2125,$$

$$d_{11} = 0.793 + 0.15\sqrt{0.667} = 0.3350,$$

$$d_2 = \left[\ln\left(\frac{95}{102}\right) + \left(0.08 - 0 - \frac{1}{2} \times 0.20^2\right)0.667\right] / (0.20\sqrt{0.667}) = -0.1904,$$

$$d_{22} = -0.206 + 0.20\sqrt{0.667} = -0.0271,$$

$$\sigma_a = \sqrt{0.15^2 \times 0.75 \times 0.15 \times 0.20 + 0.20^2} = 0.1323,$$

$$d_{12} = \left[\ln\left(\frac{95}{100}\right) + \left(0 - 0 - \frac{1}{2} \times 0.1323^2\right)0.667\right] / (0.1323\sqrt{0.667})$$
$$= -0.5557,$$

$$d_{21} = \left[\ln\left(\frac{100}{95}\right) + (0 - 0 - \frac{1}{2} \times 0.1323^2)0.667\right] / (0.1323\sqrt{0.667}) = 0.3944,$$

$$\rho_1 = \frac{0.75 \times 0.20 - 0.15}{0.1323} = 0,$$

$$\rho_2 = \frac{0.75 \times 0.15 - 0.20}{0.1323} = -0.6614,$$

$$\begin{aligned} MNC &= 100N_2(0.3350, -0.5557, 0) + 95N_2(0.2125, -0.1904, 0.75) \\ &\quad - 102e^{-0.08\times 0.667}N_2(0.2125, -0.1904, 0.75) \\ &= \$2.585\,. \end{aligned}$$

Call options written on the maximum of two assets can be similarly derived following the similar procedures as in deriving (21.3). Using the bivariate density function in (IV4) and (IV5), we can obtain the price of a call option written on the maximum (MXC) of two assets as follows:

$$\begin{aligned} MXC &= I_1 e^{-g_1\tau}N_2(d_{11}, -d_{12}, -\rho_1) + I_2 e^{-g_2\tau}N_2(d_{22}, -d_{21}, -\rho_2) \\ &\quad - Ke^{-r\tau}[1 - N_2(-d_1, -d_2, \rho)]\,, \end{aligned} \tag{21.4}$$

where all parameters are the same as in (21.3).

The following mathematical identities

$$N_2(a, b, \rho) + N_2(a, -b, -\rho) = \mathrm{N}(a) \tag{21.5a}$$

and

$$N_2(a, b, \rho) + N_2(-a, b, -\rho) = \mathrm{N}(b)\,, \tag{21.5b}$$

always hold for any real numbers a and b and any correlation coefficient $|\rho| < 1$.[1] Using the two identities (21.5a) and (21.5b), we find that the prices of the call options written on the maximum and minimum of two assets satisfy the following relationship:[2]

$$MXC + MNC = C(I_1, K, r, g_1, \tau, \sigma_1) + C(I_2, K, r, g_2, \tau, \sigma_2)\,, \tag{21.6}$$

where $C(I_1, K, r, g_1, \tau, \sigma_1)$ and $C(I_2, K, r, g_2, \tau, \sigma_2)$ are the Black-Scholes call option prices written on the first and the second assets, respectively. Equation (21.6) was first given by Stulz (1982). Stulz showed that in order for (21.6) to hold, we simply need to show that the summation of the payoffs of the two call options on the maximum and minimum of the two underlying assets is the same as that of the two separate vanilla options on the

[1] See Appendix for a proof.

[2] The first two terms on the right-hand side of (21.6) are straightforward using (21.5a) and (21.5b). The last two terms need a few extra steps of derivations using (21.5a) and (21.5b):

$$\begin{aligned} &N_2(d_1, d_2, \rho)] + 1 - N_2(-d_1, -d_2, \rho)] \\ &\quad = 1 - \{N_2(-d_1, -d_2, \rho) + N_2(-d_1, d_2, \rho)\} + \{N_2(-d_1, d_2, -\rho) + N_2(d_1, d_2, \rho)\} \\ &\quad = 1 - N(-d_1) + N(-d_1) + [1 - N(d_2)] = N(-d_1) + N(-d_2)\,. \end{aligned}$$

underlying assets. We need to consider 24 different cases, each representing one different order of the relative magnitudes of the two strike prices and the two forward asset prices at maturity. However, the number of cases can be reduced to 12 because of the symmetry argument. For example, if K_1 is the maximum of $I_1(\tau), I_2(\tau), K_1$, and K_2, the payoff of the rainbow option on the first asset is zero and MNC is also zero, thus the payoff of the out-performance option on the maximum and the single call option on the second asset become the same, therefore the relationship holds. By symmetry, it holds if K_1 is the maximum of $I_1(\tau)$, $I_2(\tau)$, K_1, and K_2. Similarly, we can show the equivalence of the payoffs in all other cases.

With the identity in (21.6), we can also obtain a pricing formula for a call option on the maximum (MXC) of two assets:

$$MXC = C(I_1, K, r, g_1, \tau, \sigma_1) + C(I_2, K, r, g_2, \tau, \sigma_2) - MNC\,, \quad (21.7)$$

where MNC is given in (21.3) and $C(I_1, K, r, \tau, \sigma_1)$ and $C(I_2, K, r, \tau, \sigma_2)$ are given in (2.12).

Example 21.2. Find the price of the call option written on the maximum of the two risky assets in Example 21.1.

We can find the two call option prices written on the two underlying assets:

$$C(I_1, K, r, g_1, \tau, \sigma_1) = C(100, 102, 0.08, 0, 0.667, 0.15) = \$6.630\,,$$

and

$$C(I_2, K, r, g_2, \tau, \sigma_2) = C(95, 102, 0.08, 0, 0.667, 0.20) = \$5.423\,.$$

Substituting the call option prices above and the price of the call option written on the minimum of the two assets in Example 21.1 into (21.7) yields

$$MXC = 6.663 + 5.423 - 2.585 = \$9.663\,.$$

21.4. SENSITIVITIES

Stulz (1982) obtained the Greeks (sensitivity analysis), for most parameters in a model. Rubinstein (1991) simplified Stulz's delta expressions. We will further simplify Stulz's chi expression using some identities we have used to simplify sensitivity expressions for other exotic options in this book.

Stulz obtained the chi or the sensitivity of the call option price with respect to the correlation coefficient as follows

$$\frac{\partial MNC}{\partial \rho} = \frac{\sigma_1\sigma_2}{\sigma_a}\left[I_1 N\left(\frac{d_{11}-\rho_1 d_{12}}{\sqrt{1-\rho_1^2}}\right)f(d_{12})\left(\sqrt{\tau}+\frac{d_{12}}{\sigma_a}\right)\right.$$
$$\left. + I_2 N\left(\frac{d_{22}-\rho_2 d_{22}}{\sqrt{1-\rho_2^2}}\right)f(d_{21})\left(\sqrt{\tau}+\frac{d_{21}}{\sigma_a}\right)\right]. \qquad (21.8)$$

Stulz showed that the chi in (21.8) is always positive for $-1 \leq \rho \leq 1$. He further showed with examples that the price of the call option on the minimum of two risky assets is indeed worthless because $d_1 + d_2 \leq 0$.[3]

We will show in the rest of this section that whereas Stulz's general statement is partly true, that it is possible for call options on the minimum of two risky assets to be worthless, they may have positive values even if $d_1+d_2 \leq 0$. Our argument is based on the intuition that whereas $d_1+d_2 \leq 0$ guarantees the third bivariate term of (21.3) to be zero, it does not always imply that the sums of the two pairs of upper bounds in the first two bivariate terms ($d_{11} + d_{12}$ and $d_{22} + d_{21}$) are smaller than or equal to zero. Whereas the two sums $d_{11}+d_{12}$ and $d_{22}+d_{21}$ are indeed not always greater than zero if $d_1 + d_2 \leq 0$ given $\sigma_1 = \sigma_2$ and $\rho = -1$ as in his examples, one of the two sums can be positive if $d_1 + d_2 \leq 0$ given $\rho = -1$ and $\sigma_1 \neq \sigma_2$.

For example, given the two underlying spot prices $I_1 = I_2 = \$95$, the payout rates $g_1 = g_2 = 0$, the volatilities of the two assets $\sigma_1 = 15\%$ and $\sigma_2 = 5\%$, the strike price $K = \$100$, the interest rate $r = 5\%$, the time to maturity $\tau = 2$ years, we can find $d_1 = -0.0591$ and $d_2 = -0.0359$, and $d_1 + d_2 = -0.0951 < 0$ is satisfied. However, $d_{11} = d_1 + \sigma_1\sqrt{\tau} = 0.1530$, $d_{22} = d_2 + \sigma_2\sqrt{\tau} = 0.0347$, $d_{12} = d_{21} = -0.1414$, thus $d_{11} + d_{12} = 0.1530 - 0.1414 = 0.0116 > 0$ and $d_{22} + d_{21} = 0.0347 - 0.1414 = -0.1067 < 0$. The intermediate volatility $\sigma_a = \sigma_1 + \sigma_2$ and the correlation coefficient $\rho_1 = -1$ when $\rho = -1$. Therefore the second and the third terms of (21.3) are indeed zero, yet the first term of (21.3) is

$$I_1 N_2(d_{11}, d_{12}, \rho_1) = 95 N_2(0.1530, -0.1414, -1)\,.$$

Using the identity in (11.43), we can find the value of the bivariate cumulative function above:

$$95[N(0.1530) - N(-0.1414)] = 95(0.5608 - 0.4438) = \$11.115$$

which is greater than zero!

[3]The value of the bivariate normal cumulative function is always zero as shown in (11.42) if the sum of the two arguments is zero or negative regardless of the correlation coefficient.

21.5. OPTIONS ON THE BEST OR WORST OF SEVERAL ASSETS

Johnson (1987) generalized the results of options on the maximum and minimum of two assets to on the maximum and minimum of $n \geq 3$ underlying assets and provided closed-form formulas for these options in terms of cumulative functions of multinormal distributions. The payoff of an option on the maximum of $n \geq 2$ underlying assets (PMXN) is given as follows:

$$PMXN = \max\{\omega \max[I_1(\tau), I_2(\tau), I_3(\tau), \ldots, I_n(\tau)] - \omega K, 0\}, \tag{21.9}$$

where $I_i(\tau)$ represents the price of the ith $(i = 1, 2, \ldots, n)$ asset at the option maturity and it is assumed to follow the stochastic process given in (IV1), K is the strike price of the option, and ω is a binary operator (1 for a call option and -1 for a put option).

All the asset returns are correlated among one another. More specifically, the ith asset is assumed to be correlated with the jth asset with the correlation coefficient ρ_{ij}, and the covariance between the returns of the ith and jth assets is thus given as

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_i^2 . \tag{21.10}$$

Similarly, the payoff of an option on the minimum of $n \geq 2$ underlying assets (PMNN) is given as

$$PMNN = \max\{\omega \min\,[I_1(\tau), I_2(\tau), I_e(\tau), \ldots, I_n(\tau)] - \omega K, 0\}, \tag{21.11}$$

where min (.,.) stands for the function giving the minimum of n numbers and all other parameters are the same as in (21.9).

With the above definitions and assumptions, Johnson (1987) obtained a pricing formula for a call option written on the minimum of n risky assets:

$$\begin{aligned} C_{\min} = {} & I_1 N_n[d_1(I_1, K, \sigma_1^2), -d_1'(I_1, I_2, \sigma_{12}^2), \ldots, \\ & - d_1'(I_1, I_n, \sigma_{1n}^2), -\rho_{112}, -\rho_{113}, -\rho_{123}, \ldots] \\ & + I_2 N_n[d_1(I_2, K, \sigma_2^2), -d_1'(I_2, I_1, \sigma_{12}^2), \ldots, \\ & - d_1'(I_2, I_n, \sigma_{2n}^2), -\rho_{212}, -\rho_{223}, -\rho_{213} \ldots] \\ & + \ldots \end{aligned}$$

$$+ I_n N[d_1(I_n, K, \sigma_n^2), -d_1'(I_n, I_1, \sigma_{1n}^2), \ldots,$$
$$- d_1'(I_{n-1}, I_n, \sigma_n^2), -\rho_{n1n}, -\rho_{n2n}, \ldots, -\rho_{n12}, \ldots]$$
$$- KN_n[d_2(I_1, K, \sigma_1^2), d_2(I_2, K, \sigma_{12}^2), \ldots,$$
$$+ d_2(I_n, K, \sigma_{1n}^2), \rho_{12}, \rho_{13}, \ldots], \qquad (21.12)$$

where

$$d_1(a, K, c) = \frac{\ln(a/K) + c\tau/2}{c\sqrt{\tau}}, d_1'(a, b, c) = \frac{\ln(a/b) + c\tau/2}{c\sqrt{\tau}}, \qquad (21.13)$$

$$d_2(a, b, c) = \frac{\ln(a/K) + (r - c/2)\tau}{c\sqrt{\tau}}, \qquad (21.14)$$

$$\rho_{ijk} = \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}}. \qquad (21.15)$$

The corresponding formula for a call option on the maximum of n risky assets is expressed similarly and we will skip it here.

21.6. SUMMARY AND CONCLUSIONS

Rainbow options are options on the maximum or minimum of two risky assets or indices. They can be used to value many financial products such as foreign currency bonds, default-free option bonds, and others. They have become popular in the OTC marketplace. We priced rainbow options using the joint density function given in (IV4) and (IV5). As a matter of fact, the pricing formulas of rainbow options can be obtained using the pricing formulas of exchange options in Chapter 13.

QUESTIONS AND EXERCISES

21.1. What are rainbow options?

21.2. Does the price of a call option on the minimum of two risky assets increase or decrease with the correlation coefficient between them? Why?

21.3. Does the price of a call option on the maximum of two risky assets increase or decrease with the correlation coefficient between them? Why?

21.4. Is it possible that a call option is worthless even when the two risky assets have positive values?

21.5. Is it true that a call option is worthless when $d_1 + d_2 \leq 0$? Why?

21.6. Show the identity in (21.6).

21.7. Find the price of a call option written on the minimum of two risky assets to expire in six months with the strike price \$105, given the spot prices of the two assets \$102 and \$98, the payout rates of the two assets 1% and 2%, the interest rate 8%, the correlation coefficient between the two assets 75%, the volatilities of the two assets 15% and 20%, respectively.

21.8. Find the chi of the call option in Exercise 21.7.

21.9. Find the price of the corresponding call option on the maximum of the two underlying assets in Exercise 21.7.

21.10. Find the chi of the call option in Exercise 21.9.

21.11. Find the price of the call option in Exercise 21.7 if the correlation coefficient between the two assets is changed to 25% and other parameters remain unchanged.

21.12. Find the price of the corresponding call option written on the maximum of the two risky assets in Exercise 21.11.

APPENDIX

Using the bivariate density function given in (IV4), we can obtain the following

$$N_2(a,b,\rho) = \int_{-\infty}^{a} f(u)N\left[(b-\rho u)/\sqrt{1-\rho^2}\right]du \tag{A21.1}$$

and

$$\begin{aligned} N_2(a,-b,-\rho) &= \int_{-\infty}^{a} f(u)N\left\{[-b-(-\rho)u]/\sqrt{1-(-\rho)^2}\right\}du \\ &= \int_{-\infty}^{a} f(u)N\left[-(b-\rho u)/\sqrt{1-\rho^2}\right]du\,. \end{aligned} \tag{A21.2}$$

Thus,

$$N_2(a,b,\rho)+N_2(a,-b,-\rho) = \int_{-\infty}^{a} f(u)\left\{N\left[(b-\rho u)/\sqrt{1-\rho^2}\right]\right\}du\,. \tag{A21.3}$$

Using the identity $N(z)+N(-z)=1$ for any real number z, we simplify (A21.3) into

$$\begin{aligned} N_2(a,b,\rho)+N_2(a,-b,-\rho) &= \int_{-\infty}^{a} f(u)(1)du \\ &= \int_{-\infty}^{a} f(u)du = N(a)\,, \end{aligned} \tag{A21.4}$$

which is (21.5a). Identity (21.5b) can be similarly proven by changing the order of integration and using (IV5) instead of (IV4).

Chapter 22

SPREAD OPTIONS

22.1. INTRODUCTION

A spread option is an option written on the difference between two indices, prices, or rates. For example, the spread between refined and crude oil prices fluctuates due to international economic and financial information. Options written on this spread can be used by oil refiners to hedge the risks of their gross profits. Other popular spread options include yield curve options which are written on yield spreads. Such a spread is normally a long-term treasury interest rate minus a short-term rate. The popular yield curve options are written on 2-year to 10-year, 2-year to 30-year, and 10-year to 30-year spreads. Spread options are among the very few kinds of exotic options trading in exchanges. The New York Mercantile Exchange launched spread options written on the spread between heating oil or unleaded gasoline and crude oil futures on October 7, 1994. Spread options are also among the few most popular exotic options trading in the OTC marketplace.

In the early stage when spread options were first used, the spread was regarded as an imaginary single asset price and the Black-Scholes formula was used to approximate the spread option price. This method is the so-called one-factor model. Garman (1992) pointed out the limitations and problems of the single-factor model and discussed how to price spread options with a two-factor model. Falloon (1992) discussed and provided examples of spread options and their uses. Ravindran (1993) attempted to price spread options within a two-factor model using statistical procedures and numerical analysis. Whereas the numerical method may be valid, it is rather inconvenient to use because additional mathematical procedures such as the Gaussian quadrature have to be used to find the expected payoffs of the spread options. What is more, integration ranges have to be partitioned first and then suitable transformations have to be used in order to use the Gaussian quadrature. Ravindran's attempt may be feasible, yet the method is not efficient within a Black-Scholes environment. There has been a continuing interest in studying spread options recently. Pearson (1995) discussed an efficient approach to price standard spread options and approximated their prices. He illustrated a replicating portfolio for standard spread options. Brooks (1995) priced interest rate spread options using a lattice approach.

We will analyze spread options in a two-factor model in a Black-Scholes environment. The purpose of this chapter is to find closed-form solutions for simple spread options written on spreads between two instruments, and then extend the results to multiple spread options written on differences of more than two instruments. We will concentrate on European-style options in this chapter as they are easier to work with and their solutions can be used as control variates for their corresponding American options. We will confine our analysis to a Black-Scholes environment for the purpose of transparency and easy comparison.

22.2. SIMPLE SPREAD OPTIONS

The payoff of a European option on the spread of two instruments (ESD) can be expressed as follows:

$$ESD = \max[a\omega I_1(\tau) + b\omega I_2(\tau) - \omega K,\ 0]\,, \tag{22.1}$$

where $a > 0$ and $b < 0$ are two real numbers, I_1 and I_2 stand for the two instruments in question, K is the strike price of the option, max $(.,.)$ is a function that gives the larger of two numbers, and ω is the option binary operator (1 for a call option and -1 for a put option).

For a standard spread option, we can simply set $a = 1$ and $b = -1$. If we specify $a = 1$, $b = -1$, and $K = 0$, the payoff of the spread option in (22.1) becomes precisely that of an exchange option given in (13.1). Therefore, exchange options studied in Chapter 13 can be considered as a special case of spread options.

In a typical Black-Scholes environment, all the underlying asset prices are assumed to follow a lognormal process. We assume that the two instruments $I_1(\tau)$ and $I_2(\tau)$ also follow the standard log-normal process given in (IV1).

22.3. ONE-FACTOR VS TWO-FACTOR MODELS IN PRICING SIMPLE SPREAD OPTIONS

In the early stage, spread options were priced as if the spread were one individual underlying instrument, and the Black-Scholes formula was used. The volatility of the spread could be estimated using historical spread data.

Since the spread is considered as one imaginary asset price, this method is called the one-factor model in pricing spread options. Whereas the one-factor model is convenient to use, it has a few limitations. The first is that the correlation coefficient between the two assets involved does not play any explicit role in the pricing formula, compared to the important role it plays in all other correlation options shown in previous chapters. The second is that the sensitivities of the spread option price with respect to the two underlying asset volatilities cannot be found in the one-factor model because the spread option price in the one-factor model is a function of the spot spread and the volatility of the spread. Another important limitation is that it implicitly assumes that the spread cannot be negative because the underlying asset price must be non-negative in a Black-Scholes model. Whereas the non-negative assumption may not be of much concern in some applications such as spread options on the spread of refined and crude oil prices, it can be problematic in other applications.

Due to the above limitations, the one-factor model has lost its popularity these days and more attention has been paid to two-factor models. All two-factor models treat the two assets individually and also consider the correlation between them explicitly as what we have done so far in Part IV. We will focus on a two-factor model within a Black-Scholes model in this chapter.

22.4. TWO-FACTOR MODEL TO PRICE SIMPLE SPREAD OPTIONS

Figure 22.1 shows the integration domain of a standard spread option with $a = 1$ and $b = -1$. The shadowed area represents the integration domain. It is convenient to integrate $I_1(\tau)$ first and then $I_2(\tau)$ because $I_2(\tau)$ is from zero to infinity. Using the density function given in (IV4) and (IV5), we can obtain the expected payoff of the spread option in (22.1) by integrating I_1 first:

$$E(ESD) = \omega a I_1 e^{\mu_1 \tau} A_1 + \omega b I_2 e^{\mu_2 \tau} A_2 - \omega K A_3 \,, \tag{22.2}$$

where

$$A_1 = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d + \rho v + \sigma_1\sqrt{\tau} + \phi(v + \rho\sigma_1\sqrt{\tau})}{\sqrt{1-\rho^2}}\right] dv \,,$$

$$A_2 = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d + \rho v + \rho\sigma_2\sqrt{\tau} + \phi(v + \sigma_2\sqrt{\tau})}{\sqrt{1-\rho^2}}\right] dv \,,$$

$$A_3 = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d + \rho v + \phi(v)}{\sqrt{1-\rho^2}}\right] dv \,,$$

$$d = d(aI_1, K, \sigma_1, g_1, \tau, r) = \left[\ln\left(\frac{aI_1}{K}\right) + \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] / (\sigma_1\sqrt{\tau}),$$

and

$$\phi(v) = -\frac{1}{\sigma_1\sqrt{\tau}} \ln\left\{1 - \frac{bI_2}{K} \exp\left[\left(\mu_2 - \frac{1}{2}\nu 2^2\right)\tau + v\sigma_2\sqrt{\tau}\right]\right\}.$$

Although the result in (22.2) is not in closed-form in a strict sense as the Black-Scholes formula, it is a significant improvement in the sense that it converts a bivariate problem into a univariate one. Whereas the three parameters A_1, A_2, and A_3 look complicated, they can be calculated numerically with any computer as both the density and cumulative functions of the standard normal distribution can be much more easily computed than the bivariate normal distribution.

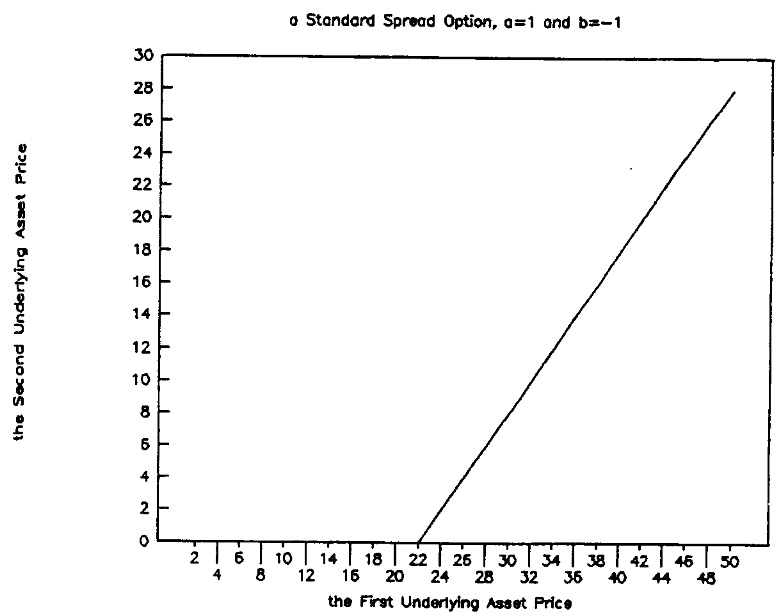

Fig. 22.1. The integration domain of a standard simple spread call option.

Arbitrage argument permits us to use the risk-neutral valuation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate. We can obtain the call option price by substituting $\mu_i = r - g_i$ (g_1 and g_2 are the payout rates of the two underlying assets) and discounting the expected payoff in (22.2) by the risk-free rate r,

$$C_{\text{sp}} = \omega aI_1 e^{-g_1\tau} A_1 + \omega bI_2 e^{-g_2\tau} A_2 - \omega K e^{-r\tau} A_3, \qquad (22.3)$$

where A_1, A_2, and A_3 are the same as in (22.2) with

$$d(aI_1, K, \sigma_1, g_1, \tau, r) = \left[\ln\left(\frac{aI_1}{K}\right) + \left(r - g_1 - \frac{1_2}{2^1}\right)\tau\right] / (\sigma_1\sqrt{\tau}),$$

and

$$\phi(v) = -\frac{1}{\sigma_1\sqrt{\tau}} \ln\left\{1 - \frac{bI_2}{K}\exp\left[\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau + v\sigma_2\sqrt{\tau}\right]\right\}.$$

Using the identity $N(x)+N(-x) = 1$ for any real number x and the two pricing formulas in (22.3) with $\omega = 1$ and $\omega = -1$, we can have the put-call parity for simple spread options

$$P_{\text{sp}} = C_{\text{sp}} - ae^{-g_1\tau}I_1 - be^{-g_2\tau}I_2 + Ke^{-r\tau}, \qquad (22.4)$$

with which we can calculate the price of the corresponding put spread option much more easily from the call option price.

Example 22.1. Find the prices of the standard spread options on the spread of the gasoline price over the crude oil price to expire in half a year, given the strike price \$2, the spot gasoline price \$24.55 per barrel, the spot crude oil price \$18.76 per barrel, the interest rate 5.8%, the payout rates of the two assets 5.8%, the correlation coefficient between the gasoline and crude oil prices 82%, and the volatilities of gasoline and crude oil prices 24% and 21%, respectively.

Substituting $\omega = 1$, $a = 1$, $b = -1$, $I_1 = 24.55$, $I_2 = \$18.76$, $g_1 = g_2 = r = 0.058$, $K = \$2$, $\rho = 0.82$, $\tau = 0.50$, $\sigma_1 = 0.24$, and $\sigma_2 = 0.21$ into (22.3) yields

$$d(aI_1, K, \sigma_1, g_1, \tau, r) = \left[\ln\left(\frac{24.55}{2}\right) + \left(0.058 - 0.058 - \frac{1}{2} \times 0.24^2\right)0.5\right] / (0.24\sqrt{0.5}) = 14.691.$$

Integrating the functions in (22.2), we can find the three coefficients as follows:

$$A_1 = 0.9696, \ A_2 = 0.9623, \ \text{and} \ A_3 = 0.9616.$$

Substituting these three parameters into (22.3) yields the call option price

$$C(a = 1, b = -1) = e^{-0.058\times0.5}(24.55 \times 0.9696 - 18.76 \times 0.9623 - 2 \times 0.9616) = \$3.718,$$

and the put option price can be obtained from the put-call parity given in (22.4):

$$P(a = 1, b = -1) = 3.718 - e^{-0.058\times 0.5}(24.55 - 18.76 - 2) = \$0.036\,.$$

22.5. APPROXIMATING THE PRICING FORMULA FOR SIMPLE SPREAD OPTIONS

An exact closed-form solution in terms of the univariate normal cumulative functions for simple spread options is very unlikely because $\phi(v)$ is a highly nonlinear function of the integrating variable v. However, if we examine the sensitivity of $\phi(v)$ with respect to v, we could easily find that $\phi(v)$ is nearly a linear function of v within the reasonable range between -5 and 5. Figure 22.2 presents the curvature of $\phi(v)$ within the range from -5 to 5 for $\sigma_1 = 20\%$, $\sigma_2 = 15\%$, given $a = 1$, $b = -1$, $I_1(0) = I_2(0) = \$100$, $K = \$1.00$, $\tau =$ one year, the payout rates $g_1 = g_2 = 0$, and $r = 10\%$. Outside the range between -5 and 5, the standard normal density function $f(v)$ is very close to zero and thus could be neglected without affecting accuracy.

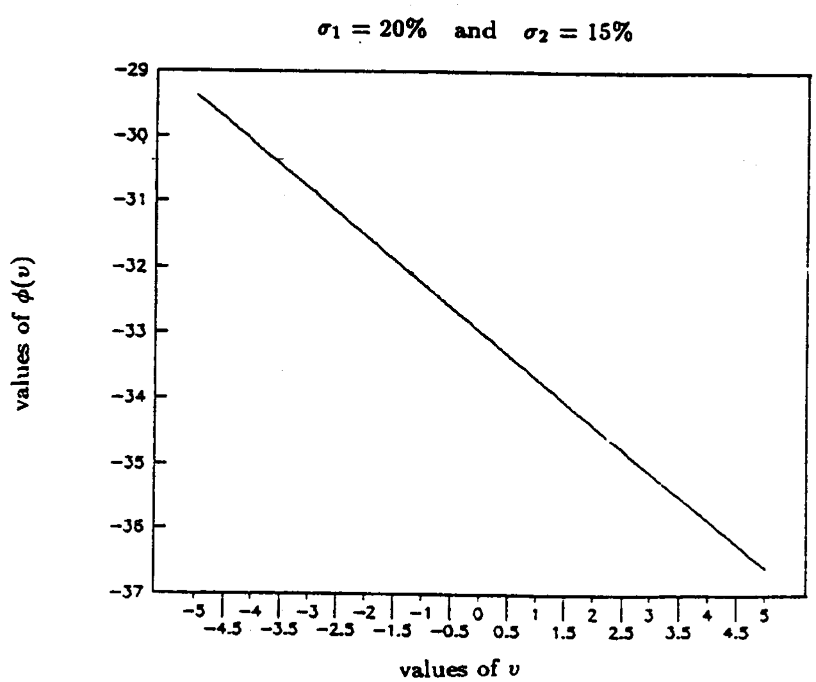

Fig. 22.2. The linearity of the function $\phi(v)$.

The approximation of the three parameters A_1, A_2, and A_3 is the same as approximating the function $\phi(v)$ in (22.3). We have the following approximation results for the function $\phi(v)$.

Proposition 22.1. If $d(-bI_2, K, \sigma_2, g_2, \tau, r) > 4$, then the function $\phi(v)$ can be approximated as follows

$$\phi(v) = \alpha - \beta v\,, \tag{22.5}$$

where

$$\alpha = -\beta d(-bI_2, K, \sigma_2, g_2, \tau, r) - \frac{1}{\sigma_1\sqrt{\tau}}\sum_{l=1}^{\infty} i!(-1)^{i-1}l^i\,,$$

$$l = \frac{1}{-bI_2}\exp\left[-\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau + 5\sigma_2\sqrt{\tau}\right],$$

$$\beta = \frac{\sigma_2}{\sigma_1}\,,$$

and

$$d(-bI_2, K, \sigma_2, g_2, \tau, r) = \left[\ln\left(\frac{-bI_2}{K}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau\right]/(\sigma_2\sqrt{\tau})$$

is the same argument as in the extended Black-Scholes formula with the spot and strike prices, volatility, payout and interest rates, and time to maturity $-bI_2, K, \sigma_2, g_2, r$, and τ, respectively.

Proof. The following identity is always true

$$\ln\left[1 + \left(\frac{1}{l}\right)\right] = \ln\left(\frac{1}{l}\right) + \ln(1 + l), \text{ for any } |l| < 1\,.$$

When $d(-bI_2, K, \sigma_2, g_2, \tau, r) > 4$, we can show that $l < 1$. The approximation results in (22.5) is immediately from Taylor's series expansion around zero. □

The linearization process is very important here as it determines the accuracy level of the approximation. Better linearization results are possible, yet they are not the focus of this chapter and we choose not to illustrate these linearization processes here.

With the linear approximation of $\phi(v)$ in (22.5) and the mathematical identity in (13.6), the three parameters in (22.2) can be simplified as follows:

$$A_1 = N\{\omega[\gamma + (1 - \beta\rho)w\sigma_1\sqrt{\tau}]\}\,, \tag{22.6a}$$

$$A_2 = N\{\omega[\gamma + (\rho - \beta)w\sigma_2\sqrt{\tau}]\}\,, \tag{22.6b}$$

and

$$A_3 = N(\omega\gamma)\,, \tag{22.6c}$$

where

$$w = 1/\sqrt{1-\rho^2+(\beta-\rho)^2} = 1/\sqrt{1-2\rho\beta+\beta^2}\,,$$

and

$$\gamma = [\alpha + d(aI_1, K, \sigma_1, g_1, \tau, r)]w\,.$$

The call option price given in (22.3) can thus be approximated using the approximations in (22.6a), (22.6b), and (22.6c):

$$\begin{aligned} C_{sp}(\omega) \simeq\ & \omega a e^{-g_1\tau} I_1 N\{\omega[\gamma + (1-\beta\rho)w\sigma_1\sqrt{\tau}]\} \\ & + \omega b e^{-g_2\tau} I_2 N\{\omega[\gamma + (\rho-\beta)w\sigma_2\sqrt{\tau}]\} \\ & - \omega K e^{-r\tau} N(\omega\gamma)\,. \end{aligned} \tag{22.7}$$

We can check that the Black-Scholes formula is a special case of (22.5) when $a = 1$ and $b = \rho = 0$, $\alpha = \beta = 0$ [because $\phi(v) = 0$], thus $w = 1$ and $\gamma = d$ and the second term in (22.7) disappears.

Formula (22.7) is of the same complexity as the Black–Scholes formula for options on a single underlying asset or index, with the exception of more parameters because two correlated underlying assets or indices are involved. The trick in obtaining the formula given in (22.7) is the order of integration. Cumulative functions of bivariate normal distributions have to be used to express the option prices if the integration order is changed.

Example 22.2. Find the prices of the standard spread options in Example 22.1 using the approximation in (22.27).

Substituting $\omega = 1$, $a = 1$, $b = -1$, $I_1 = 24.55$, $I_2 = \$18.76$, $g_1 = g_2 = r = 0.058$, $K = \$5$, $\rho = 0.82$, $\sigma_1 = 0.24$, and $\sigma_2 = 0.21$ into (22.3) yields

$$\begin{aligned} d(aI_1, K, \sigma_1, g_1, \tau, r) &= \left[\ln\left(\frac{24.55}{5}\right) + \left(0.058 - 0.058 \right.\right. \\ &\quad \left.\left. -\frac{1}{2}\times 0.24^2\right)0.5\right]/(0.24\sqrt{0.5}) = 14.691\,, \\ d(-bI_2, K, \sigma_2, g_2, \tau, r) &= \left[\ln\left(\frac{18.76}{5}\right) + \left(0.057 - 0.057 \right.\right. \\ &\quad \left.\left. -\frac{1}{2}0.21^2\right)0.50\right]/(0.21\sqrt{0.50}) = 15.001\,, \end{aligned}$$

$$\beta = \frac{0.21}{0.24} = 0.875\,,$$

$$l = \frac{1}{18.76}\exp\left[-\left(0.058 - 0.058 - \frac{1}{2}\times 0.21^2\right)0.50 \right.$$

$$\left. +5\times 0.21\sqrt{0.50}\right] = 0.1132\,,$$

$$\alpha = -0.875\times 15.001 - \frac{1}{0.24\sqrt{0.50}}(0.1132 - 0.1132^2$$

$$+2\times 0.1132^3 - 6\times 0.1132^4) = -13.729\,,$$

$$w = 1/\sqrt{1 - 0.82^2 + (0.875 - 0.82)^2} = 1.7391\,,$$

$$\gamma = [\alpha + d(aI_1, K, \sigma_1, g_1, \tau, r)]w$$

$$= (-13.729 + 14.691)1.7392 = 1.6732\,,$$

$$\gamma + (1 - \beta\rho)w\sigma_1\sqrt{\tau} = 1.6732 + (1 - 0.875\times 0.82)1.7391\times 0.24\sqrt{0.50}$$

$$= 1.7566\,,$$

$$\gamma + (\rho - \beta)w\sigma_2\sqrt{\tau} = 0.8426 + (0.82 - 0.875)1.7392\times 0.21\sqrt{0.50}$$

$$A_1 = N(1.7566) = 0.9605,\ A_2 = N(1.659) = 0.9514,$$

$$= 1.6590\,,$$

and

$$A_3 = N(1.6732) = 0.9529\,,$$

the call spread option price is

$$C(a = 1, b = -1) = e^{-0.058\times 0.5}(24.55\times 0.9605 - 18.76\times 0.9514$$

$$- 2\times 0.9529) = \$3.717\,,$$

and the put spread option price is

$$P(a = 1, b = -1) = e^{-0.058\times 0.5}[24.55(1 - 0.9605) - 18.76(1 - 0.9514)$$

$$- 2(1 - 0.9529)] = \$0.035\,.$$

Comparing the results in Examples 22.1 and 22.2, we can find that the difference between the spread call and put option prices obtained using the integration method in Example 22.1 and those obtained using the approximation method in Example 22.2 is only $0.001. However, the time used in each approach is significantly different: whereas the approximation method takes less than one second, the integration method takes nearly two minutes to solve the problem. Since the accuracy level is high and time saving, the approximation method has significant advantages over the integration method.

22.6. SPREAD GREEKS

The Greeks of spread options can be conveniently obtained using the approximation formula in (22.27), because it is of the Black-Scholes type and any sensitivity of a simple spread option can be obtained by taking partial derivative of (22.27) with respect to the corresponding factor directly. In order to illustrate the procedure, we try to find the chi (sensitivity with respect to the correlation coefficient) of the spread option price by taking partial derivative of (22.27) with respect to ρ:

$$\begin{aligned} \text{chi} = \frac{\partial}{\partial \rho} C_{\text{sp}}(\omega) &= -aw(1+\beta w^2)\beta\sigma_1\sqrt{\tau}e^{-g_1\tau}I_1 f[\gamma + (1-\beta\rho)w\sigma_1\sqrt{\tau}] \\ &\quad + bw[1+(\rho-\beta)\beta w^2]\sigma_2\sqrt{\tau}e^{-g_2\tau}I_2 f[\gamma+(\rho-\beta)w\sigma_2\sqrt{\tau}] \\ &\quad - \beta w^3 K e^{-r\tau} f(\gamma)\,, \end{aligned} \tag{22.8}$$

where all parameters are the same as in (22.27).

Formula (22.8) is in explicit form and can be used very conveniently. One thing worth noticing about the formula is that it is free of the option operator ω, implying that a spread call option has exactly the same chi as its corresponding put option.

The chi of a spread option can also be obtained using the exact solution in (22.3). We simply need to find the partial differentiation of each of the three parameters A_1, A_2, A_3 with respect to the correlation coefficient. To illustrate the procedure, we show how the partial differentiation of A_3 with respect to ρ can be found:

$$\begin{aligned} \frac{\partial}{\partial \rho} A_3 = \frac{1}{(1-\rho^2)^{3/2}} \int_{-\infty}^{\infty} & f(v) f\left[\frac{d+\rho v+\phi(v)}{\sqrt{1-\rho^2}}\right]\{(1-\rho^2)[\rho-\phi'(v)] \\ & + \rho[d+\rho v+\phi(v)]\}dv\,, \end{aligned} \tag{22.9}$$

where $d = d(aI_1, K, \sigma_1, g_1, \tau, r)$ and $\phi'(v)$ represents the first-order derivative of the function $\phi(v)$ with respect to v.

Part of the integration in (22.9) can be obtained in closed-form using the integration results given in Appendix of Chapter 15 because the brace in the integration function in (22.9) contains $(d+1-\rho^2)\rho + \rho^2 v$. The other part of the integration can be obtained using numerical integration as in calculating the spread option price using (22.3).

Calculating time is not really an issue using either the exact formula in (22.3) or the approximation formula in (22.7). This is because the approximation formula (22.7) is expressed in terms of univariate normal cumulative functions as in the Black-Scholes formula and it should take about the same time as in using the Black-Scholes formula, and the exact formula involves only univariate integrations within a limited integration range (from -4 to $+4$, for example) and these integrations can be done within a few seconds with most personal computers. The calculating time using the exact formula, however, should always be shorter than that using the method described by Ravindran (1993) as no other mathematical procedures such as the Gaussian quadrature are necessary so that the time for finding the possible integration range partitions and suitable transformations employed by Ravindran is avoided.

22.7. MULTIPLE SPREAD OPTIONS

Although spread options written on more than two underlying assets or indices are not as popular as simple spread options written on spreads of two instruments, many institutions have been trading these products and they have become more and more popular. For example, options can be written on the spread between $X_1 + X_2$ and $X_3 + X_4$, where X_i is either an asset price or index. We may call this kind of spread options multiple spread options. By multiple spreads, we mean spreads among at least three underlying assets. With further development of OTC derivatives, increasing sophistication in risk management and accelerating globalization of international capital market, multiple spread options will certainly rise in popularity.

Intuitively, it is almost impossible to find closed-form solutions for multiple spread options for the same reason that makes it difficult to find exact closed-form solutions for arithmetic Asian options even in a Black-Scholes environment, because the sum of two or more than two lognormal variables is simply not lognormal. However, we may use the existing method of approximating arithmetic Asian options from their corresponding geometric Asian options to find an approximation pricing formula for multiple spread options. We will spend the rest of this chapter to find an approximating pricing formula for multiple spread options in a Black-Scholes environment using the analytical approximation results developed in Chapter 6. For simplicity, we concentrate on a single instrument rather than instruments multiplied by some coefficients such as a spread between X_1+2X_2 and $3X_3+X_4$. However, the method can be easily extended to analyze such multiple spread options. We will show more specific steps on how to extend this method at the end of Section 22.8.

Suppose that there are $n + m$ underlying assets or indices, $X_i, i = 1, 2, \ldots, n+m$, each following the standard stochastic process given in (IV1), and the correlation coefficient between the returns of the ith asset and the jth asset is assumed to be constant ρ_{ij}.

Let us define the following expressions:

$$L_1(\tau) = \sum_{l=1}^{n} S_i(\tau)\,, \tag{22.10}$$

and

$$L_2(\tau) = \sum_{j=n+1}^{n+m} S_j(\tau)\,, \tag{22.11}$$

where n and m are positive integers and $n + m \geq 3$.

The payoff of a European call option on a multiple spread (EMSD) can now be written as

$$EMSD = \max[a\omega L_1(\tau) + \omega b L_2(\tau) - \omega K,\ 0]\,, \tag{22.12}$$

where a, b, and K are the same as in (21.1).

We may call the multiple spread option defined in (22.10) to (22.12) simple multiple spread option as the two sums L_1 and L_2 are sums of single-instrument prices at maturity. If either L_1 or/and L_2 are sums of instrument prices at maturity multiplied by some coefficients, for example $L_1 = X_1 + 2X_2$ and $L_2 = 3X_3 + X_4$ as discussed above, we may call this kind of multiple spread options complex multiple spread options.

22.8. APPROXIMATING THE EQUALLY WEIGHTED SUMS

The sums in (22.10) and (22.11) can be rewritten as products of the corresponding arithmetic averages and the numbers in each summation. As the arithmetic averages can be approximated from their corresponding geometric averages as shown in Chapter 6, we can approximate the sums from their corresponding arithmetic averages using the results of Chapter 6.

Proposition 22.2. The sum L_1 in (22.10) can be approximated as follows

$$L_1(\tau) \simeq n\kappa_n G_1(\tau)\,, \tag{22.13a}$$

where

$$G_1(\tau) = \left[\prod_{i=1}^{n} S_i(\tau)\right]^{1/n}, \tag{22.13b}$$

$$\kappa_n = 1 + \frac{1}{2}E(v_n) + \frac{1}{4}[E(v_n)]^2 + Var(v_n)], \tag{22.13c}$$

$$E(v_n) = \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\left[\ln\left(\frac{S_i}{S_j} - \frac{\tau}{2}(\sigma_i^2 - \sigma_j^2)\right]^2 + \tau\sigma_{ij}^2\right\}, \tag{22.13d}$$

$$Var(v_n) = \frac{4\tau}{n^4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sigma_{ij}^2\left[\ln\left(\frac{S_i}{S_j}\right) - \frac{\tau}{2}(\sigma_i^2 - \sigma_j^2)\right]^2, \tag{22.13e}$$

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2,$$

and $G_1(\tau)$ is the geometric average of the first n asset prices.

Proof. Substituting the n underlying asset prices of the n different instruments at maturity and following a similar procedure to derive Theorem 6.1 yields the results in (22.13). □

The results in Proposition 22.2 are different from those in Theorem 6.1 in Chapter 6 although the analyzing method is the same. This is because the averages in (22.13) are for different asset prices at the same time (the time to maturity) and the averages for Asian options in Chapter 6 are for different observations of one underlying asset price.

In the trivial case of $n = 1$, it can be easily verified that $E(v_n) = Var(v_n) = 0$, thus $\kappa_1 = 1$. Substituting $\kappa = 1$ into (22.13a) yields $L_1(\tau) = G_1(\tau)$ which is obviously consistent with the intuition that the arithmetic and geometric means are the same when there is only one asset. From (22.13c), (22.13d), and (22.13e), we can observe that κ_n is a deterministic function of the n current prices, the volatilities, and the correlation coefficients among the n asset returns. As the geometric average is lognormally distributed, the sum approximation in (22.13a) is thus approximately lognormal.

Similarly, the second summation L_2 of the last m underlying asset prices, each following the Brownian motion specified in (IV1), can be approximated from its corresponding geometric mean as follows:

$$L_2(\tau) \cong m\kappa_m G_2(\tau), \tag{22.14a}$$

where

$$G_2(\tau) = \left\{ \prod_{i=n+1}^{n+m} S_i(\tau) \right\}^{1/m}, \tag{22.14b}$$

$$\kappa_m = 1 + \frac{1}{2}E(v_m) + \frac{1}{4}[E(v_m)]^2 + Var(v_m), \tag{22.14c}$$

$$E(v_m) = \frac{1}{m^2} \sum_{i=n+1}^{n+m-1} \sum_{j=i+1}^{n+m} \left\{ \left[\ln\left(\frac{S_i}{S_j}\right) - \frac{\tau}{2}(\sigma_i^2 - \sigma_l^2) \right]^2 \right.$$

$$\left. + \tau[\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2] \right\}, \tag{22.14d}$$

$$Var(v_m) = \frac{4\tau}{m^4} \sum_{i=n+1}^{n+m-1} \sum_{j=i+1}^{n+m} \left[\ln\left(\frac{S_i}{S_j}\right) - \frac{\tau}{2}(\sigma_i^2 - \sigma_j^2) \right] (\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2), \tag{22.14e}$$

and $G_2(\tau)$ is the geometric average of the last m asset prices.

The approximated summations in (22.13) and (22.14) can be easily shown to be lognormally distributed. We present the results in the following proposition.

Proposition 22.3. Let $x = \ln[L_1(\tau)/L_1(0)]$ and $y = \ln[L_2(\tau)/L_2(0)]$, x is normally distributed with mean $\mu_x = \tau\mu_{a1}$ and variance $\sigma_x^2 = \tau\sigma_{a1}^2$; y is normally distributed with mean $\mu_y = \tau\mu_{a2}$ and variance $\sigma_y^2 = \tau\sigma_{\sigma 2}^2$; and x and y are jointly normally distributed with the correlation coefficient ρ_a, where

$$\mu_{a1} = r - \frac{1}{n}\sum_{i=1}^{n}\left(g_i + \frac{1}{2}\sigma_i^2\right),$$

$$\mu_{a2} = r - \frac{1}{m}\sum_{i=n+1}^{n+m}\left(g_i + \frac{1}{2}\sigma_i^2\right), \tag{22.15a}$$

$$\sigma_{a1} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j},$$

$$\sigma_{a2} = \frac{1}{m}\sqrt{\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m}\rho_{ij}\sigma_i\sigma_j}, \tag{22.15b}$$

and

$$\rho_a = \frac{1}{\sigma_{a1}\sigma_{a2}}\frac{1}{mn}\left(\sum_{i=1}^{n}\sum_{j=n+1}^{n+m}\rho_{ij}\sigma_i\sigma_j\right). \tag{22.15c}$$

Proof. Immediate from Proposition 22.2 and (22.15). □

22.9. PRICING MULTIPLE SPREAD OPTIONS

Consider $L_1(\tau)$ and $L_2(\tau)$ as two asset prices with means, and volatilities and the correlation coefficient given in Proposition 22.2, we can obtain the expected payoff of the European multiple spread call option in (22.12):

$$E(EMSD) = \omega a L_1(0) e^{\tau(\mu_{a1}+\sigma_{a1}^2/2)} A_{m1} + \omega b L_2(0) e^{\tau(\mu_{a2}+\sigma_{a2}^2/2)} A_{m2} - \omega K A_{m3}\,, \tag{22.16}$$

where

$$A_{m1} = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d_m + \rho_a V + \sigma_{a1}\sqrt{\tau} + \phi(v + \rho_a \sigma_{a1}\sqrt{\tau})}{\sqrt{1-\rho_a^2}}\right] dv\,, \tag{22.16a}$$

$$A_{m2} = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d_m + \rho_a v + \rho_a \sigma_{a2}\sqrt{\tau} + \phi(v + \sigma_{a2}\sqrt{\tau})}{\sqrt{1-\rho_a^2}}\right] dv\,, \tag{22.16b}$$

$$A_{m3} = \int_{-\infty}^{\infty} f(v) N\left[\omega \frac{d_m + \rho_a v + \phi(v)}{\sqrt{1-\rho_a^2}}\right] dv\,, \tag{22.16c}$$

$$d[-bL_2(0),\ K, \sigma_{a2},\ g_{a2},\ \tau,\ r] = \frac{\ln\left[\frac{-bI_2(0)}{K}\right] + \left(r - g_{a2} - \frac{1}{2}\sigma_{a2}^2\right)\tau}{\sigma_{a2}\sqrt{\tau}}\,, \tag{22.16d}$$

$$\phi(v) = -\frac{1}{\sigma_{a1}\sqrt{\tau}} \ln\left[1 - \frac{bL_2(0)}{K}\exp(\tau\mu_{a2} + v\sigma_{a2}\sqrt{\tau})\right].$$

The three parameters A_{m1}, A_{m2}, and A_{m3} are extensions of the three parameters A_1, A_2, and A_3 for simple spread options given in (22.2) and thus can be calculated as conveniently as them using numerical integration with any computer. We can obtain the multiple spread option price (MSOP) by discounting the expected payoff in (22.16) by the risk-free rate r,

$$MSOP = \omega a L_1(0) e^{-D_{d1}\tau} A_{m1} + \omega b L_2(0) e^{-D_{d2}\tau} A_{m2} - \omega K e^{-r\tau} A_{m3}\,, \tag{22.17}$$

where

$$D_{d1} = \frac{\tau}{2}\left[-\sigma_{a1}^2 + \frac{1}{n}\sum_{i=1}^{n}\left(g_i - \frac{1}{2}\sigma_i^2\right)\right],$$

$$D_{d2} = \frac{\tau}{2}\left[-\sigma_{a2}^2 + \frac{1}{m}\sum_{i=n+1}^{n+m}\left(g_i - \frac{1}{2}\sigma_i^2\right)\right],$$

and A_{m1}, A_{m2}, and A_{m3} are the same as in (22.16).

We can also find an approximation formula for multiple spread options using the approximation formula for simple spread options in (22.7) and the means, variances, and correlation coefficient in Proposition 22.2:

$$A_{m1} = N[\gamma_m + (1 - \beta_m \rho_a) w_m \sigma_{a1} \sqrt{\tau}], \tag{22.18a}$$

$$A_{m2} = N[\gamma m + (\rho_a - \beta_m) w_m \sigma_{a2} \sqrt{\tau}], \tag{22.18b}$$

$$A_{m3} = N(\gamma m), \tag{22.18c}$$

where

$$w_m = 1/\sqrt{1 - 2\rho_a \beta_m + \beta_m^2},$$

$$\gamma_n = \{d[L_1(0), K, \sigma_{a1}, g_{a1}, \tau, r] + \alpha_m\} w_m,$$

$$g_{a1} = \frac{1}{n} \sum_{i=1}^{n} \left(g - i + \frac{1}{2} \sigma_i^2 \right),$$

$$g_{a2} = \frac{1}{m} \sum_{i=n+1}^{n+m} \left(g_i + \frac{1}{2} \sigma_i^2 \right),$$

$$\beta_m = \frac{\sigma_{a2}}{\sigma_{a1}},$$

$$\alpha_m = -\beta_m d[-bL_2, K, \sigma_{a2}, g_{a2}, \tau, r) - \frac{1}{\sigma_a \sqrt{\tau}} \sum_{l=1}^{\infty} i!(-1)^{i-1} l_m^i,$$

$$l_m = \frac{1}{-bI_2} \exp\left[-\left(r - g_2 - \frac{1}{2}\sigma_2^2 \right) \tau + 5\sigma\sqrt{\tau} \right],$$

and other parameters are the same as in (22.16) and (22.17).

22.10. PRICING COMPLEX SPREAD OPTIONS

The results in the previous section are for simple multiple spread options with unitary coefficient for each individual asset. To extend the analysis of simple multiple spread options to complex multiple spread options in this section, we can simply make some algebraic changes. Let $L_1(\tau)$ in (22.10) be substituted with $L_{c1}(\tau) = \sum_{l=1}^{n} \mathrm{cef}_i S_i(\tau)$ for a complex multiple spread option, where cef_i stands for the coefficient for the ith asset price. The sum $L_{c1}(\tau)$ can be alternatively expressed as follows:

$$L_{c1}(\tau) = \left(\sum_{l=1}^{n} \mathrm{cef}_i \right) \sum_{l=1}^{n} W_i S_i(\tau), \tag{22.19}$$

where

$$W_i = \text{cef}_i \Big/ \Big(\sum_{l=1}^{n} \text{cef}_i \Big) \quad \text{and} \quad \sum_{l=1}^{n} W_i = 1\,.$$

With W_i understood as the weight given to the ith underlying asset at maturity, the sum in (22.19) can be interpreted as the weighted sum of the n underlying asset prices at maturity multiplied by the sum of all the n coefficients. Since all the coefficients are given when the contract is signed, the sum of all the coefficients are known. The weighted sum of the underlying asset prices at maturity can be approximated with a lognormal distribution following a similar method developed in Chapter 7 and the substitution used in Proposition 22.2 to replace different observations of one underlying asset price with various underlying asset prices at maturity.

We can extend the results of Proposition 22.2 to approximate the weighted sum in (22.19). We need to construct a weighted geometric "index" according to the weighted sum in (22.19) and find its distribution in order to express the approximated distribution of the weighted sum in (22.19). We can construct the following weighted geometric "index" (WGI) as follows according to the weighted sum in (22.19)

$$WGI(\tau) = [S_1(\tau)]^{W_1}[S_2(\tau)]^{W_2} \cdots [S_n(\tau)]^{W_n}\,, \tag{22.20}$$

where $W_i, S_i(\tau)$ are the weight assigned to the ith underlying asset price, $i = 1, 2, \ldots, n$.

Proposition 22.4. Let $x = \ln[WGI(\tau)/WGI(t)]$, where $WGI(t)$ stands for the weighted geometric average of the n spot asset prices. The variable x is normally distributed with mean and variance

$$\tau \sum_{i=1}^{n} W_i \Big(r - g_i - \frac{1}{2}\sigma_i^2 \Big) \quad \text{and} \quad \tau \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j \rho_{ij} \sigma_i \sigma_j\,, \quad \text{respectively}\,.$$

Proof. Substituting the solution in (5.3) for the standard geometric Brownian motion given in (IV1) into the weighted geometric "index" in (22.20) yields Proposition 22.4 after simplifications. □

Proposition 22.5. The weighted sum in (22.19) can be approximated as follows

$$\sum_{l=1}^{n} W_i S_i(\tau) \cong \kappa_{nw} WGI(\tau)\,, \tag{22.21}$$

where

$$\kappa_{nw} = 1 + \frac{1}{2}E(v_{nw} + \frac{1}{4}[E(v_{nw})]^2 + Var(v_{nw})] , \tag{22.21a}$$

$$E(v_{nw}) = \sum_{i=1}^{n} W_i[\sigma_i^2\tau + (v_i\tau + \ln S_i)^2]$$

$$- \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_iW_j\rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln S_i)(v_j\tau + \ln S_j) , \tag{22.21b}$$

$$Var(v_n) = 4V_1 - 4V_2 + V_3 , \tag{22.21c}$$

$$V_1 = \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_iW_j\rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln A_i)(v_j\tau + \ln S_j) ,$$

$$V_2 = \tau \sum_{k=1}^{n} W_k(v_k\tau + \ln S_k) \sum_{i=1}^{n}\sum_{j=1}^{n} W_iW_j[\rho_{ik}\sigma_i(v_j\tau + \ln S_j) + \rho_{jk}\sigma_j(v_i\tau + \ln S_i)] ,$$

$$V_3 = \tau \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} W_iW_jW_kW_l[\rho_{ik}\sigma_i\sigma_k(v_j\tau + \ln S_j)(v_l\tau + \ln S_l) + \rho_{il}\sigma_i\sigma_l(v_j\tau + \ln S_j)(v_k\tau + \ln S_k) + \rho_{jk}\sigma_j\sigma_k(v_i\tau + \ln S)(\sigma_l\tau + \ln S_l) [+\rho_{jl}\sigma_j\sigma_l(v_i\tau + \ln S_i)(v_k\tau + \ln S_k)] ,$$

$WGI(\tau)$ is the weighted geometric "index" defined in (22.19) and ρ_{ij} is the correlation coefficient between the returns of the ith and the jth assets.

Proof. Substituting the solution in (5.3) for the standard geometric Brownian motion given in (IV1) into the weighted sum in (22.19) yields Proposition 22.5 after simplifications. □

Although the mean and variance parameters in the approximation expression in Proposition 22.5 are expressed in multiple summations, they are in closed-form and these summations can be carried out conveniently. The closed-form variance expression is better than the variance parameter approximation in Theorem 7.3. The reason that a closed-form expression is obtained in Proposition 22.5 and not in Theorem 7.3 is that the n underlying asset prices are all at the maturity time and there is only the structure of correlation coefficients among these n assets involved in Proposition 22.5.

where

$$\kappa_{nw} = 1 + \frac{1}{2}E(v_{nw} + \frac{1}{4}[E(v_{nw})]^2 + Var(v_{nw})] \,, \tag{22.21a}$$

$$E(v_{nw}) = \sum_{i=1}^{n} W_i[\sigma_i^2\tau + (v_i\tau + \ln S_i)^2]$$

$$- \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln S_i)(v_j\tau + \ln S_j) \,, \tag{22.21b}$$

$$Var(v_n) = 4V_1 - 4V_2 + V_3 \,, \tag{22.21c}$$

$$V_1 = \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln A_i)(v_j\tau + \ln S_j) \,,$$

$$V_2 = \tau \sum_{k=1}^{n} W_k(v_k\tau + \ln S_k) \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j[\rho_{ik}\sigma_i(v_j\tau + \ln S_j)$$

$$+ \rho_{jk}\sigma_j(v_i\tau + \ln S_i)] \,,$$

$$V_3 = \tau \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j W_k W_l[\rho_{ik}\sigma_i\sigma_k(v_j\tau + \ln S_j)(v_l\tau + \ln S_l)$$

$$+ \rho_{il}\sigma_i\sigma_l(v_j\tau + \ln S_j)(v_k\tau + \ln S_k)$$

$$+ \rho_{jk}\sigma_j\sigma_k(v_i\tau + \ln S)(\sigma_l\tau + \ln S_l)$$

$$[+\rho_{jl}\sigma_j\sigma_l(v_i\tau + \ln S_i)(v_k\tau + \ln S_k)] \,,$$

$WGI(\tau)$ is the weighted geometric "index" defined in (22.19) and ρ_{ij} is the correlation coefficient between the returns of the ith and the jth assets.

Proof. Substituting the solution in (5.3) for the standard geometric Brownian motion given in (IV1) into the weighted sum in (22.19) yields Proposition 22.5 after simplifications. □

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where

$$\kappa_{nw} = 1 + \frac{1}{2}E(v_{nw} + \frac{1}{4}[E(v_{nw})]^2 + Var(v_{nw})], \tag{22.21a}$$

$$E(v_{nw}) = \sum_{i=1}^{n} W_i[\sigma_i^2\tau + (v_i\tau + \ln S_i)^2]$$

$$- \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln S_i)(v_j\tau + \ln S_j), \tag{22.21b}$$

$$Var(v_n) = 4V_1 - 4V_2 + V_3, \tag{22.21c}$$

$$V_1 = \tau \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j(v_i\tau + \ln A_i)(v_j\tau + \ln S_j),$$

$$V_2 = \tau \sum_{k=1}^{n} W_k(v_k\tau + \ln S_k) \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j [\rho_{ik}\sigma_i(v_j\tau + \ln S_j)$$

$$+ \rho_{jk}\sigma_j(v_i\tau + \ln S_i)],$$

$$V_3 = \tau \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j W_k W_l [\rho_{ik}\sigma_i\sigma_k(v_j\tau + \ln S_j)(v_l\tau + \ln S_l)$$

$$+ \rho_{il}\sigma_i\sigma_l(v_j\tau + \ln S_j)(v_k\tau + \ln S_k)$$

$$+ \rho_{jk}\sigma_j\sigma_k(v_i\tau + \ln S)(\sigma_l\tau + \ln S_l)$$

$$[+\rho_{jl}\sigma_j\sigma_l(v_i\tau + \ln S_i)(v_k\tau + \ln S_k)],$$

$WGI(\tau)$ is the weighted geometric "index" defined in (22.19) and ρ_{ij} is the correlation coefficient between the returns of the ith and the jth assets.

Proof. Substituting the solution in (5.3) for the standard geometric Brownian motion given in (IV1) into the weighted sum in (22.19) yields Proposition 22.5 after simplifications. □

Although the mean and variance parameters in the approximation expression in Proposition 22.5 are expressed in multiple summations, they are in closed-form and these summations can be carried out conveniently. The closed-form variance expression is better than the variance parameter approximation in Theorem 7.3. The reason that a closed-form expression is obtained in Proposition 22.5 and not in Theorem 7.3 is that the n underlying asset prices are all at the maturity time and there is only the structure of correlation coefficients among these n assets involved in Proposition 22.5.

Complex spread options can be priced using the approximation results in Propositions 22.4 and 22.5 and the pricing formulas for simple spread options if we specify

$$a = \sum_{l=1}^{n} \mathrm{cef}_l , \equiv - \sum_{l=n+1}^{n+m} \mathrm{cef}_l ,$$

$$I_1 = \kappa_{nw} WGI(t) , \; I_2 = \kappa_{nw} \left(\prod_{j=n+1}^{n+m} S_i^{W_i} \right) ,$$

$$\sigma_{l1} = \tau \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j \rho_{ij} \sigma_i \sigma_j ,$$

$$\sigma_{l2} = \tau \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} W_i W_j \rho_{ij} \sigma_i \sigma_j ,$$

$$g_{l1} = r - \tau \sum_{i=1}^{n} W_i \left(r - g_i - \frac{1}{2}\sigma_i^2 \right) ,$$

$$g_{l2} = r - \tau \sum_{i=n+1}^{n+m} W_i \left(r - g_i - \frac{1}{2}\sigma_i^2 \right) .$$

22.11. SOME SPECIAL MULTIPLE SPREAD OPTIONS

Although the approximation formula for multiple spread options in (22.17) and (22.18) is of the Black-Scholes type, it involves many initial as well as intermediate parameters. In order to illustrate how to use this formula, we show one simple example in this section for the special case of $n = m = 2$.

Substituting $n = m = 2$ into Proposition 22.4, we obtain the following:

$$\mu_x = \tau \mu_{a1} = \tau \left[r - \frac{1}{2}(g_1 + g_2) - \frac{1}{4}(\sigma_1^2 + \sigma_2^2) \right] , \tag{22.22a}$$

$$\mu_y = \tau \mu_{a2} = \tau \left[r - \frac{1}{2}(g_3 + g_4) - \frac{1}{4}(\sigma_3^2 + \sigma_4^2) \right] , \tag{22.22b}$$

$$\sigma_{a1} = \frac{1}{2}\sqrt{\sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2} , \tag{22.22c}$$

$$\sigma_{a2} = \frac{1}{2}\sqrt{\sigma_3^2 + 2\rho_{34}\sigma_3\sigma_4 + \sigma_4^2} , \tag{22.22d}$$

$$\rho_a = \frac{\rho_{13}\sigma_1\sigma_3 + \rho_{14}\sigma_1\sigma_4 + \rho_{23}\sigma_2\sigma_3 + \rho_{24}\sigma_2\sigma_4}{\sqrt{\sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}\,\sqrt{\sigma_3^2 + 2\rho_{34}\sigma_3\sigma_4 + \sigma_4^2}} , \tag{22.22e}$$

$$E(v_2) = \frac{\tau}{4}\left[\ln\left(\frac{S_1}{S_2}\right) - \frac{\tau}{2}(\sigma_1^2 - \sigma_2^2)\right]^2 + \tau(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2), \quad (22.22f)$$

$$Var(v_2) = \frac{\tau}{4}\left[\ln\left(\frac{S_1}{S_2}\right) - \frac{\tau}{2}(\sigma_1^2 - \sigma_2^2)\right]^2 (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2). \quad (22.22g)$$

The effective standard deviations in (21.22c) and (21.22d) are obviously smaller than $(\sigma_1 + \sigma_2)/2$ and $(\sigma_3 + \sigma_4)/2$, respectively, because the correlation coefficients are always smaller than 1. This result holds in general for $n > 2$ and $m > 2$ because σ_{a1} and σ_{a2} can be shown to be smaller than the corresponding arithmetic average standard deviations of the first n asset returns and the last m asset returns involved because the correlation coefficients among all asset returns are always smaller than 1.

We can simplify the expressions in (22.22) further if we assume:

$$g_i = 0,\ \sigma_i = \sigma \text{ for all } i = 1,2,3,4, \text{ and } \rho_{ij} = \rho \text{ for } i \neq j = 1,2,3,4. \quad (22.23)$$

Substituting the above assumptions into (22.22) yields

$$\mu_x = \mu_y = \tau\mu_{a1} = \tau\left(r - \frac{1}{2}\sigma^2\right), \quad (22.23ab)$$

$$\sigma_{a1} = \sigma_{a2} = \sigma\sqrt{(1+\rho)/2}, \quad (22.23cd)$$

$$\rho_a = 2\rho/(1+\rho), \quad (22.23e)$$

$$E(\phi_2) = \frac{\tau}{4}\left\{\left[\ln\left(\frac{S_1}{S_2}\right)\right]^2 + 2(1-\rho)\sigma^2\right\}, \quad (22.23f)$$

$$Var(\phi_2) = \frac{\pi}{2}\left[\ln\left(\frac{S_1}{S_2}\right)\right]^2 (1-\rho)\sigma^2. \quad (22.23g)$$

We further suppose that the first two assets have the same current price $S_1 = S_2 = \$25$, the last two assets have the same current price $S_3 = S_4 = \$20$, the interest rate $r = 6\%$, the volatility $\sigma = 15\%$, and the time to maturity of the option $\tau =$ half a year. To calculate the option price, we have to follow a four-step procedure:

(i) To calculate the effective standard deviations of the two summations (σ_{a1} and σ_{a2}) using (22.15b), the effective means of the two summations (μ_{a1} and μ_{a2}) using (22.14a), the correlation coefficient between the returns of the two summations ρ_a using (22.15c), the two initial geometric averages $G_1(0)$ and $G_2(0)$, and the means and variances of ϕ for both n and m [$E(\phi)$ and $Var(\phi)$] using (22.13d) and (22.13e).

(ii) To calculate the parameters κ_n and κ_m using (22.13c) and then to calculate the two initial indices $I_1(0) = n\kappa_n G_1(0)$ and $I_2(0) = m\kappa_m G_2(0)$.

(iii) To calculate the parameter $d[L_1(0), K, \sigma_{a1}, g_{a1}, \tau, r]$ using (22.17d), the approximation parameters α_m and β_m in (22.19), and then parameters w_m and γ_m in (22.19).

(iv) To calculate the three parameters A_{m1}, A_{m2}, and A_{m3} in (22.19), and finally the option price using (22.18).

Although there are quite a few steps involved in calculating the multiple spread option price, the formula in (22.18) is of the Black-Scholes type. These steps can be carried out very conveniently with Lotus 1-2-3 or EXCEL. The multiple spread options are complex products involving many underlying assets. Our solutions have tremendously reduced the time which otherwise has to be taken if Monte-Carlo simulations are used.

22.12. SUMMARY AND CONCLUSIONS

Using the method to approximate arithmetic averages from their corresponding geometric averages developed in Chapter 6, we have found closed-form approximation formulas for multiple spread option prices. As in Chapter 6 for arithmetic Asian options, these closed-form solutions can be well approximated with formulas of the Black-Scholes type. As the effective standard deviations of the sums are always smaller than the arithmetic average standard deviations of the individual asset returns, the approximation formulas should be very accurate. These approximation formulas significantly reduce the time necessary to price multiple spread options using Monte-Carlo simulations in which the paths of all the $n + m$ assets have to be simulated. These formulas also make it possible to obtain closed-form expressions for the sensitivities of various parameters on the multiple spread option prices. Due to these characteristics, the results in this chapter can significantly facilitate the analysis and trading of these options.

QUESTIONS AND EXERCISES

22.1. What is a simple spread option?

22.2. How are spread options related to exchange options?

22.3. What are the similarities and differences between spread options and ratio options?

22.4. What are multiple spread options?

22.5. What are simple multiple spread options?

22.6. What are complex multiple spread options?

22.7. What are the one-factor and two-factor models in pricing spread options?

22.8. What are the limitations of the one-factor model in pricing spread options?

22.9. What is the most important advantage of the approximation formula over the exact formula using integration? What is its disadvantage?

22.10. Is the chi of spread options always positive, zero, or negative? Why?

22.11. Find the prices of the standard spread options on the spread of the gasoline price over the crude oil price to expire in one year, given the strike price \$3, the spot gasoline price \$25 per barrel, the spot crude oil price \$19 per barrel, the interest rate 6.2%, then payout rates of the two assets 6.2%, the correlation coefficient between the gasoline and crude prices 86%, and the volatilities of gasoline and crude oil prices 25% and 20%, respectively.

22.12. Find the prices of the spread options in Exercise 22.11 using the approximation formula.

22.13. Find the chi of the spread options in Exercise 22.11 using the approximation formula in (22.8).

22.14. Find the prices of the spread options in Exercise 22.11 if the volatility of gasoline is changed to 15% and other parameters remain unchanged.

22.15. Find the chi of the spread options in Exercise 22.14.

22.16. Show Proposition 22.4.

22.17. Show the mean expression in Proposition 22.5.

22.18.* Show the variance expression in Proposition 22.5.

Chapter 23

SPREAD OVER THE RAINBOWS

23.1. INTRODUCTION

We studied options paying the best or worst of two underlying assets and cash in Chapter 14 and options on the maximum or minimum of two underlying assets, or rainbow options in Chapter 21. Whereas two-color rainbow options are popular, little attention has been paid to the relationship between the two-color rainbows. The purpose of this chapter is to study the correlation between the two-color rainbows. To have a good understanding of this correlation can certainly help us understand rainbow options better. We will introduce and price spread options written on the spread between the two-color rainbows. A spread option written on the spread between the two-color rainbows can be understood as an option written on the absolute value of the difference between the two asset prices at maturity. It can also be understood as a pair of vanilla call and put options, the former written on the maximum and the latter on the minimum of the two underlying assets with the two strike prices between the two-color rainbows. Due to the equivalent payoffs of a spread option on the spread over the two rainbows and an option written on the corresponding absolute difference between the two asset prices at maturity, we may simply call these options absolute options.

Due to the unique nature of the spread over the two rainbows, options written on this spread have particular potential applications because they have characteristics all other correlation options do not possess. Spread options over the rainbows should be used when the difference between two underlying asset prices is not clear.

23.2. SPREAD OPTIONS OVER THE RAINBOWS

Suppose there are two assets or indices I_1 and I_2, both following the standard stochastic process in (IV1) and the two asset returns are correlated with a constant correlation coefficient ρ. We can formally define two-color rainbows as follows:

$$\max tc(\tau) = \max\left[I_1(\tau), I_2(\tau)\right] , \tag{23.1}$$

and

$$\min tc(\tau) = \min\left[I_1(\tau), I_2(\tau)\right] , \tag{23.2}$$

where max (., .) and min (., .) are functions that give the larger and smaller of the two prices involved, respectively.

The payoff of a European-style option written on the spread of the two-color rainbows in (23.1) and (23.2) can be expressed as

$$RC = \max\left\{\omega[a \times maxtc(\tau) + b \times mintc(\tau)] - \omega K, 0\right\}, \tag{23.3}$$

where K is the strike price of the spread option, ω is a binary operator (1 for a call and -1 for a put), $a > 0$ and $b < 0$ are the same parameters as in (22.1) for a simple spread option. When $a = 1$ and $b = -1$, the spread over the rainbows becomes the standard spread over the rainbows.

It is straightforward to check that the spread between the two rainbows can be alternatively expressed as follows:

$$\max(I_1, I_2) - \min(I_1, I_2) = |I_1 - I_2|, \tag{23.4}$$

where $|z|$ gives the absolute value of the real number z.

From the relationship in (23.4), we can interpret spread options with payoffs given in (23.3) as options written on the absolute price difference of the two assets involved. This is a very interesting interpretation. It connects the spread between the maximum and minimum of two assets and the absolute price difference of the same two underlying assets. Options written on the absolute price difference are the same as options written on the spread between the maximum and minimum prices of the two assets. These options possess characteristics which all other known correlation options do not have because the option buyer can receive the absolute price difference regardless of which asset price ending up higher. Due to the relationship in (23.4), we call spread options over the two-color rainbows in (23.3) absolute options on the absolute price difference of the two underlying assets.

23.3. RELATIONSHIP BETWEEN THE TWO RAINBOWS

Before we start to price options on the spread over the two-color rainbows, it is better for us to know how the two rainbows are correlated. The expected values of the two-color rainbows are given in (23.1) and (23.2).

Using the same method as to derive the means in (14.5) and (14.6) using the two bivariate density functions, we can obtain the standard deviations of the two rainbows:

$$\sigma_{\max} = e^{r\tau}\sqrt{\psi_1 I_1^2 - qI_1I_2N(d_{12}) + \psi_2 I_2^2}\,, \tag{23.5}$$

and

$$\sigma_{\min} = e^{r\tau}\sqrt{\psi_3 I_1^2 - 2I_1I_2N(-d_{12})N(-d_{21}) + \psi_4 I_2^2}\,, \tag{23.6}$$

where

$$\psi_1 = e^{\sigma_1^2\tau}N\Big[d_{21} + \frac{\sigma_1(\sigma_1 - \rho\sigma_2)}{\sigma}\Big] - N^2(d_{21}),$$

$$\psi_2 = e^{\sigma_2^2\tau}N\Big[d_{12} + \frac{\sigma_2(\sigma_\rho\sigma_1)}{\sigma}\Big] - N^2(d_{12})\,,$$

$$\psi_3 = e^{\sigma_1^2\tau}N\Big[-d_{21} - \frac{\sigma_1(\sigma_1 - \rho\sigma_2)}{\sigma}\Big] - N^2(-d_{21})\,,$$

$$\psi_4 = e^{\sigma_2^2\tau}N\Big[-d_{12} - \frac{\sigma_2(\sigma_2 - \rho\sigma_1)}{\sigma}\Big] - N^2(-d_{12})\,,$$

and

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,.$$

The covariance between the two-color rainbows can be similarly obtained:

$$cov(maxtc, mintc) = I_1I_2e^{2r\tau+\rho\sigma_2(\sigma_1-\rho\sigma_2)\tau)}\,. \tag{23.7}$$

With the covariance in (23.7) and the two standard deviations in (23.5) and (23.6), we can easily obtain the correlation coefficient between the two rainbows:

$$\rho_{\tau b} = \frac{I_1I_2e^{2r\tau+\rho\sigma_2(\sigma_1-\rho\sigma_2)\tau}}{\sigma_{\max}\sigma_{\min}}\,, \tag{23.8}$$

where the two standard deviations in the denominator are given in (23.5) and (23.6), respectively.

Equation (23.8) indicates that the correlation coefficient between the two rainbows is determined by the correlation coefficient between the two underlying assets, the time to maturity of the option, the spot prices of the two instruments, and the volatilities of these two assets.

23.4. PRICING ABSOLUTE OPTIONS

As in pricing all correlation options in previous chapters, we first need to depict the integration domain of the option. Figure 23.1 depicts the integration domain of an absolute option, and it shows that there are two regions in which the absolute option has a positive expected payoff. If we compare Figure 23.1 with Figure 22.1, we can easily find that the area marked $\langle 1 \rangle$ in Figure 23.1 is the same as the integration domain of the standard spread option of the first asset price over the second in Figure 22.1. Using the symmetry property, we can find that the area marked $< 2 >$ in Figure 23.1 is the integration domain of the spread option of the second asset price over the first.

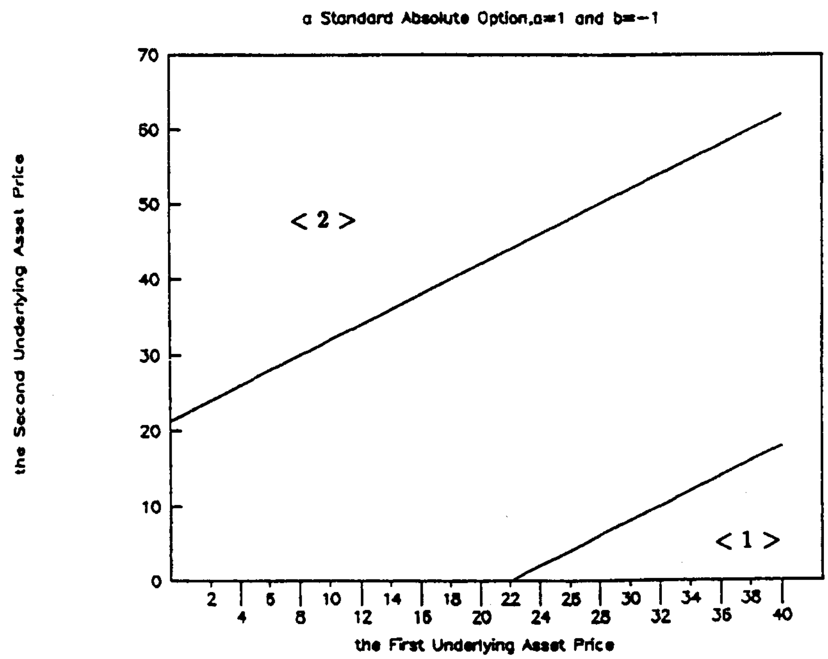

Fig. 23.1.

Since the two integration domains $< 1 >$ and $< 2 >$ are non-overlapping, they are additive. Thus, the expected payoff of the absolute option is the sum of the expected payoff of the spread option of the first asset price over the second, and that of the spread option of the second asset price over the first. Using the one-asset price law, we know that the price of an absolute option equals the sum of the prices of two spread options: one on the spread of the first asset price over the second and the other on the spread of the second asset price over the first with the same strike price.

Let $PSSD(I_1, I_2, \sigma_1, \sigma_2, K, \tau, \rho, \omega)$ stand for the price of the standard spread option on the spread of the first asset price over the second as given in (22.3) with $a = 1$, $b = -1$, and the option operator ω. The price of the absolute option ($PABS$) can thus be expressed as follows:

$$PABS(\omega) = PSSD(I_1, I_2, \sigma_1, \sigma_2, K, \tau, \rho, \omega) + PSSD(I_2, I_1, \sigma_2, \sigma_1, K, \tau, \rho, \omega)\,. \quad (23.9)$$

where $PSSD(A, B, \dots)$ stands for the price of a simple spread option of A over B.

Let $APSSD(I_1, I_2, \sigma_1, \sigma_2, K, \tau, \rho, \omega)$ stand for the approximated price of the standard spread option on the spread of the first over the second asset price as given in (22.8) with $a = 1$ and $b = -1$. The approximated price of the absolute option ($APABS$) can be expressed as follows:

$$PABS(\omega) = APSSD(I_1, I_2, \sigma_1, \sigma_2, K, \tau\rho, \omega) + APSSD(I_2, I_1, \sigma_2, \sigma_1, K, \tau\rho, \omega)\,. \quad (23.10)$$

Example 23.1. Find the prices of the absolute options to expire in one year with the strike price \$3, given the spot prices of the two stocks with $I_1 = \$100$, $I_2 = \$95$, the volatilities $\sigma_1 = 20\%$ and $\sigma_2 = 15\%$, the yields on the two stocks are $g_1 = 5\%$ and $g_2 = 4\%$, the interest rate 10%, and the two stock returns are correlated with the correlation coefficient $\rho = 75\%$.

Substituting $I_1 = \$100$, $I_2 = \$95$, $K = \$3$, $\sigma_1 = 20\%$, $\sigma_2 = 15\%$, $g_1 = 5\%$ and $g_2 = 4\%$, and $\rho = 0.75$ into (22.3) yields the price of the option on the spread of the first asset price over the second

$$PSSD(100, 95, 0.20, 0.15, 3, 1, 0.75, 1) = \$6.679\,.$$

The price of the option on the spread of the second asset price over the first can be similarly obtained

$$PSSD(95, 100, 0.20, 0.15, 3, 1, 0.75, 1) = \$2.821\,.$$

Thus, the price of the absolute call option is the sum of the prices of the above two spread call options

$$PABS(1) = 6.679 + 2.281 = \$9.500\,.$$

Similarly, the price for the absolute put option is the sum of the prices of two spread put options

$$\begin{aligned} PABS(-1) &= PSSD(100, 95, 0.20, 0.15, 3, 1, 0.75, -1) \\ &\quad + PSSD(100, 95, 0.20, 0.15, 3, 1, 0.75, -1) \\ &= 4.870 + 10.060 = \$14.930\,. \end{aligned}$$

23.5. AN ALTERNATIVE INTERPRETATION

The rainbow spread options introduced and studied in this chapter can be interpreted in a few different ways. We will introduce an alternative interpretation and find possible applications for them in this section.

It is straightforward to check that the following relationship always holds for any I_1 and I_2:

$$\max\ (maxtc - K_1, 0) + \max\ (K_2 - mintc, 0) = \max(maxtc - mintc - K, 0)\,, \tag{23.11}$$

where K_1 and K_2 are the two strike prices of the call option on the maximum and the put option on the minimum of the two assets, respectively, and $\min\ (I_1, I_2) < K_2 < K_1 < \max\ (I_1, I_2)$, $K = K_1 - K_2$.

Equation (23.11) indicates that the total payoff of a call option written on the maximum of two assets together with a put written on the minimum of the same two assets is the same as that of a call option written on the spread of the maximum and minimum of the two assets with the strike price equal to the difference between the strike prices of the call and the put options. Due to the equivalence of the payoffs, a call option written on the spread of the maximum and minimum can be understood as the sum of a call option and a put option written on the maximum and the minimum of the two assets, respectively, with appropriate strike prices. These appropriate strike prices, or $\min\ (I_1, I_2) < K_2 < K_1 < \max\ (I_1, I_2)$, make both the call and put options in-the-money.

23.6. SUMMARY AND CONCLUSIONS

We have studied the correlation coefficient between the two-color rainbows and found that it is affected by parameters such as the time to maturity of the option, the current prices, the volatilities of the two assets, and especially the correlation coefficient between the returns of the two assets. Due to the uncertain nature between them, it makes sense to introduce options on the spread between the two rainbows. We have introduced spread options written on the two-color rainbows. A call option written on the spread between the two-color rainbows can be understood as a call option written on the absolute value of the difference between the two asset prices at maturity. It can also be understood as the sum of a pair of vanilla call and put options, the former written on the maximum and the latter on the minimum of the two underlying assets with the two strike prices between the two-color rainbows.

The price of an absolute option equals the sum of the price of the spread option on the spread of the first asset price over the second, and the price of the spread option on the spread of the second asset price over the first. Due to the unique nature of spread options over the rainbows, these options certainly have their particular appeals to participants in the exotic field. Our approximated closed-form solutions for their prices can certainly help us understand these options better and facilitate pricing and trading of these options.

QUESTIONS AND EXERCISES

23.1. What is a spread over the rainbows?

23.2. What are spread options over the rainbows?

23.3. Why can an option on the spread over the rainbows be considered as an option on the absolute difference of the two asset prices?

23.4. What is the relationship between an absolute option and its two corresponding spread options?

23.5. Under what conditions should absolute options be used?

23.6. What are the factors which affect the correlation coefficient between the two rainbows?

23.7. Find the price of the absolute call option to expire in one year with the strike price \$5, given the spot prices of the two stocks with $I_1 = \$100$, $I_2 = \$95$, the volatilities $\sigma_1 = 15\%$ and $\sigma_2 = 25\%$, the yields on the two stocks are $g_1 = 2\%$ and $g_2 = 4\%$, the interest rate 8%, and the two stock returns are correlated with the correlation coefficient $\rho = 50\%$.

23.8. Find the chi of the absolute call option in Exercise 23.7 using the approximaton formula in (23.10).

23.9. Find the price of the absolute put option with the same information given in Exercise 23.7.

23.10. Find the price of the absolute call option in Exercise 23.7 if the correlation coefficient is changed to -50% and other parameters remain unchanged.

Chapter 24

DUAL-STRIKE OPTIONS

24.1. INTRODUCTION

As their name implies, a dual-strike option is an option with two strike prices written on two underlying instruments. Alternative options to be studied in Chapter 26 are dual-strike options because there are two strike rates in each alternative option. Dual-strike options also include options on the maximum or minimum as a special case when the two strike prices are the same. They can be used in a variety of situations. We will price dual-strike options and find applications for them in this chapter.

In a typical Black-Scholes environment, the underlying asset returns are assumed to follow a lognormal process. Suppose that the two assets or indices $I_1(\tau)$ and $I_2(\tau)$ both follow the standard stochastic process given in (IV1) and are correlated with the correlation coefficient ρ. Let $x = \ln[I_1(\tau)/I_1]$ and $y = \ln[I_2(\tau)/I_2]$. It can be easily proven that both x and y are normally distributed with means $\mu_x = (\mu_1 - \sigma_1^2/2)\tau$ and $\mu_y = (\mu_2 - \sigma_2^2/2)\tau$ and variances $\sigma_x^2 = \sigma_1^2\tau$ and $\sigma_y^2 = \sigma_2^2\tau$, respectively. It can also be shown that x and y are bivariate normally distributed with the correlation coefficient ρ.

The payoff of a European-style dual-strike option of two assets can be expressed as

$$R = \max\{w_1[I_1(\tau) - K_1],\ w_2[I_2(\tau) - K_2],\ 0\}, \tag{24.1}$$

where max (.,.) is a function that gives the larger of two numbers; K_1 and K_2 are the two strike prices of the option; and w_1 and w_2 are two binary operators (1 for a call option and -1 for a put option).

24.2. PRICING DUAL-STRIKE OPTIONS

Since there are four combinations of the two binary operators w_1 and w_2, namely

$$(1,1),\ (1,-1),\ (-1,1),\ \text{and}\ (-1,-1),$$

there are four types of dual-strike options each for one particular combination of w_1 and w_2 specified above. Although we price dual-strike options with $w_1 = w_2 = 1$ in this chapter, the other three types of dual-strike options can be similarly obtained and we leave them as exercises at the end of this chapter.

Using the density functions given in (IV4) and (IV5), we can obtain the expected payoff of a European-style dual-strike option given in (24.1) with $w_1 = w_2 = 1$ by double integration:

$$E(R) = I_1 e^{\mu_1 \tau} A_1 + I_2 e^{\mu_2 \tau} A_2 - K_1 A_3 - K_2 A_4 \,, \tag{24.2}$$

where

$$A_1 = \int_{-\infty}^{d_1 + \sigma_1\sqrt{\tau}} f(u) N\left[\frac{q_1(u + \sigma_1\sqrt{\tau}) - \rho\sigma_1\sqrt{\tau} + \rho u}{\sqrt{1-\rho^2}}\right] du \,, \tag{24.2a}$$

$$A_2 = \int_{-\infty}^{d_2 + \sigma_2\sqrt{\tau}} f(v) N\left[\frac{q_2(v + \sigma_2\sqrt{\tau}) - \rho\sigma_2\sqrt{\tau} + \rho v}{\sqrt{1-\rho^2}}\right] dv \,, \tag{24.2b}$$

$$A_3 = \int_{-\infty}^{d_1} f(u) N\left[\frac{q_1(u) + \rho u}{\sqrt{1-\rho^2}}\right] du \,, \tag{24.2c}$$

$$A_4 = \int_{-\infty}^{d_2} f(s) N\left[\frac{q_2(v) + \rho v}{\sqrt{1-\rho^2}}\right] dv \,, \tag{24.2d}$$

$$q_1(u) = \frac{1}{\sigma_2\sqrt{\tau}}\left(\ln\left\{\frac{K_2 - K_1 + I_1 \exp[(\mu_1 - \sigma_1^2/2)\tau - u\sigma_1\sqrt{\tau}]}{I_2}\right\} - \left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\tau\right), \tag{24.2e}$$

$$q_2(v) = \frac{1}{\sigma_1\sqrt{\tau}}\left(\ln\left\{\frac{K_1 - K_2 + I_2 \exp[(\mu_2 - \sigma_2^2/2)\tau - v\sigma_2\sqrt{\tau}]}{I_1}\right\} - \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau\right), \tag{24.2f}$$

$$d_1 = \left[\ln\left(\frac{I_1}{K_1}\right) + \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] / (\sigma_1\sqrt{\tau}) \,, \tag{24.2g}$$

$$d_2 = \left[\ln\left(\frac{I_2}{K_2}\right) + \left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\tau\right] / (\sigma_2\sqrt{\tau}) \,. \tag{24.2h}$$

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate r. We can obtain the price of a dual-strike call option

(DUSTK) by discounting the expected payoff in (24.2) by the risk-free rate r:

$$DUSTK = I_1 e^{-g_1\tau} A_1 + I_2 e^{-g_2\tau} A_2 - e^{-r\tau}(K_1 A_3 + K_2 A_4)\,, \tag{24.3}$$

where

$$q_1(u) = \frac{1}{\sigma_2\sqrt{\tau}}\left(\ln\left\{\frac{K_2 - K_1 + I_1\exp[(r - g_1 + \sigma_1^2/2)\tau - u\sigma_1\sqrt{\tau}]}{I_2}\right\} - \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau\right), \tag{24.3a}$$

$$q_2(v) = \frac{1}{\sigma_1\sqrt{\tau}}\left(\ln\left\{\frac{K_1 - K_2 + I_2\exp[(r - g_2 + \sigma_2^2/2)\tau - v\sigma_2\sqrt{\tau}]}{I_1}\right\} - \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right), \tag{24.3b}$$

$$d_1 = \left[\ln\left(\frac{I_1}{K_1}\right) + \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right]/(\sigma_1\sqrt{\tau})\,, \tag{24.3c}$$

$$d_2 = \left[\ln\left(\frac{I_2}{K_2}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau\right]/(\sigma_2\sqrt{\tau})\,, \tag{24.3d}$$

and all other parameters are the same as in (24.2).

Formula (24.3) appears somewhat complicated, yet it is in terms of univariate integration as the cumulative function of the standard univariate normal distribution can be approximated with polynomial functions to a very high accuracy level. Therefore, dual-strike option prices can be calculated rather conveniently using (24.3) following the standard numerical integration methods as in pricing spread options.

Example 24.1. Find the price of a dual-strike option to expire in one year with the strike prices $K_1 = \$98$ and $K_2 = \$92$, given the spot prices of the two assets \$100 and \$95, the payout rates of the two assets zero, the interest rate 8%, the correlation coefficient between the two assets 45%, the volatilities of the two assets 20%.

Substituting $I_1 = \$100$, $I_2 = \$95$, $K_1 = \$98$, $K_2 = \$92$, $\tau = 1$, $g_1 = g_2 = 0$, $r = 0.08$, $\rho = 0.45$, $\sigma_1 = \sigma_2 = 0.20$ into (24.2) yields

$$A_1 = 0.5739,\ A_2 = 0.4520,\ A_3 = 0.4699,\ A_4 = 0.3466\,,$$

$$\begin{aligned} DUSTK &= 100 \times 0.5739 + 95 \times 0.4520 - e^{-0.08\times 1}(98 \times 0.4699 \\ &\quad + 92 \times 0.3466) \\ &= \$28.39\,. \end{aligned}$$

24.3. APPROXIMATING THE PRICING FORMULA

Although the general expression for the dual-strike option prices given in (24.3) can be calculated rather conveniently using standard numerical integration methods, they cannot, in general, be expressed in terms of the cumulative bivariate normal distribution function. However, in the special case of $K_1 = K_2$, we can find a surprisingly good approximation of (24.3) in terms of the cumulative bivariate normal distribution function. We will illustrate the approximation in this section.

Substituting $K_1 = K_2$ into (24.3e) and (24.3f) yields

$$g_1(u) = \alpha_1 - \beta_1 u\,, \tag{24.4a}$$

and

$$g_2(v) = \alpha_2 - \beta_2 v\,, \tag{24.4b}$$

where

$$\alpha_1 = \frac{1}{\sigma_2\sqrt{\tau}}\left\{ \ln\left(\frac{I_1}{I_2}\right) + \left[(g_2 - g_1) + \frac{(\sigma_2^2 - \sigma_1^2)}{2}\right]\tau \right\}, \quad \beta_1 = \frac{\sigma_1}{\sigma_2}\,,$$

$$\alpha_2 = \frac{1}{\sigma_1\sqrt{\tau}}\left\{ \ln\left(\frac{I_2}{I_1}\right) + \left[(g_1 - g_2) + \frac{(\sigma_1^2 - \sigma_2^2)}{2}\right]\tau \right\}, \quad \beta_2 = \frac{\sigma_2}{\sigma_1}\,.$$

Substituting (24.4a) and (24.4b) into (24.3) and simplifying the result using (A11.4) yields the following

$$A_1 = N_2\left(d_1 + \sigma_1\sqrt{\tau},\ \frac{\alpha_1 + (\rho + \beta_1)\sigma_1\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_1 + \beta_1^2}},\ \frac{\beta_1 - \rho}{\sqrt{1 - 2\rho\beta_1 + \beta_1^2}} \right), \tag{24.5a}$$

$$A_2 = N_2\left(d_2 + \sigma_2\sqrt{\tau},\ \frac{\alpha_2 + (\rho + \beta_2)\sigma_2\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_2 + \beta_2^2}},\ \frac{\beta_2 - \rho}{\sqrt{1 - 2\rho\beta_2 + \beta_2^2}} \right), \tag{24.5b}$$

$$A_3 = N_2\left(d_1,\ \frac{\alpha_1}{\sqrt{1 - 2\rho\beta_1 + \beta_1^2}},\ \frac{\beta_1 - \rho}{\sqrt{1 - 2\rho\beta_1 + \beta_1^2}} \right), \tag{24.5c}$$

and

$$A_4 = N_2\left(d_2,\ \frac{\alpha_2}{\sqrt{1 - 2\rho\beta_2 + \beta_2^2}},\ \frac{\beta_2 - \rho}{\sqrt{1 - 2\rho\beta_2 + \beta_2^2}} \right), \tag{24.5d}$$

where $N_2(A, B, C)$ is the cumulative function of the standard bivariate normal distribution with the upper limits A and B and the correlation coefficient C.

The crucial point in approximating (24.3) is the two intermediate functions q_1 and q_2 given in (24.2e) and (24.2f). Comparing the general expressions of q_1 and q_2 in (24.2e) and (24.2f) and their simplified forms in (24.4a) and (24.4b) naturally leads us to the following simplifications:

$$q_1(u) = \left[1 + \eta_1 \ln\left(\frac{K_2}{K_1}\right)\right](\alpha_1 - \beta_1 u), \tag{24.6a}$$

and

$$q_2(v) = \left[1 + \eta_2 \ln\left(\frac{K_1}{K_2}\right)\right](\alpha_2 - \beta_2 v), \tag{24.6b}$$

where $\eta_1 > 0$ and $\eta_2 > 0$ are two positive numbers which can be specified in various problems according to various accuracy levels.

It is obvious that both $q_1(u)$ and $q_2(v)$ in (24.6) become the same as those given in (24.4) when $K_1 = K_2$, regardless of the values of η_1 and η_2. Therefore, the approximations in (24.6) include the special case of $K_1 = K_2$ in (24.4). The positiveness of η_1 (resp. η_2) captures the effects of the strike price difference $K_2 - K_1$ (resp. $K_1 - K_2$) on $q_1(u)$ [resp. $q_2(v)$] and its magnitude should reflect the sensitivity of the effects of the strike price difference $K_2 - K_1$ (resp. $K_1 - K_2$) on $q_1(u)$ [resp. $q_2(v)$]. For different problems, the magnitude of the $\eta_1 + \eta_2$ can be different for different accuracy requirements.

Since the approximations in (24.6) are linear functions of the integration variables, we can find the approximating expression of the dual-strike option price following the same steps as in obtaining (24.5). Substituting (24.6a) and (24.6b) into (24.3) and simplifying the result yields the following

$$ADUSTR = I_1 e^{-g_1\tau} A_1 + I_2 e^{-g_2\tau} A_2 - e^{-r\tau}(K_1 A_3 + K_2 A_4), \tag{24.7}$$

where

$$A_1 = N_2\left(d_1 + \sigma_1\sqrt{\tau},\ \frac{\alpha_1' + (\rho + \beta_1')\sigma_1\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_1' + \beta_1'^2}},\ \frac{\beta_1' - \rho}{\sqrt{1 - 2\rho\beta_1' + \beta_1'^2}}\right), \tag{24.7a}$$

$$A_2 = N_2\left(d_2 + \sigma_2\sqrt{\tau},\ \frac{\alpha_2' + (\rho + \beta_2')\sigma_2\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_2' + \beta_2'^2}},\ \frac{\beta_2' - \rho}{\sqrt{1 - 2\rho\beta_2' + \beta_2'^2}}\right), \tag{24.7b}$$

$$A_3 = N_2\left(d_1,\ \frac{\alpha_1'}{\sqrt{1 - 2\rho\beta_1' + \beta_1'^2}}\ \frac{\beta_1' - \rho}{\sqrt{1 - 2\rho\beta_1' + \beta_1'^2}}\right), \tag{24.7c}$$

$$A_4 = N_2\left(d_2,\ \frac{\alpha_2'}{\sqrt{1 - 2\rho\beta_2' + \beta_2'^2}}\ \frac{\beta_2' - \rho}{\sqrt{1 - 2\rho\beta_2' + \beta_2'^2}}\right), \tag{24.7d}$$

$$\alpha_1' = \alpha_1\left[1 + \eta_1 \ln\left(\frac{K_2}{K_1}\right)\right], \quad \beta_1' = \beta_1\left[1 + \eta_1 \ln\left(\frac{K_2}{K_1}\right)\right], \tag{24.7e}$$

$$\alpha_2' = \alpha_2\left[1 + \eta_2 \ln\left(\frac{K_1}{K_2}\right)\right], \quad \beta_2' = \beta_2\left[1 + \eta_2 \ln\left(\frac{K_1}{K_2}\right)\right], \tag{24.7f}$$

and all other parameters are the same as in (24.4), (24.5), and (24.6).

24.4. SUMMARY AND CONCLUSIONS

A dual-strike option is an option written on two underlying assets or instruments with two strike prices. There are four types of dual-strike options resulting from the four combinations of the two option binary operators w_1 and w_2, (1, 1), (1, −1), (−1, 1), and (−1, −1), respectively. We obtained a pricing formula in closed-form for dual-strike options with the binary operators $w_1 = w_2 = 1$ and approximated the formula with Black-Scholes type expressions.

QUESTIONS AND EXERCISES

24.1. What is a dual-strike option?

24.2. How many types of dual-strike options are there?

24.3. Find the price of a dual-strike option to expire in nine months with the strike prices K_1 = \$96 and K_2 = \$93, given the spot prices of the two assets \$100 and \$95, the payout rates of the assets zero, the interest rate 7%, the correlation coefficient between the two assets 65%, the volatilities of the two assets 15% and 20%, respectively.

24.4. Find the price of the dual-strike option in Exercise 24.3 if the correlation coefficient is changed to −25% and other parameters remain unchanged.

24.5. Find the price of the dual-strike option in Exercise 24.3 if the payout rates of the two assets are 2% and 3% and other parameters remain unchanged.

24.6.* Show that $\alpha_1 = -\beta_1\alpha_2$ in (24.4).

24.7.* Find a general pricing formula for dual-strike option with $\omega_1 \neq 1$ and $\omega_2 \neq 1$.

Chapter 25

OUT-PERFORMANCE OPTIONS

25.1. INTRODUCTION

An out-performance option is a special call option which allows investors to take advantage of the expected difference in the relative performance of two underlying assets or indices. Its payoff at maturity is the performance of one instrument minus that of a second instrument and a prespecified strike rate multiplied by a fixed notional or face amount. The performance is normally measured by the rate of return in percentages. The underlying instruments can be any combination of stocks, bonds, currencies, commodities, or indices based on any of these instruments. One popular combination of out-performance instruments is a bond index and a stock index or vice versa. A particularly popular out-performance option in recent years has been the yield spread option, which pays off on the basis of the difference between the yields on the fixed income securities in two different countries or at two different points on one country's yield curve.

Out-performance options are also used to capitalize the relative performance of two stock markets such as the US stock market measured by Standard & Poor's 500 relative to the Japanese market measured by Nikkei 225.[1] Actually, an out-performance option can be viewed as a spread option between the returns of two instruments rather than the actual instrument values in the case of spread options. Gastineau (1993a) first discussed out-performance options and described how they work. An investor of an out-performance option should have confidence not only in the good performance of one instrument, but also expect the poor performance from the other instrument.

[1]Nikkei 225 is a Japanese stock market index composed of the stocks of 225 companies listed in the First Section of the Tokyo Stock Exchange (TSE). These stocks are highly liquid and account for approximately 60% of the total market value of the 1100 plus issues listed in the First Section. See Zhang (1995e) for other indices of the Japanese stock market.

An out-performance option is different from most other options covered so far in this book. The difference lies in the fact that it gives a return rate rather than an actual payoff in dollar values as in most other options. From this important difference, an out-performance option contract always contains one additional factor or parameter called notional amount or face amount. The total payoff of an out-performance option in dollar terms is simply the product of the notional amount and the payoff rate at maturity. Since the notional amount is prespecified, we can simply concentrate on the payoff rate of an out-performance option.

We will price out-performance options and show how they can be used in this chapter. As performance can be measured using either net returns or log-returns, the out-performance option pricing formulas can be rather different depending on what performance measure is used. We will provide an exact pricing formula for out-performance options in a Black-Scholes environment with log-returns as the performance measure. Although strict closed-form solutions for out-performance options do not exist if actual returns are used as the performance measure, we will provide an exact pricing formula in the form of univariate integrations and approximate these integrations with closed-form formulas. We confine our analysis to a Black-Scholes environment for the purpose of transparency as well as easy comparison.

25.2. OUT-PERFORMANCE OPTIONS

The payoff rate of a European-style out-performance option (PROUT) written on the spread of the relative performance of two underlying instruments can be expressed as

$$PROUT = \max\left\{w\left[\frac{I_1(\tau)}{I_1} - \frac{I_2(\tau)}{I_2}\right] - wk,\ 0\right\}, \tag{25.1}$$

where I_1 and I_2 represent the current values of the two underlying instruments, $I_1(\tau)$ and $I_2(\tau)$ are the values of the two instruments at the option maturity, k is the strike rate of the option; $\tau = t^* - t$ is the time to maturity of the option, and w is a binary operator (1 for a call option and -1 for a put option).

The payoff rate expression in (25.1) appears very similar to the payoff expressions of all other options covered so far in this book, yet it is different from them all. The difference lies in the fact that it gives a return rate rather than an actual payoff in dollar values as in all other options. All out-performance option contracts contain one additional factor or parameter called notional amount or face amount as in swap contracts.[2] The total payoff of an out-performance option in dollar terms is simply the product of the notional amount and the payoff rate at maturity given in (25.1). Since the notional amount is prespecified, we can simply concentrate on the payoff rate.

Assume that the two underlying instruments follow the same stochastic process in (IV1) specified at the beginning of Part IV and the two instruments are correlated with the correlation coefficient ρ. Let $x = \ln[I_1(\tau)/I_1]$ and $y = \ln[I_2(\tau)/I_2]$. It can be easily proven that both x and y are normally distributed with means $\mu_x = (\mu_1 - \sigma_1^2/2)\tau$ and $\mu_y = (\mu_2 - \sigma_2^2/2)\tau$, and variances $\sigma_x^2 = \sigma_1^2\tau$ and $\sigma_y^2 = \sigma_2^2\tau$, respectively. It can also be shown that x and y are bivariate normally distributed with the correlation coefficient ρ. The payoff rate of an out-performance option in (25.1) can be expressed in terms of the gross returns of the two instruments:

$$PROUT = \max\left[w(e^x - e^y) - wk,\ 0\right], \tag{25.2}$$

where k is the same strike rate as in (25.1).

The gross return of any asset always equals one plus its corresponding net return, i.e., the gross return

$$\frac{S(\tau)}{S} = 1 + \frac{S(\tau) - S}{S} = 1 + \text{net return},$$

where $S(\tau)$ and S stand for the forward price and the spot price of the underlying asset, respectively. Thus, the difference between the two gross returns in (25.1) or (25.2) is the same as that between the two corresponding net returns. Besides the gross return and the net return, there is another return called the log-return — the logarithm of the gross return. Since net returns are normally small percentages, log-returns can be approximated using the Taylor series expansion as follows:

$$\ln(1+\nu) \cong \nu + \mathrm{O}(\nu^2), \tag{25.3}$$

where ν represents the net return and $\mathrm{O}(\nu^2)$ represents a summation of all ν terms of powers equal to or higher than two.

[2]Swaps are another kind of financial derivatives which have been heavily trading in the OTC marketplace in the past two decades. In a standard or vanilla swap, one party agrees to pay the other party a fixed rate and the other party agrees to pay a floating rate such as the LIBOR (London Interbank Offered Rate) for a prespecified period of time, normally a few years with certain payment frequency. The fixed rate payee is called the buyer of the swap as he pays a fixed rate in exchange for a floating rate. The net payment or receivable for each party is simply the product of a prespecified notional amount usually in millions of dolars and the difference between his promised pay rate and receive rate.

Due to the relationship in (25.3), log-returns are often not distinguished from net returns in practical use. Actually, the volatility of the underlying asset σ in the Black-Scholes model is not the volatility of the return of the underlying asset but that of the log-return. If we use log-returns rather than gross returns, the payoff rate of an out-performance option (PROUTLG) given in (25.2) can be expressed alternatively:

$$PROUTLG = \max\left[w(x-y) - wk,\ 0\right], \tag{25.4}$$

where k is the same strike price as in (25.1) and (25.2).

25.3. PRICING OUT-PERFORMANCE OPTIONS WITH GROSS RETURNS AS PERFORMANCE MEASURE

Assume that the two underlying asset prices are lognormally distributed as in (IV1) and the log-returns of the two assets are then bivariate normally distributed with a bivariate density function given in (IV4) or (IV5). Figure 25.1 depicts the integration domain of an out-performance option. The straight line starts from $I_1 = k$ and its slope is the ratio of the current price of the second instrument over that of the first. The integration domain is very similar to that of a spread call option.

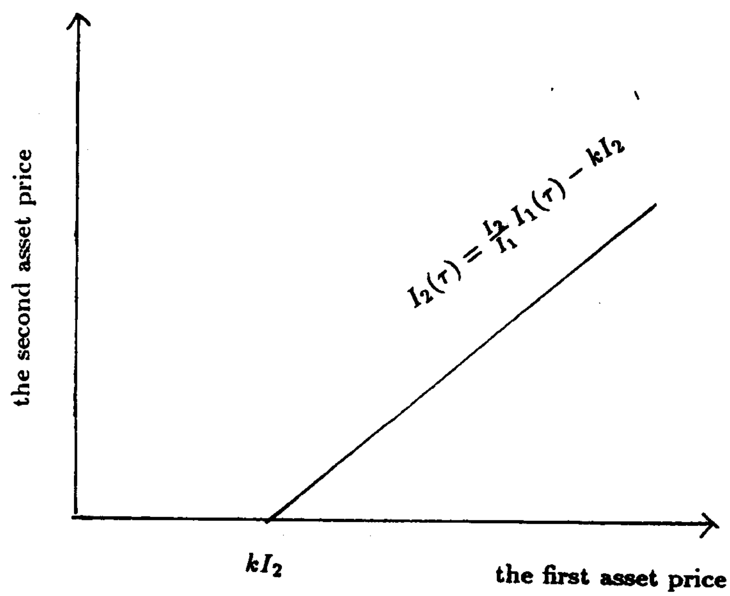

Fig. 25.1. The integration domain for an out-performance option.

Using the bivariate normal density function given in (IV4) or (IV5) and the integration domain shown in Figure 25.1, we can obtain the expected payoff rate of an out-performance call option given in (25.2) by double integration:

$$E(PROUT) = e^{\mu_1 \tau} A_{o1} - e^{\mu_2 \tau} A_{o2} - k A_{o3}\,, \tag{25.5}$$

where

$$A_{o1} = \int_{-\infty}^{\infty} f(v) N\left[\frac{\zeta_1 + \sigma_1\sqrt{\tau} + \rho v + \phi_o(v + \rho\sigma_1\sqrt{\tau})}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$A_{o2} = \int_{-\infty}^{\infty} f(v) N\left[\frac{\zeta_1 + \rho\sigma_2\sqrt{\tau} + \rho v + \phi_o(v + \sigma_2\sqrt{\tau})}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$A_{o3} = \int_{-\infty}^{\infty} f(v) N\left[\frac{\zeta_1 + \rho v + \phi_o(v)}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$\zeta_1 = \frac{\mu_x}{\sigma_x} = \sqrt{\tau}\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)/\sigma_1\,,$$

and

$$\phi_o(v) = -\frac{1}{\sigma_1\sqrt{\tau}} \ln\left\{k + e^{(\mu_2 - \sigma_2^2/2)\tau + v\sigma_2\sqrt{\tau}}\right\}.$$

The expected payoff rate in (25.5) is similar to the expected payoff expression of a spread option given in (22.4). From this similarity, we let X_o stand for the parameter in the out-performance option corresponding to any parameter X in the spread option expression. Although the three parameters A_{o1}, A_{o2}, and A_{o3} look complicated, they can be calculated very conveniently using any numerical integration method with computers as both the density and cumulative functions of the standard normal distribution can be much more easily computed than the bivariate normal distribution.

If we compare the three parameters A_{o1}, A_{o2}, and A_{o3} in (25.5) with the corresponding three parameters A_1, A_2, and A_3 in (22.2) for simple spread options, we can find that the functional forms of these parameters are exactly the same. Thus, the former can be obtained simply by substituting the parameter d and the function $\phi(v)$ with ζ_1 and $\phi_0(v)$, respectively.

The arbitrage argument permits us to use the risk-neutral valuation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate. Hence, we can obtain the out-performance call option price (OUTP) by substituting $\mu_i = r - g_i$ (g_1 and g_2 are the payout rates of the two underlying assets) and discounting the expected payoff rate given in (25.5) by the risk-free rate r,

$$OUTP = e^{-g_1 \tau} A_{o1} - e^{-g_2 \tau} A_{o2} - k e^{-r\tau} A_{o3}\,, \tag{25.6}$$

where A_{o1}, A_{o2}, and A_{o3} are the same as in (25.5) and

$$\zeta_1 = \frac{\mu_x}{\sigma_x} = \sqrt{\tau}\Big(r - g_1 - \frac{1}{2}\sigma_1^2\Big)/\sigma_1\,,$$

and

$$\phi_o(v) = -\frac{1}{\sigma_1\sqrt{\tau}}\ln\left[k + e^{(r-g_2-\sigma_2^2/2)\tau + v\sigma_2\sqrt{\tau}}\right].$$

There is one important property of the pricing formula in (25.6) — level-free. We can check that the spot values or levels of the two instruments I_1 and I_2 do not appear in neither (25.6) nor any of the parameters it requires. This level-free property is not surprising because out-performance options determine payoff rates rather than actual payoffs. Therefore, the theoretical risk-neutral pricing formula for out-performance options is independent of the two spot level prices and is a function of (i) the interest rate r, (ii), the payout rates of the two underlying assets g_1 and g_2, (iii) the strike rate k, (iv) the time to maturity τ, (v) the volatilities of the returns of the underlying assets, and (vi) the correlation coefficient ρ between the returns of the two assets. However, it should be kept in mind that the actual payout rates are normally calculated at option maturity using (25.1) with both the spot prices and the prices of the underlying assets at maturity because the returns are not directly observable.

Example 25.1. Find the prices of the out-performance options of the US stock market over the Japanese stock market to expire in six months with the strike rates 2% and 1%, given the interest rate 6%, the payout rate of the S&P 500 Index 5%, the payout rate of the Nikkei 225 Index 2%, the correlation coefficient between the returns of the two indexes 50%, the volatilities of the two indexes 20% and 18%, respectively.

Substituting $\tau = 0.50$, $k = 0.02$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (25.6) yields

$$\zeta_1 = \frac{\mu_x}{\sigma_x} = \sqrt{0.50}\left(0.06 - 0.05 - \frac{1}{2}\times 0.18^2\right)/0.20 = -0.0219\,,$$

$$A_{o1} = 0.3957,\ A_{o2} = 0.3448,\ A_{03} = 0.3745\,,$$

and the call option

$$\begin{aligned} OUTP(k = 0.02) &= e^{-0.05\times 0.5}0.3957 - e^{-0.05\times 0.5}0.3448 \\ &\quad - 0.02e^{-0.06\times 0.5}0.3745 \\ &= 3.73\%\,, \end{aligned}$$

and substituting $\tau = 0.50$, $k = 0.01$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (25.6) yields

$$A_{o1} = 0.4242, \ A_{o2} = 0.3722, \ A_{o3} = 0.4030,$$

and the call option

$$\begin{aligned} OUTP(k = 0.01) &= e^{-0.05\times 0.5}0.4242 - e^{-0.05\times 0.5}0.3722 \\ &\quad - 0.01e^{-0.06\times 0.5}0.4030 \\ &= 4.13\%. \end{aligned}$$

The pricing formula given in (25.6) is not in closed-form in a strict sense because the three parameters A_{o1}, A_{o2} and A_{o3} cannot, in general, be expressed in terms of the cumulative functions of the standard normal distribution. For special out-performance options with zero strike rate ($k = 0$), however, these three parameters can be expressed in terms of standard normal cumulative functions. Substituting $k = 0$ into the ϕ_o function given in (25.6) yields

$$\phi_{o0}(v) = -\zeta_2 - v\beta_{o0}, \tag{25.7}$$

where

$$\zeta_2 = \frac{\mu_y}{\sigma_x} = \sqrt{\tau}\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)/\sigma_1 \text{ and } \beta_{o0} = \frac{\sigma_2}{\sigma_1}.$$

Substituting the linear expression in (25.7) into (25.6) yields the following

$$OUTP(k = 0) = e^{-g_1\tau}A_{o10} - e^{-g_2\tau}A_{o20}, \tag{25.8}$$

where

$$A_{o10} = N\left[\frac{\zeta_1 - \zeta_2 + (1 - \rho\beta_{o0})\sigma_1\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_{o0} + \beta_{o0}^2}}\right],$$

and

$$A_{o20} = N\left[\frac{\zeta_1 - \zeta_2 + (\rho - \beta_{o0})\sigma_2\sqrt{\tau}}{\sqrt{1 - 2\rho\beta_{o0} + \beta_{o0}^2}}\right].$$

The formula in (25.8) is in closed-form and it is of the Black-Scholes type. We will give an example to show how to use this formula immediately.

Example 25.2. Find the prices of the out-performance options in Example 25.1 with zero strike rate.

Since the strike rate $k = 0$, we can use the closed-form solution in (25.8) to find the option prices. Substituting the given parameters into (25.8) yields

$$\zeta_1 = \frac{\mu_x}{\sigma_x} = \sqrt{0.50}\left(0.06 - 0.05 - \frac{1}{2} \times 0.18^2\right)/0.18 = -0.0243\,,$$

$$\zeta_2 = \frac{\mu_y}{\sigma_x} = \sqrt{0.50}\left(0.06 - 0.02 - \frac{1}{2} \times 0.20^2\right)\Big/0.18 = 0.0786\,,$$

$$\beta_{o0} = \frac{\sigma_2}{\sigma_1} = \frac{0.20}{0.18} = 1.1111\,,$$

$$A_{o10} = N\left[\frac{-0.0243 - 0.0786 + (1 - 0.50 \times 1.1111) \times 0.18\sqrt{0.50}}{\sqrt{1 - 2 \times 0.50 \times 1.1111 + 1.1111^2}}\right] = 0.4826\,,$$

$$A_{o20} = N\left[\frac{-0.0219 - 0.0707 + (0.50 - 1.1111) \times 0.20\sqrt{0.50}}{\sqrt{1 - 2 \times 0.50 \times 1.1111 + 1.1111^2}}\right] = 0.4291\,,$$

$$\begin{aligned} OUTP(k = 0) &= e^{-g_1\tau}A_{o10} - e^{-g_2\tau}A_{o20} \\ &= e^{-0.05\times0.5}0.4826 - e^{-0.02\times0.5}0.4291 \\ &= 4.59\%\,. \end{aligned}$$

25.4. AN APPROXIMATING CLOSED-FORM FORMULA

An exact closed-form solution in terms of the cumulative univariate normal distribution functions for out-performance options is very unlikely in general because $\phi_o(v)$ is a nonlinear function of the integrating variable v when $k \neq 0$. Since the strike rate k is normally in percentages, the volatilities of the two underlying assets are also in percentages, and the time to maturity of most out-performance options is around one year, the function $\phi_o(v)$ is nearly a linear function of v between -4 and 4 for most reasonable parameters as for spread options. Outside the range, the standard normal density function $f(v)$ is very close to zero and thus could be neglected without affecting accuracy.

Using the approximation method, we may linearize $\phi_o(v)$ as follows

$$\phi(v) = \alpha_o - \beta_o v\,, \tag{25.9}$$

where

$$\alpha_o = \frac{5}{4}\ln(1 + k) - \zeta_2 \text{ and } \beta_o = \frac{\sigma_2}{\sigma_1}\,.$$

It is obvious that the linearization in (25.9) becomes the special linearization in (25.7) when $k = 0$. Better linearization results are possible, yet they are not the focus of this chapter and we choose not to illustrate these linearization processes here.

The three parameters in (25.5) can be simplified using (25.9) as follows:

$$A_{o1} = N[\gamma_o + (1 - \beta_o\rho)w_o\sigma_1\sqrt{\tau}]\,, \tag{25.10a}$$

$$A_{o2} = N[\gamma_o + (\rho - \beta_o)w_o\sigma_2\sqrt{\tau}]\,, \tag{25.10b}$$

and

$$A_{o3} = N(\gamma_o)\,, \tag{25.10c}$$

where

$$w_o = \frac{1}{\sqrt{1 - 2\rho\beta_o + \beta_o^2}} \quad \text{and} \quad \gamma_o = (\alpha_o + \zeta_1)w_o\,.$$

The call option price in (25.6) can thus be approximated using the approximations given in (25.10a)–(25.10c):

$$OUTP \cong e^{-g_1\tau}A_{o1} - e^{-g_2\tau}A_{o2} - ke^{-r\tau}A_{o3}\,. \tag{25.11}$$

25.5. PRICING OUT-PERFORMANCE OPTIONS WITH LOGARITHM RETURNS AS PERFORMANCE MEASURE

Using the bivariate density function given in (IV4) and (IV5), we can obtain the expected payoff rate of an out-performance option in (25.4) by double integration:

$$E(PROUTLG) = w(\mu_x - \mu_y - k)N(w\gamma_{lr}) + w\sigma_a\sqrt{\tau}f(\gamma_{lr})\,, \tag{25.12}$$

where

$$\gamma_{lr} = \frac{\mu_x - \mu_y - k}{\sigma_a\sqrt{\tau}}\,,$$

and $N(.)$ and $f(.)$ are the cumulative and density functions of a standard normal distribution and σ_a is the same as in (14.5).

The derivation of (25.12) is rather long, thus we skip it here. Interested readers may find an outline of the derivation in Appendix of this chapter. The expected payoff rate given in (25.12) is essentially the expected payoff for each unit of the notional amount of the out-performance option. We can obtain the out-performance option price (OUTOP) by substituting $\mu_x =$

$(r - g_1 - \sigma_1^2/2)\tau$, $\mu_y = (r - g_2 - \sigma_2^2/2)\tau$, $\sigma_x = \sigma_1\sqrt{\tau}$, and $\sigma_y = \sigma_2\sqrt{\tau}$ into (25.12) and discounting it by the risk-free rate r:

$$OUTOP = we^{-r\tau}\left(\left\{\left[g_2 - g_1 + \frac{(\sigma_2^2 - \sigma_1^2)}{2}\right]\tau - k\right\}N(w\gamma_{lr}) + w\sigma_a\sqrt{\tau}f(\gamma_{lr})\right), \tag{25.13}$$

where

$$\gamma_{lr} = \left[\left(g_1 - g_2 + \frac{\sigma_1^2 - \sigma_2^2}{2}\right)\tau - k\right]/(\sigma_a\sqrt{\tau}).$$

The pricing formula in (25.13) is very similar in functional form to that in Brennan (1979), which assumes a normal distribution of the underlying assets. This is not surprising because in (25.13), performance is measured with logarithm returns which are normally distributed. Since individual logarithm returns are normally distributed, the performance spread is also normally distributed. Since, an out-performance option is written on the performance spread of two assets, its pricing formula should be similar to Brennan's pricing formula with a normal distribution assumption.

The pricing formula in (25.13) is an exact closed-form solution. It possesses the level-free property of the pricing formula in (25.6) when gross returns are used as the performance measure. Therefore, the theoretical risk-neutral pricing formula for out-performance options is independent of the two spot level prices. Although both (25.6) and (25.13) are functions of the same parameters, the latter is much simpler than the former. It is even simpler than the approximated formula in (25.11).

Example 25.3. Given the information in Example 25.1, what are the out-performance call option prices if log-return are used as the performance measure?

Substituting $\tau = 0.50$, $k = 0.02$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (25.13) yields

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = 0.1908,$$

$$\gamma_{lr} = \left[\left(g_2 - g_1 + \frac{\sigma_2^2 - \sigma_1^2}{2}\right)\tau - k\right]\frac{1}{\sigma_a\sqrt{\tau}} = -0.2453,$$

$$\begin{aligned} OUTOPLR &= e^{-r\tau}\left(\left\{[g_2 - g_1 + (\sigma_2^2 - \sigma_1^2)/2]\tau - k\right\}N(\gamma_{lr}) + \sigma_a\sqrt{\tau}f(\gamma_{lr})\right) \\ &= e^{-0.06\times0.5}\left(\left\{[0.02 - 0.05 + (0.20^2 - 0.18^2)/2]0.5 - 0.02\right\}\right. \\ &\quad \left.\times N(-0.2453) + 0.1908\sqrt{0.50}f(-0.2453)\right) \\ &= 3.79\%; \end{aligned}$$

and substituting $\tau = 0.50$, $k = 0.01$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (25.13) yields

$$\begin{aligned}\gamma_{lr} &= \left[\left(g_2 - g_1 + \frac{\sigma_2^2 - \sigma_1^2}{2}\right)\tau - k\right]\frac{1}{\sigma_a\sqrt{\tau}} \\ &= -0.1711\,,\end{aligned}$$

$$\begin{aligned}OUTOPLR &= e^{-r\tau}\Big(\Big\{[g_2 - g_1 + (\sigma_2^2 - \sigma_1^2)/2]\tau - k\Big\}N(\gamma_{lr}) + \sigma_a\sqrt{\tau}f(\gamma_{lr})\Big) \\ &= e^{-0.06\times0.5}\Big(\Big\{[0.02 - 0.05 + (0.20^2 - 0.18)/2]0.5 - 0.01\Big\} \\ &\quad \times N(-0.1711) + 0.1908\sqrt{0.50}f(-0.1711)\Big) \\ &= 4.19\%\,.\end{aligned}$$

Comparing the results in Examples 25.1 and 25.3, we can find that the difference between the out-performance option prices using net returns as the performance measure in Example 25.1 and those using log-returns in Example 25.3 is

$$0.0419 - 0.0413 = 0.0379 - 0.0373 = 0.0006 = 0.06\%\,,$$

which is less than one tenth of a percentage. The small difference is expected because the difference between net returns and log-returns is rather small in general as shown in (25.3). Yet the pricing formula in (25.13) is much more convenient than the one in (25.6), because the former is strictly in closed-form and the latter is in terms of univariate integrations. We will discuss more about (25.13) in the following section.

25.6. GREEKS FOR OUT-PERFORMANCE OPTIONS

The closed-form solution in (25.13) using log-returns as the performance measure is very close to the one in (25.6) using net returns as the performance measure because the two measures are very similar. In general, the closed-form solution in (25.13) is much more convenient to use than that in (25.6). Due to its simplicity, we can obtain the Greeks very conveniently using (25.13). As an example to show how they can be obtained using (25.13), we will find the chi of an out-perfomance option.

Taking partial derivative of (25.13) with respect to the correlation coefficient ρ yields the chi of an out-performance option:

$$\text{chi} = \left(e^{-r\tau}\left\{\left[g_2 - g_1 + \frac{(\sigma_2^2 - \sigma_1^2)}{2}\right]\tau - k - w\gamma_{lr}\sigma_a\sqrt{\tau}\right\}\right) f(\gamma_{lr})\frac{\sigma_1\sigma_2\gamma_{lr}}{\sigma_a^2}\,, \tag{25.14}$$

where all parameters are the same as in (25.13).

Example 25.4. Find the chi of the out-performance option with the strike rate $k = 2\%$ in Example 25.3.

Substituting $\tau = 0.50$, $k = 0.02$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (25.14) yields

$$OUTOPLR = \left(e^{-r\tau}\left\{[g_2 - g_1 + (\sigma_2^2 - \sigma_1^2)/2]\tau - k - \gamma_{lr}\sigma_a\sqrt{\tau}\right\}\right)f(\gamma_{lr})\frac{\sigma_1\sigma_2\gamma_{lr}}{\sigma_a^2}$$

$$= \left(e^{-0.06\times0.5}\left\{[0.02 - 0.05 + (0.20^2 - 0.18^2)/2\right)0.5 - 0.02$$

$$- (-0.2453) \times 0.1908\sqrt{0.50}\Big]f(-0.2453)$$

$$\times (-0.2453 \times 0.18 \times 0.20)/0.1908^2 = -0.006\%.$$

25.7. SUMMARY AND CONCLUSIONS

Out-performance options can be used to take advantage of investors' perception of the relative performance of two underlying assets. A buyer needs to specify which asset or index performs better and which performs worse so that the performance spread can be made significant. We have provided closed-form solutions for out-performance options when log-returns are used as the performance measure. We have also found nearly closed-form solutions for out-performance options in terms of univariate integrations. These univariate integrations can be calculated easily using any numerical methods with computers. We have also found appropriate approximations for these univariate integrations in terms of univariate normal cumulative functions.

QUESTIONS AND EXERCISES

Questions

25.1. What is an out-performance option?

25.2. How is performance normally measured?

25.3. Why are the results using the pricing formula in (25.6) very similar to those using the pricing formula in (25.13)?

25.4. What are the advantages of the pricing formula in (25.13) over that in (26.6)?

25.5. Why are the pricing formulas of out-performance options level-free (or independent of the spot prices of the two underlying assets)?

25.6. Why is the pricing formula in (25.13) similar to the option pricing formula with the normal distribution assumption in Brennan (1979)?

Exercises

25.1. Find the prices of the out-performance options of the US stock market over the Japanese stock market to expire in nine months with the strike rates 2% and 1%, respectively given the interest rate 7%, the payout rate of the S&P 500 Index 4%, the payout rate of the Nikkei 225 Index 2.5%, the correlation coefficient between the returns of the two indexes 35%, and the volatilities of the two indexes are 20% and 18%, respectively.

25.2. Find the prices of the out-performance options in Exercise 25.1 using the closed-form solution given in (25.13).

25.3. Find the chi of the out-performance options in Exercise 25.2 using the closed-form expression given in (25.14).

25.4. Find the prices of the out-performance options in Exercise 25.1 if the volatility of the Japanese stock return is changed to 25% and other parameters remain unchanged.

25.5. Find the chi for the out-performance options in Exercise 25.4.

25.6. Show the chi expression given in (25.14).

25.7. Find the vega of an out-perfomance option with respect to the volatility of the first asset.

25.8. Find the vega of an out-perfomance option with respect to the volatility of the second asset.

25.9. Find the theta of an out-perfomance option using (25.13).

25.10. Find the prices of the out-performance options in Exercise 25.1 if the correlation coefficient between the two stock indexes is changed to 75% and other parameters remain unchanged.

APPENDIX

Since the derivation is very long, we will only list the important steps of the derivation and interested readers may check the derivation in more detail with these steps. The following two formulas about the bivariate normal distribution are used to derive the results in (25.12):

$$\int_a^\infty v f(v|u)dv = \rho u\, N\left(\frac{\rho u - a}{\sqrt{1-\rho^2}}\right) + \sqrt{1-\rho^2} f\left(\frac{\rho u - a}{\sqrt{1-\rho^2}}\right), \qquad \text{(A25.1)}$$

and

$$\int_{-\infty}^a v f(v|u)dv = \rho u\, N\left(\frac{a - \rho u}{\sqrt{1-\rho^2}}\right) - \sqrt{1-\rho^2} f\left(\frac{\rho u - a}{\sqrt{1-\rho^2}}\right), \qquad \text{(A25.2)}$$

where f(.) and N(.) are the density and cumulative functions of a standard normal distribution, respectively.

For a call option written on the performance spread $x - y$, the expected payoff rate is given from (25.4) by integrating y first:

$$E(PROUTLR) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(x - y - k,\ 0) f(x, y) dy dx$$
$$= \int_{-\infty}^{\infty} \int_{y-k}^{\infty} (x - y - k) f(x, y) dx dy\,. \tag{A25.3}$$

Making the standard substitution $u = (x - \mu_x)/\sigma_x$ and $v = (y - \mu_y)/\sigma_y$ and the integration in (A25.3) becomes

$$E(PROUTLR) = \int_{-\infty}^{\infty} \int_{low\, b}^{\infty} [(\mu_x - \mu_y - k) + (\sigma_x u - \sigma_y v)] f(u, v) du dv\,, \tag{A25.4}$$

where $low\ b = (\mu_y - \mu_x - k + \sigma_y v)/\sigma_x$.

Carrying out the bivariate integration using the two formulas in (A25.1) and (A25.2), and the results in Appendix of Chapter 15 yields (25.12) after several steps of simplifications.

Chapter 26

ALTERNATIVE OPTIONS

26.1. INTRODUCTION

Alternative options are also called either-or options or best-of-two options. They pay the best performing of two distinct security or index call options at maturity. Any payoff is equal to the positive percentage change from the strike on the best performing of the two assets multiplied by a prespecified notional or face amount. Alternative options are similar to out-performance options in the sense that they are both used to quantify the relative performance of two assets or indices and that they both provide payoff rates rather than payoff values. A notional or face amount is necessary for both out-performance and alternative options. An alternative option contract consists of two call options with different strike rates on different underlying instruments. The strike prices are usually at-the-money. When an alternative option expires, the option holder receives the payout of the most valuable of the two component options. There is no payout on the other less valuable option even if it is in-the-money at maturity.

The properties we have discussed about alternative options are for alternative call options which pay the best performing of two call options on two underlying assets. An alternative put option is sometimes called a worst-of-two option with the payout rate based on the worst performing of two underlying instruments. Although an alternative option is similar to an out-performance option in some respects, they are quite different in others. Whareas the payout rate of the latter is based on the relative performance of two underlying instruments, that of the former is based on the performance of the best or worst of two options. An alternative option can have a significant payout when the out-performance option is valueless, and the latter may have an attractive payout when the former is valueless.

Alternative options are used by asset managers who want exposure to the best performing of two asset classes in one market or to the best performing of two markets. The asset classes can be either stocks or bonds, and the markets can be US, Japanese, or German stock markets. Applications in the foreign exchange market can give a liability manager optional exposure to the best performing of two currencies relative to a third or to the best performing of two interest rates denominated in different currencies.

Gastineau (1993a) first discussed how alternative options work and how to apply them in practice, yet no efforts have been made to price these options. We will price alternative options in a Black-Scholes environment and find applications for them in this chapter.

26.2. ALTERNATIVE OPTIONS

In a typical Black-Scholes environment, the underlying asset returns are assumed to follow a lognormal process. Suppose there are two assets or indices $I_1(\tau)$ and $I_2(\tau)$ both following the standard stochastic process given in (IV1) and the two instruments are correlated with the correlation coefficient ρ. Let $x = \ln[I_1(\tau)/I_1]$ and $y = \ln[I_2(\tau)/I_2]$. It can be proven that both x and y are normally distributed with means $\mu_x = (\mu_1 - \sigma_1^2/2)\tau$ and $\mu_y = (\mu_2 - \sigma_2^2/2)\tau$ and variances $\sigma_x^2 = \sigma_1^2\tau$ and $\sigma_y^2 = \sigma_2^2\tau$, respectively. It can also be shown that x and y are bivariate normally distributed with the correlation coefficient ρ.

The payoff rate of a European-style alternative option on the best performing of two call options can be expressed as

$$PRAT = \max\left\{\max\left[\frac{I_1(\tau)}{I_1} - k_1,\ 0\right],\ \max\left[\frac{I_2(\tau)}{I_2} - k_2,\ 0\right]\right\}, \quad (26.1)$$

where $PRAT$ stands for the payoff rate of an alternative option, max (., .) is a function that gives the larger of two numbers; k_1 and k_2 are the two strike rates of the alternative option.

The payoff rate in (26.1) can be shown to be equal to the following:

$$PRAT = \max\left[\frac{I_1(\tau)}{I_1} - k_1,\ \frac{I_2(\tau)}{I_2} - k_2,\ 0\right], \quad (26.2)$$

where all parameters are the same as in (26.1).

As in out-performance option in Chapter 25 the payoff rate of an alternative option based on the best of two options ($PRBT$) given in (26.2) can be expressed in terms of the gross returns of the two instruments:

$$PRBT = \max\left(e^x - k_1,\ e^y - k_2,\ 0\right), \quad (26.3)$$

where x and y are the log-returns of the two underlying assets, and k_1 and k_2 are the same strike rates as in (26.1) and (26.2).

If we use log-returns rather than gross returns, the payoff rate of an alternative option based on the best of two options given in (26.3) can also be expressed alternatively as follows:

$$APRLR = \max(x - k_1,\ y - k_2,\ 0)\,, \tag{26.4}$$

where k_1 and k_2 are the same strike rates as in (26.1) to (26.3).

26.3. A CLOSED-FORM SOLUTION FOR THE BEST-OF-TWO OPTIONS

We provided a closed-form solution for out-performance options using log-returns as the performance measures in the previous chapter. We will do the same for alternative options in this section. The integration domain of alternative options is the most complicated of all correlation options covered so far in this book. The integration domain of x and y is shown in Figures 26.1 to 26.3, for $k_1 = k_2$, $k_1 < k_2$, and $k_1 > k_2$, respectively. From the figures, we can observe that the domain is the simplest when $k_1 = k_2$. Actually, it is the same as the one for options on the maximum or the minimum of two assets. In general, when $k_1 \neq k_2$, the forty-five degree line does not go through the origin. As shown in Figure 26.2 (resp. Figure 26.3), the forty-five degree line intersects the two axes above (resp. below) the origin for $k_1 < k_2$ (resp. $k_1 > k_2$). When k_1 becomes the same as k_2, the forty-five degree line moves toward the origin and the two points Q and W in Figures 26.2 and 26.3 will become one Q as in Figure 26.1.

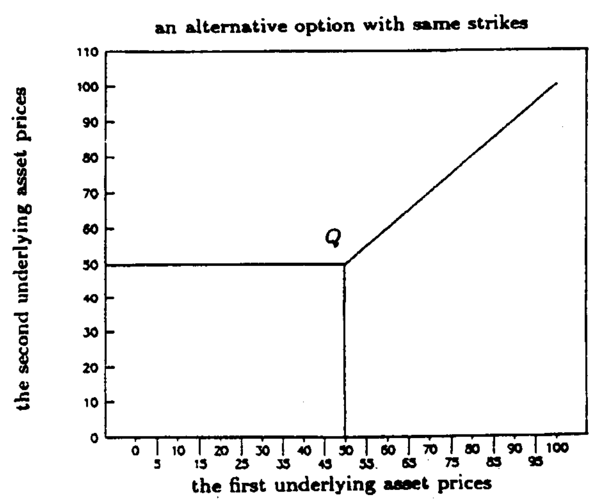

Fig. 26.1. The integration domain for an alternative option with the same strike price for the two call options.

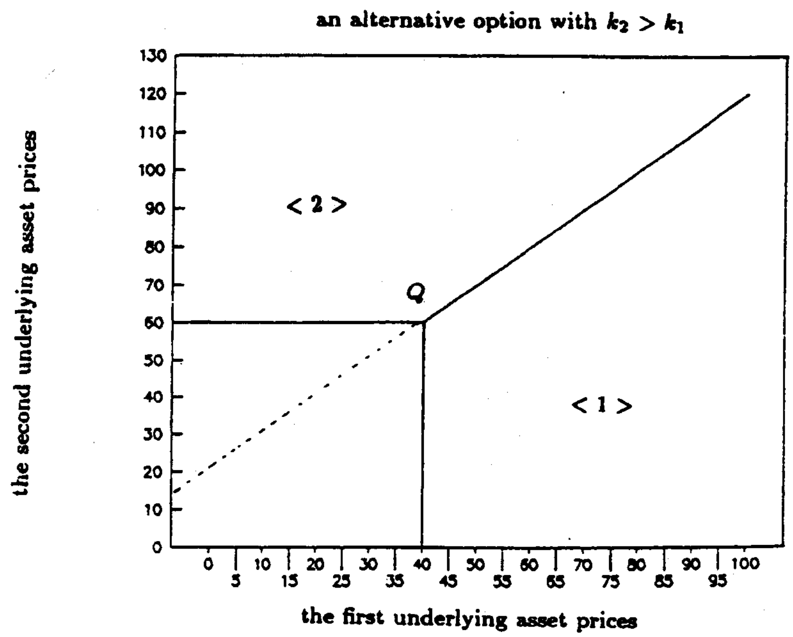

Fig. 26.2.

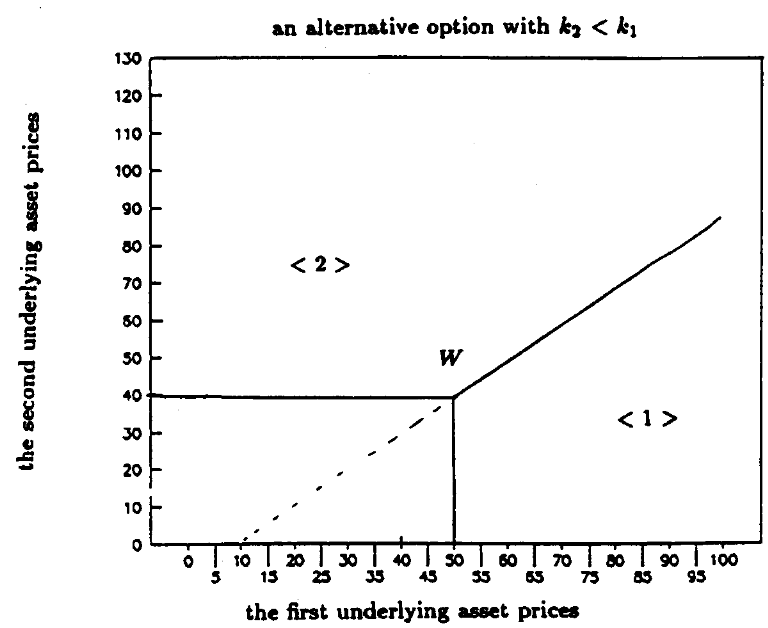

Fig. 26.3.

In the area marked $\langle 1\rangle$ in both Figures 26.2 and 26.3, a call option written on the first underlying asset with the strike rate k_1 is in-the-money and the payoff rate $x - k_1$ is always higher than the payoff rate $y - k_2$ of a call option written on the second asset with the strike rate k_2. In the area marked $\langle 2\rangle$, a call option written on the second asset with the strike rate k_2 always has a higher payoff rate than that on the first asset with the strike rate k_1. Using the joint density functions given in (IV4) and (IV5), we can obtain the following expected payoff rate of a best-of-two option (POFBT):

$$\begin{aligned} E(POFBT) = {} & (\mu_x - k_1)N_2(d_{k1}, d_{k12}, \rho_1) + (\mu_y - k_2)N_2(d_{k2}, d_{k12}, \rho_2) \\ & + (\sigma_1 Q_1 + \sigma_2 Q_2)\sqrt{\tau}\,, \end{aligned} \tag{26.5}$$

where

$$\begin{aligned}
Q_1 &= \frac{\sigma_2}{\sigma_a}\sqrt{1-\rho^2}\left[\frac{\sigma_2}{\sigma_a}\sqrt{1-\rho^2}\mathrm{f}(d_{k1})N(agm_{11}) - \rho_1 d_{k12} f(d_{k12})\mathrm{N}(agm_{12})\right], \\
Q_2 &= \frac{\sigma_1}{\sigma_a}\sqrt{1-\rho^2}\left[\frac{\sigma_1}{\sigma}\sqrt{1-\rho^2}\mathrm{f}(d_{k2})N(agm_{21}) - \rho_2 d_{k12} f(d_{k12})\mathrm{N}(agm_{22})\right],
\end{aligned}$$

$$agm_{11} = \frac{\sigma_a}{\sigma_2}\frac{d_{k12} + \rho_1 d_{k1}}{\sqrt{1-\rho^2}}, \quad agm_{12} = \frac{\sigma_a}{\sigma_2}\frac{d_{k1} - \rho_1 d_{k12}}{\sqrt{1-\rho^2}},$$

$$agm_{21} = \frac{\sigma_a}{\sigma_1}\frac{d_{k12} + \rho_2 d_{k2}}{\sqrt{1-\rho^2}}, \quad agm_{22} = \frac{\sigma_a}{\sigma_1}\frac{d_{k2} - \rho_2 d_{k12}}{\sqrt{1-\rho^2}},$$

$$d_{k1} = \frac{\mu_x - k_1}{\sigma_x}, \ d_{k2} = \frac{\mu_y - k_2}{\sigma_y}, \ d_{k12} = \frac{(\mu_x - k_1) - (\mu_y - k_2)}{\sigma_a\sqrt{\tau}},$$

$$\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sigma_a}, \ \rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\sigma_a},$$

$$\mu_x = \mu_1 - g_1 - \frac{1}{2}\sigma_1^2, \quad \mu_y = \mu_2 - g_2 - \frac{1}{2}\sigma_2^2,$$

$$\sigma_a = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

and $N_2(a, b, \theta)$ is the cumulative function of the standard bivariate normal distribution, $f(.)$ and $N(.)$ are the density and cumulative functions of the standard univariate normal distribution, respectively.

The derivation of (26.5) is very lengthy and we will not show it here. Interested readers may check Appendix at the end of this chapter for an outline of the proof.

Substituting $\mu_x = (r - g_1 - \sigma_1^2/2)\tau$ and $\mu_y = (r - g_2 - \sigma_2^2/2)\tau$ into (26.5) and discounting it at the risk-free rate of return r yields the best-of-two option price (PBTOP)

$$PBTOP = e^{-r\tau}\left\{\left[\left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau - k_1\right] N_2(d_{k1}, d_{k12}, \rho_1) + \left[\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau - k_2\right] N_2(d_{k2}, -d_{k12}, \rho_2) + (\sigma_1 Q_1 + \sigma_2 Q_2)\sqrt{\tau}\,, \right. \tag{26.6}$$

where

$$\mu_x = r - g_1 - \frac{1}{2}\sigma_1^2, \quad \mu_y = r - g_2 - \frac{1}{2}\sigma_2^2,$$

and all other parameters are the same as in (26.5).

Example 26.1. Find the price of the alternative option of the relative performance of the US stock market measured by S&P 500 and the Japanese stock market measured by Nikkei 225, given the the time to maturity of the option is six months, the interest rate 6%, $g_1 = 5\%$ and $g_2 = 2\%$, the strike rates $k_1 = k_2 = 5\%$, the volatilities of the two indices are $\sigma_1 = 18\%$ and $\sigma_2 = 20\%$, and the correlation coefficient between the returns of the two stocks ρ is 50%.

Substituting $\tau = 0.50$, $k_1 = k_2 = 0.05$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.20$ into (26.6) yields

$$\mu_x = 0.06 - 0.05 - 0.18^2/2 = -0.0062\,,$$

$$\mu_y = 0.06 - 0.02 - 0.20^2/2 = -0.02\,,$$

$$\sigma_a = \sqrt{0.18^2 - 2 \times 0.50 \times 0.18 \times 0.20 + 0.20^2} = 0.1908\,,$$

$$\rho_1 = \frac{0.50 \times 0.20 - 0.18}{0.1908} = -0.4193\,,$$

$$\rho_2 = \frac{0.50 \times 0.18 - 0.20}{0.1908} = -0.5765\,,$$

$$d_{k1} = \frac{-0.0062 - 0.05}{0.18\sqrt{0.50}} = -0.4415\,,$$

$$d_{k2} = \frac{0.02 - 0.05}{0.20\sqrt{0.50}} = -0.2121\,,$$

$$d_{k12} = \frac{(-0.0062 - 0.05) - (0.02 - 0.05)}{0.1908\sqrt{0.50}} = -0.1942\,,$$

$$agm_{11} = \frac{0.1908}{0.20}\frac{[-0.1942 - 0.4193(-0.4415)]}{\sqrt{1-0.50^2}} = -0.0100\,,$$

$$agm_{12} = \frac{0.1908}{0.20}\frac{[-0.4415 + 0.4193(-0.1942)]}{\sqrt{1-0.50^2}} = -0.5761\,,$$

$$agm_{21} = \frac{0.1908}{0.18}\frac{[-0.1942 - (-0.5765)(-0.2121)]}{\sqrt{1-0.50^2}} = -0.3874\,,$$

$$agm_{22} = \frac{0.1908}{0.18}\frac{[-0.2121 + (-0.5765)(-0.1942)]}{\sqrt{1-0.50^2}} = -0.1226\,,$$

$$\begin{aligned} Q_1 = {} & \frac{0.20}{0.1908}\sqrt{1-0.50^2}\left[\frac{0.20}{0.1908}\sqrt{1-0.50^2}f(-0.4451)N(-0.0100)\right. \\ & \left. - (-0.4193)(-0.1942)f(-0.1942)N(-0.5761)\right] \\ & = 0.1394\,, \end{aligned}$$

$$\begin{aligned} Q_2 = {} & \frac{0.18}{0.1908}\sqrt{1-0.50^2}\left\{\frac{0.18}{0.1908}\sqrt{1-0.50^2}f(-0.2121)N(-0.3874)\right. \\ & - (-0.5765)(-0.1942)f(-0.1942)N(-0.1226) \\ & = 0.0748\,, \end{aligned}$$

$$\begin{aligned} PBTOP = {} & e^{-0.06\times 0.5}[(-0.0062 \times 0.50 - 0.05)N_2(-0.4415, -0.1942, \\ & - 0.4193) + (0.02 \times 0.50 - 0.05)N_2(-0.2121, 0.1942, -0.5765) \\ & + (0.18 \times 0.1394 + 0.20 \times 0.0748)\sqrt{0.50}] \\ & = 2.35\%\,. \end{aligned}$$

Although the example above is rather long since there are many steps involved, it takes only a few seconds for computers to calculate because all the steps are straightforward calculations.

26.4. A CLOSED-FORM SOLUTION FOR THE WORST-OF-TWO OPTIONS

In the previous section, we provided a closed-form solution for an alternative option on the best performing of two call options. Using the same integration domains shown in Figures 26.2 and 26.3, we can find a similar solution for an alternative put option or worst-of-two option. If we use log-returns as the performance measures, the payoff rate of an alternative option based on the worst of two options (PRWT) can be expressed as follows:

$$PRWT = \min[\max(x - k_1,\ 0),\ \max(y - k_2,\ 0)]\,, \tag{26.7}$$

where k_1 and k_2 are the same strike rates as in (26.4).

In the area marked ⟨1⟩ in both Figures 26.2 and 26.3, a call option written on the second underlying asset with the strike rate k_2 is in-the-money and the payoff rate $y - k_2$ is always higher than the payoff rate $x - k_1$ of a call option written on the first asset with the strike rate k_1. In the area marked ⟨2⟩, a call option written on the first asset with the strike rate k_1 always has a higher payoff rate than that on the second asset with the strike rate k_2. Using the joint density functions given in (IV4) and (IV5) and discounting the expected payoff rate at the risk-free rate of return yields the price of an alternative option based on the worst of the two options (PRWT):

$$\begin{aligned} PRWT = e^{-r\tau}\Bigg\{&\left[\left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau - k_1\right]N_2(d_{k2}, -d_{k12}, \rho_2) \\ &+ \left[\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau - k_2\right]N_2(d_{k1}, d_{k12}, \rho_1) \\ &+ (\sigma_1 Q_{w1} + \sigma_2 Q_{w2})\sqrt{\tau}\Bigg\}\,, \end{aligned} \tag{26.8}$$

where

$$Q_{w1} = \rho Q_1 - \rho_1\sqrt{1-\rho^2}\frac{\sigma_1}{\sigma_a}f(d_{k12})N(agm_{22})\,,$$

$$Q_{w2} = \rho Q_2 - \rho_2\sqrt{1-\rho^2}\frac{\sigma_2}{\sigma_a}f(d_{k12})N(agm_{12})\,,$$

Q_1 and Q_2 are given in (26.5) and all other parameters are the same as in (26.5) and (26.6).

Example 26.2. Find the price of the worst-of-two option with the same informatin as in Example 26.1.

Substituting $\tau = 0.50$, $k_1 = k_2 = 0.05$, $r = 0.06$, $g_1 = 0.05$, $g_2 = 0.02$, $\rho = 0.50$, $\sigma_1 = 0.18$, $\sigma_2 = 0.20$, and other intermediate paramters obtained in Example 26.1 into (26.8) yields

$$\begin{aligned} Q_{w1} &= 0.50 \times 0.1394 - (-0.4193) \times 0.866 \times \frac{0.18}{0.1908}f(-0.1942)N(-0.1226) \\ &= 0.1302\,, \end{aligned}$$

$$\begin{aligned} Q_{w2} &= 0.50 \times 0.0748 - (-0.5765) \times 0.866 \times \frac{0.20}{0.1908}f(-0.1942)N(-0.5761) \\ &= 0.0952\,, \end{aligned}$$

$$\begin{aligned} PATPUT &= e^{-0.06\times 0.5}[(-0.0062\times 0.50 - 0.05)N_2(-0.2121, 0.1942,\\ &\quad -0.5765) + (0.02\times 0.50 - 0.05)N_2(-0.4415, -0.1942,\\ &\quad -0.4193) + 0.18\times 0.1302 + 0.20\times 0.0952\sqrt{0.50}]\\ &= 1.85\%\,. \end{aligned}$$

26.5. SUMMARY AND CONCLUSIONS

Alternative options include the best-of-two options and the worst-of-two options. They are a special type of dual-strike options because two strike rates are involved in each alternative option. They are also similar to out-performance options in the sense that they are both used to quantify the relative performance of two assets or indices and that they both provide payoff rates rather than payoff values. Yet, alternative options is rather different from the corresponding out-performance options, because the latter are mainly based on the relative performance of the two underlying instruments involved and the former are based on the relative values of the two options involved.

Using logarithm returns as the performance measure as in pricing out-performance options in the previous chapter, we have obtained closed-from pricing formulas for both the best and the worst-of-two options. These formulas are in terms of the density and cumulative functions of the standard univariate normal distribution and can be used very conveniently. Sensitivities of these pricing formulas can be obtained readily, and we leave their derivations as exercises. Pricing formulas using gross returns as the performance measure can be similarly obtained, and we also leave it as an exercise.

QUESTIONS AND EXERCISES

26.1. What are alternative options?

26.2. Why are alternative options similar to out-performance options?

26.3. What is the difference between out-performance options and best-of-two options?

26.4. Find the price of the best-of-two option of the relative performance of the US stock market measured by S&P 500 and the Japanese stock market measured by Nikkei 225, given the the time to maturity of the option is six months, the interest rate 7%, $g_1 = 4\%$ and $g_2 = 3\%$, the strike rates $k_1 = 4\%$, $k_2 = 5\%$, the volatilities of the two indices $\sigma_1 = 15\%$ and $\sigma_2 = 25\%$, and the correlation coefficient between the returns of the two stocks ρ is 45%.

26.5. Find the price of the alternative option in Exercise 26.4 if the correlation coefficient is changed to 75% and other parameters remain unchanged.

26.6. Find the price of the corresponding worst-of-two option in Exercise 26.4.

26.7. Find the price of the corresponding worst-of-two option in Exercise 26.5.

26.8.* Derive the closed-from pricing formula of alternative options using gross returns as the performance measures.

26.9.* Find the chi (the sensitivity with respect to the correlation coefficient) expression for best-of-two options given in (26.6).

26.10.* Find the value of chi using the formula obtained in Exercise 26.9 and other parameters in Exercise 26.4.

26.11.* Find the vega expression for best-of-two options in (26.6) with respect to the volatility of the first asset.

26.12.* Find the value of vega using the formula obtained in Exercise 26.11 and other parameters in Exercise 26.4.

APPENDIX

The results in (26.5) can be obained directly from the closed-form solution of the integration of the product of two density functions of the standard normal distribution and the integration variable given in Appendix of Chapter 15. The results in (26.8) can be obtained by first making the substitution $s = u - \rho v$ and the integration can be found using the results in (26.5) and the closed-form solution of the integration of the product of two density functions of the standard normal distribution given in Appendix of Chapter 15.

Chapter 27

BASKET OPTIONS

27.1. INTRODUCTION

Trades on underlying such as Latin American telecommunications stocks, portfolios of US junk bonds, or groups of Southeast Asian currencies have long been popular among investors. Basket options on the UK and Japanese banking stocks were popular last fall because investors thought that these sectors would rebound. Hedgers, likewise, have found instruments such as interest-rate basket options useful for their purposes. Basket options, or portfolio options, are options written on baskets or portfolios of risky assets. They generally allow end users to avoid the cost of single-asset/underlying options, because an option on a basket of underlying assets with a negative or imperfect correlation will have a lower volatility. Thus, basket options not only save a lot of work for fund managers from monitoring individual assets in the basket, but also reduce costs significantly. They are useful for protecting against movements in a market sector such as oil company stocks or technology stocks. They are also useful for managing risk exposure to a specific combination of portfolios made up of different sectors of the market.

Basket options have become better known among the derivatives professionals, although they still remain "exotic" to many who are not familiar with the exotic territory. Options written on baskets of risky assets can be used by portfolio managers to hedge the risks of their portfolios. Although basket options can be written on essentially any portfolios, those on currencies and commodities have been more popular. The currency market crisis of the European Rate Mechanism (ERM) in 1993 made currency basket options popular because investors needed to hedge a particular currency such as US dollar against a basket of European currencies.

As a matter of fact, index options, the most actively traded options in most exchanges, are a form of basket options. These options are normally bought by large institutional investors to protect their portfolios against adverse market movement. For index options written on value-weighted indexes such as S&P 500, the baskets actually consist of assets with weights proportional to their market values. As we will argue in this chapter, the approximation method in this chapter is more appropriate theoretically to price index options.

Grannis (1992) explained how options written on baskets of currencies work. Dembo and Patel (1992) studied synthetic basket options of stocks. Gentle (1993) priced basket options using Vorst (1992)'s method to approximate arithmetic Asian options from their corresponding Geometric Asian options based on the true expectation of the basket spot price at maturity. Huynh (1994) priced basket options using a generalized Edgeworth series expansion based also on the first two moments. The purpose of this chapter is to to price basket options in a Black-Scholes environment. Using the analytical approximation techniques developed for arithmetic Asian options and flexible arithmetic Asian options in Chapters 5 and 6, we will find an approximated basket option pricing formula with both even and uneven weights in the baskets.

27.2. BASKET OPTIONS

In a typical Black-Scholes environment, the underlying asset prices are assumed to follow a lognormal process. Suppose that there are n underlying assets or indices, I_i, $i = 1, 2, \ldots, n$, each following the standard stochastic process given in (IV1) or,

$$dI_i = (\mu_i - g_i)I_i dt + \sigma_i I_i dz_i(t), \quad i = 1, 2, \ldots, n\,, \tag{27.1}$$

where $z_i(t)$ is the standard Gauss-Wiener process; μ_i and σ_i are the instantaneous mean and standard deviation of the ith asset or index, respectively; g_i is the payout rate of the ith asset; and $z_i(t)$ and $z_j(t)$ are assumed to be correlated with the correlation coefficient $-1 \leq \rho_{ij} \leq 1$ and $\rho_{ii} = 1$, for all $i, j = 1, 2, \ldots, n$.

Solving the equation in (27.1) using the standard method (see Appendix of Chapter 2) yields

$$I_i(\tau) = I_i \exp\left[\left(r - g_i - \frac{1}{2}\sigma_i^2\right)\tau + \sigma_i z_i(\tau)\right], \quad i = 1, 2, \ldots, n\,, \tag{27.2}$$

where I_i is the spot price of the ith asset, $\tau = t^* - t$, and t and t^* are the current and the expiration time of the option, respectively.

Let $x_i = \ln[I_i(\tau)/I_i]$. It can be shown that x_i is normally distributed with mean $\mu_{xi} = (r - g_i - \sigma_i^2/2)\tau$ and variance $\sigma_{xi}^2 = \sigma_i^2\tau$, respectively.

Let us define a basket as follows

$$I(\tau) = \sum_{i=1}^{n} W_i I_i(\tau)\,, \tag{27.3}$$

where $I_i(\tau)$ is the price of the ith asset at the option maturity and W_i is the percentage of total investment in the ith asset and $\sum_{i=1}^{n} W_i = 1$.

The payoff of a European option on the basket defined in (27.3) can be expressed as follows:

$$PBSKT = \max\{w[I(\tau) - K],\ 0\}\,, \tag{27.4}$$

where K is the exercise price of the option and max $(.,.)$ is a function that gives the larger of two numbers, and w is a binary operator (1 for a call option and -1 for a put option).

27.3. TWO-ASSET BASKET OPTIONS

The payoff of a European call option on a basket or portfolio of two assets or indexes can be expressed exactly in the same way as that of a spread option given in (22.1) with $a = W_1 > 0$ and $b = W_2 > 0$. Figure 27.1 depicts the integration domain of a two-asset basket call option. It is very different from Figure 22.1, because the sign of b are opposite of a simple spread option ($b < 0$) and ($b = w_2 > 0$) for a two-asset basket option.

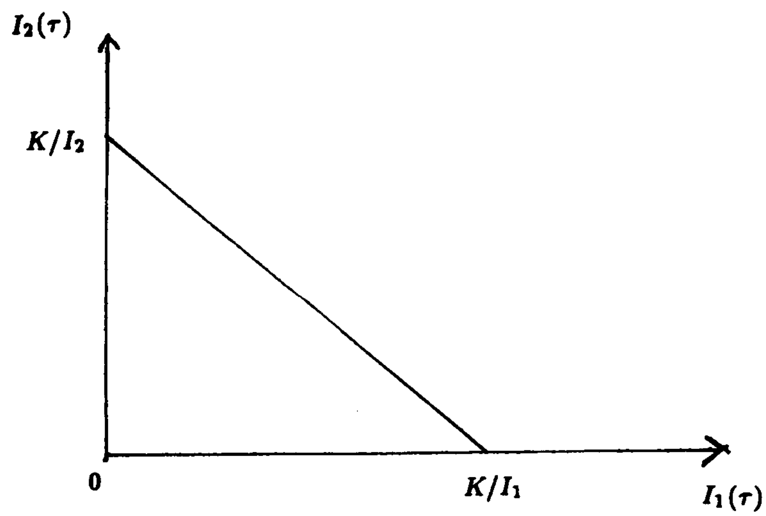

Fig. 27.1. Integration domain for a two-asset basket option.

Using the density functions given in (IV4) and (IV5) and the integration domain in Figure 27.1, we can obtain the European put option price on the basket of two assets after discounting the expected payoff at the risk-free rate of return by double-integration:

$$P_{bk} = -aI_1 e^{-g_1\tau} A_{b1} - bI_2 e^{-g_2\tau} A_{b2} + Ke^{-r\tau} A_{b3}\,, \tag{27.5}$$

where

$$A_{b1} = \int_{-\infty}^{-d(b)-\rho\sigma_1\sqrt{\tau}} f(v) N\left[\frac{-\phi(v+\rho\sigma_1\sqrt{\tau}) - d(a) - \sigma_1\sqrt{\tau} - \rho v}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$A_{b2} = \int_{-\infty}^{-d(b)-\sigma_2\sqrt{\tau}} f(v) \left[\frac{-\phi(v+\sigma_2\sqrt{\tau}) - d(a) - \rho\sigma_2\sqrt{\tau} - \rho v}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$A_{b3} = \int_{-\infty}^{-d(b)} f(v) N\left[\frac{-\phi(v) - d(a) - \rho v}{\sqrt{1-\rho^2}}\right] dv\,,$$

$$\phi(v) = \frac{-1}{\sigma_1\sqrt{\tau}} \ln\left\{1 - \frac{bI_2}{K} \exp\left[\left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau + v\sigma_2\sqrt{\tau}\right]\right\},$$

$$d(a) = d(aI_1, K, \sigma_1, g_1, \tau, r) = \left[\ln\left(\frac{aI_1}{K}\right) + \left(r - g_1 - \frac{1}{2}\sigma_1^2\right)\tau\right] / (\sigma_1\sqrt{\tau})\,,$$

$$d(b) = d(bI_2, K, \sigma_2, g_2, \tau, r) = \left[\ln\left(\frac{bI_2}{K}\right) + \left(r - g_2 - \frac{1}{2}\sigma_2^2\right)\tau\right] / (\sigma_2\sqrt{\tau})\,.$$

The functions d(a) and d(b) in (27.5) are the same as the d function in the extended Black-Scholes formula, and the function $\phi(v)$ is the same as in the simple spread call option pricing formula in (22.2) and (22.3). The pricing formula in (27.5) is a closed-form solution because the parameters are expressed in univariate integrations.

The price of a call option written on the same basket of two assets can be obtained following a similar procedure as to obtain (27.5) by double-integration following the integration domain in Figure 27.1. Yet it can be more conveniently obtained from the put-call parity as follows:

$$C_{bk} = P_{bk} + aI_1 e^{-g_1\tau} + bI_2 e^{-g_2\tau} - Ke^{-r\tau}\,, \tag{27.6}$$

where all parameters are the same as in (27.5).

Formula (27.5) can be approximated using the linearization results for the function $\phi(v)$ in Proposition 22.1. We leave this as an exercise.

Example 27.1. Find the prices of the basket options on a basket with 90% weight for the first stock and 10% for the other stock, given the spot prices of the two stocks \$100, dividend yield of the two stocks the same as 0.00%, the volatility of the two stocks 10%, interest rate 10%, strick price of the options \$100, time to maturity one year, and the correlation coefficient between the returns of the two stocks 50%, 0.0%, and 50%, respectively.

Substituting $a = W_1 = 0.90$, $b = W_2 = 0.10$, $I_1 = I_2 = K = 100$, $g_1 = g_2 = 0.0$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.10$, $\tau = 1.00$, and $\rho = -0.50$ into (27.5) yields the spread put option price $\rho(\rho = -0.50) = \$0.858$, and the corresponding call option price can be readily found from (27.6)

$$C(\rho = -0.50) = 0.858 + 0.9 \times 100 + 0.1 \times 100 - 100e^{-0.1\times 1}$$
$$= 9.085.$$

Similarly, the put option prices with correlation coefficient $\rho = 0$ and 0.50 can be found as $\rho(\rho = 0.00) = \$0.633$ and $\rho(\rho = -0.50) = \$0.345$, and their corresponding call option prices are $C(\rho = 0.0) = \$9.363$ and $C(\rho = 0.50) = \$9.651$.

27.4. BASKET OPTIONS WITH MORE THAN TWO ASSETS

We found a near closed-form solution for basket options with two assets in the previous section. It is more difficult to find a closed-form solution for basket options with more than two assets. The payoff of a basket option with n assets given in (27.3) is actually a flexibly weighted average of the n asset prices at the option maturity, where the flexible weight of each asset at maturity is exactly the percentage of total investment in that asset. In this section, we will approximate basket option prices using the results in Chapter 22.

Before we examine the general case of arbitrary weights for the assets involved, we want to price basket options written on baskets with equal weights for all assets using the lognormalization results for multiple spread options in Chapter 22. Using the density function of the standard normal distribution, we can obtain the expected payoff of the European basket option given in (27.3) and (27.4) with $W_i = 1/n$ for $i = 1, 2, \ldots, n$:

$$E(PBSKTO) = wI(0)e^{\tau(\mu_{a1}+\sigma_{a1}^2/2)}N(wd + w\sigma_a\sqrt{\tau}) - wKN(wd)\,, \quad (27.7)$$

where

$$d = \left\{\ln\left[\frac{I(0)}{K}\right] + \tau\left[r - \frac{1}{n}\sum_{i=1}^{n}\left(g_i + \frac{1}{2}\sigma_i^2\right)\right]\right\}/(\sigma_a\sqrt{\tau})\,,$$

$$\sigma_a = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j}\,,$$

and other parameters are the same as in (22.15).

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate r. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate, we can obtain the call option price by discounting the expected payoff in (27.7) by the risk-free rate r,

$$BSKTOP = wI(0)\mathrm{e}^{-\tau(r-\mu_{a1}-\sigma_{a1}^2/2)}N(wd + w\sigma_a\sqrt{\tau}) - wK\mathrm{e}^{-r\tau}N(wd)\,, \tag{27.8}$$

where d is the same as in (26.7).

Formula (27.8) appears a little complicated compared to the vanilla option pricing formula. However, the basic idea is simple. As we did in Chapter 22 in pricing multiple spread options, we could lognormalize the basket using Propositions 22.1 and 22.2. Once the basket is lognormalized, to price basket options is as easy as to price vanilla options.

To extend the analysis of simple multiple spread options to complex multiple spread options, we can directly make use of the approximations given in Propositions 22.2 and 22.3. We can interpret W_i as the weight given to the ith asset in the basket at maturity, and the sum in (27.3) can be interpreted as the weighted sum of the n asset prices at maturity. Since all the weights are known in a given basket, the weighted sum of the underlying asset prices at maturity can be approximated to a lognormal distribution using the results given in Propositions 22.2 and 22.3. We need to rearrange the results in Proposition 22.2 and 22.3 in order to use them conveniently.

Proposition 27.1. The forward value of the basket given in (27.3) can be approximated with the following standard geometric process

$$dI = (r - g_b)Idt + \sigma_b I dz_b(t)\,, \tag{27.9}$$

where

$$\sigma_b = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j}\,, \tag{27.9a}$$

$$g_b = \sum_{i=1}^{n} W_i\left(g_i - \frac{1}{2}\sigma_i^2\right) - \frac{1}{2}\sigma_b^2\,, \tag{27.9b}$$

$$I(t) = \kappa_{nw} WGI(t)\,, \tag{27.9c}$$

and $z_b(t)$ is a standard Gauss-Wiener process, ρ_{ij} is the correlation coefficient between the ith and jth asset returns, W_i is the weight assigned to the ith asset in the basket, $WGI(t)$ is the weighted geometric index of the n spot prices defined in (22.20), and κ_{nw} is the approximation coefficient given in (22.21).

Proof. Immediately from Propositions 22.2 and 22.3. □

Proposition 27.1 obviously indicates that the forward value of the basket defined in (27.3) can be approximated with a standard geometric Brownian motion with the spot price, payout rate, and volatility all expressed in terms of the individual spot prices, payout rates, volatilities, the correlation coefficients among the assets, and the weight assigned to each asset. With this approximation, we can obtain the expected payoff of the basket option given in (27.4) following the same procedures as in Chapter 2 in obtaining the Black-Scholes pricing formula:

$$E(PBSKT) = wI(t)e^{(r-g_b)\tau}N[w(d+\sigma_b\sqrt{\tau})] - wKN(wd)\,, \qquad (27.10)$$

where

$$d = \frac{\ln[I(t)/K] - (r - g_b - \sigma_b^2/2)\tau}{\sigma_b\sqrt{\tau}}\,,$$

and all parameters are the same as in (27.9).

Discounting the expected payoff in (27.10) yields the pricing formula of a basket option (BSKT):

$$BSKT = wI(t)e^{-g_b\tau}N(wd + w\sigma_b\sqrt{\tau}) - we^{-r\tau}KN(wd)\,, \qquad (27.11)$$

where all parameters are the same as in (27.9) and (27.10).

It can be shown that the pricing formula in (27.11) degenerates to that in (27.8) when the weights for all individual assets are equalized. The pricing formula in (27.11) can be applied directly to price stock index options because most stock indexes are weighted averages of a certain number of stocks and thus can be regarded as portfolios or baskets of stocks. As a matter of fact, the pricing formula in (27.11) is theoretically consistent with the assumption that individual stock prices follow a lognormal process as we have assumed throughout this book. The traditional way to price a stock index option using the lognormal assumption of the index is somewhat inconsistent with the lognormal assumption of individual stock prices, because the weighted sum of a certain number of lognormally distributed prices is not a lognormal process. Thus, the pricing formula in (27.11) is a more appropriate formula for stock index options.

27.5. BASKET DIGITAL OPTIONS

A basket digital option is a hybrid of basket and digital options by combining correlation digital options studied in Chapter 15 with basket options. Ordinary digital options provide payoffs which jump at strike prices. Zhang (1995d) argued that the underlying asset is not always necessarily the payoff asset of a binary option, and introduced and priced correlation digital options. We will discuss briefly basket digital options in this section.

The payoff of a European-style basket digital option (BDG) can be expressed:

$$BDG = w[I(\tau) - X], \quad \text{if } S_i(\tau) > K \tag{27.12a}$$

$$= 0, \quad \text{if otherwise}, \tag{27.12b}$$

where $I(\tau)$ stands for the forward value of the basket including n assets as defined in (27.3), X is the gap parameter, and K is the exercise price of the option.

Using the approximation in Proposition 27.1 and the results in Zhang (1995d), we can find the pricing formula immediately for a basket digital option. We leave this as an exercise.

27.6. BASKET BARRIER OPTIONS

In describing outside Asian barrier options in Chapter 11, we listed four combinations of spot and average in Table 11.1. The combination spot-spot stands for a vanilla barrier option because both the payment asset and the measurement asset are the same as the underlying asset price; the combination average-average stands for a simple Asian barrier option because both the payment asset and the measurement asset are the same as an average of the underlying asset prices; and the combination spot-average stands for an outside Asian barrier option studied in Section 11.9 because the payment asset is the same as the underlying asset and the measurement asset is an average of the underlying asset prices. After studying how to approximate weighted baskets of lognormally distributed assets in this chapter, we may regard the combination average-spot as a basket barrier option with the trigger as some constituent assets in the basket or some popular indexes such as LIBOR. Basket barrier options may be particularly attractive to fund managers because their portfolios are normally sensitive to some specific indexes.

27.7. SUMMARY AND CONCLUSIONS

Basket options are options written on baskets or portfolios of risky assets. With the increasing popularity of various kinds of funds, hedging risks is getting more and more important in fund management. As most individual assets in portfolios are highly correlated, it would be certainly inappropriate simply to examine the risks of individual assets and derivatives. Due to their unique characteristic of treating a basket as a whole, basket options are the natural candidate for fund managers to hedge their risks and also enhance their performance based on their perspectives of the baskets.

Using analytical approximation techniques developed for arithmetic Asian options in Chapters 5 and 6, we have obtained approximated expressions for baskets of lognormally distributed risky assets and provided Black-Scholes type formulas for basket options. The simplicity of these pricing formulas should reduce the amount of time necessary to price basket options using the standard Monte-Carlo simulations and facilitate the pricing and trading of basket options. The closed-form solutions make it possible to have closed-form expressions for the sensitivities of basket options with respect to various parameters of the individual assets and to the correlation coefficients among these assets.

The pricing formulas developed for basket options in this chapter can be applied immediately to price stock index options because most stock indexes are weighted averages of a certain number of stocks and thus can be regarded as portfolios or baskets of stocks. Basket options can be combined with correlation digital options to form basket digital options, and can also be combined with outside Asian barrier options to form basket barrier options.

QUESTIONS AND EXERCISES

27.1. What are basket options?

27.2. Give the names of two popular basket options.

27.3. Why is the traditional way to price stock index options assuming that the stock index is lognormally distributed inconsistent with the pricing formulas of individual stock options?

27.4.* Find the corresponding pricing formula of the call option written on the basket consisting of two assets in (27.6) without using the put-call parity.

27.5.* Show that the put-call parity in (27.6) for basket options on two assets holds using the results given in (27.10) and (27.5).

27.6.* Find the approximation formula of a put option on a basket consisting of two assets using the approximation results in Proposition 22.1 for spread options. [Hint: use the method to express integrations of products of standard normal density and cumulative functions in bivariate normal cumulative functions in (A11.4) in Appendix of Chapter 11.]

27.7.* Show Proposition 27.1.

27.8.* Find the pricing formula of a European-style basket digital option using the approximation result in Proposition 27.1 and the result in Chapter 15 for correlation digital options.

Chapter 28

PRICING CORRELATION OPTIONS WITH UNCERTAIN CORRELATION COEFFICIENTS

28.1. INTRODUCTION

In all the studies so far in this book, especially in Part IV, and in the vast majority of economic and financial literature, the correlation coefficient between any two correlated variables has traditionally been assumed to be constant.[1] However, anyone with some experience of real market data knows that the correlation coefficient between any two risky assets can be anything but constant. In general, correlation coefficients vary dramatically, and they tend to have cyclic movements. As an example, if we use the daily exchange rates of the four major currencies against the US dollar, the correlation coefficient between daily Japanese yen/US dollar and daily British pound/US dollar exchange rates dropped from 0.89 in August 1992 to −0.58 in September 1992, and jumped to 0.77 in the following month. It may be argued that this example is not very representative because of the Black Wednesday[2] in 1992, but we can give countless other examples. The same correlation coefficient dropped from 0.86 to −0.8 from March to April in 1990 and jumped to 0.67 in the following month. Treating correlation coefficients as constant or taking the long-term estimates of constant correlation may be misleading and can underestimate or overestimate the current correlation and thus creates serious problems in risk-taking, pricing, and hedging.

[1]Through an experiment, Frankfurter, Phillips, and Seagle (1971) showed the impact of estimation errors in estimating means, variances, and covariances on portfolio selection. They, as in a few studies related to their paper, focused on the uncertainty resulting from estimation errors in their experiment rather than the uncertainty of the correlation coefficient itself.
[2]Compared to Black Monday, which most often refers to the October stock market crash in 1987, Black Wednesday refers to the currency market crisis in the European Rate Mechanism (ERM) in the European Monetary System (EMS) in November 1992.

The purpose of this chapter is to find the limitations and errors in pricing correlation options with constant correlation coefficients so that the errors in risk-taking, pricing correlation options, and other related problems can be estimated. The uncertain correlation coefficient between two factors (the factors can be stocks, indices, foreign exchange rates, etc.) may be understood as the result of asymmetric sensitivities or different degrees of reactions of the two factors toward the same piece of international, macroeconomic, financial, industry, and other kinds of information. As various kinds of information are uncertain in nature, different factors most often possess different sensitivities toward the same information. These sensitivities themselves are also likely to change over time and/or toward various information. The correlation coefficient is therefore uncertain. Incorporating existing methods in estimating single-point correlation coefficients, the stochastic correlation coefficient method introduced in this chapter can improve the accuracy in pricing correlation options and differential swaps, measuring portfolio risks, and analyzing cross-market products.

28.2. METHODS TO ESTIMATE THE CORRELATION COEFFICIENT

Correlation coefficients, like most other parameters in financial studies, are most often estimated using historical data. Different analysts may use either daily, weekly, monthly, or annual data to calculate correlation coefficients depending on their specific needs. The problem with this method is that the estimations are very often different with different observation frequencies and different number of observations used. The problem is not surprising because correlation itself is not constant over time for the reasons we argued in the previous section of this chapter.

Besides the historical estimation method, there is another popular method called the implied correlation coefficient. This method is similar to the implied volatility in vanilla option analysis using actual market option prices. Using the price of an actual correlation option and the specific pricing formula of this option, the correlation coefficient can be solved inversely. The problem with the implied correlation coefficient rises from the liquidity of the correlation option. With rather low liquidity resulting from the highly customized nature of the OTC market, the actual market price of any correlation option may not be specified appropriately, therefore the implied correlation coefficient may not be an appropriate correlation coefficient.

The third method is to estimate the correlation coefficient using a time-series analysis, or more specifically, using a recently developed econometric model — the GARCH model. GARCH stands for generalized auto-regressive conditional heteroscedasticity. As its name implies, a GARCH model is an auto-regressive model which searches for a pattern of volatility fluctuations using historical data and forecasts future volatilities based on the historical pattern. A bivariate GARCH model forecasts volatilities of the individual factors and their covariance as well. Thus, the correlation coefficient can be calculated directly using the estimated volatilities of the two factors and their covariance. The GARCH model is an advanced model, yet its serious shortcoming is that its estimations are highly unstable.

Although the three methods differ significantly, they share one common characteristics: a single correlation coefficient value is either forecasted or calculated and then used as a constant for different purposes. Portfolio managers may prefer historical correlation coefficients because they do not change their positions frequently. The front office and traders may use up-to-the-minute forecasts rather than historical correlation coefficients, using the last two methods as more precision is needed at the transaction level. In general, either one or a combination of these three methods may be used. Some traders may simply look at the historical correlation coefficient patterns and choose one value roughly according to his expectation for future correlation coefficient fluctuations.

28.3. SOME EMPIRICAL EVIDENCE

Before we start to introduce the sample distribution of coefficients, it is highly useful for us to examine the correlation coefficient between two factors using real market data. Using the daily exchange rates of US dollar/German mark, US dollar/Japanese yen, US dollar/Swiss franc, and US dollar/ British pound from January 3, 1984 to February 14, 1994, we calculated the monthly correlation coefficients among these four major rates. We then divided the range of possible correlation coefficients from -1 to 1 into twenty equal intervals and calculated the frequency distributions of these correlation coefficients. Table 28.1 lists the average correlation coefficients among the four actual exchange rates and those among the returns of the four exchange rates. It is obvious from Table 28.1 that the higher the correlation coefficient between any two exchange rates, the higher the correlation coefficient between the corresponding returns of the two exchange rates. Figures 28.1–28.6 depict the six monthly frequency distributions of the actual exchange rates. It can be observed from Figures 28.1–28.6 that

Corr. coeff.: DLR–DM/DLR–YEN.

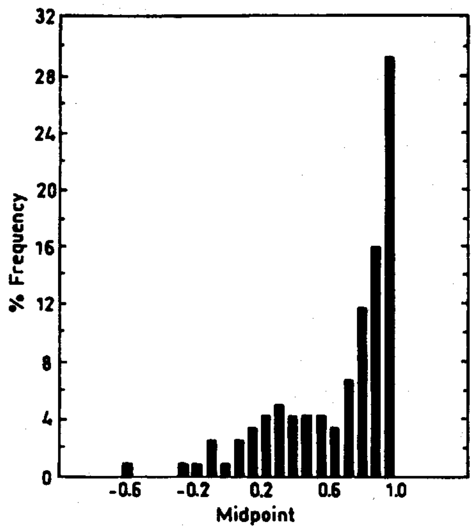

Fig. 28.1

Corr. coeff.: DLR–DM/DLR–GBP.

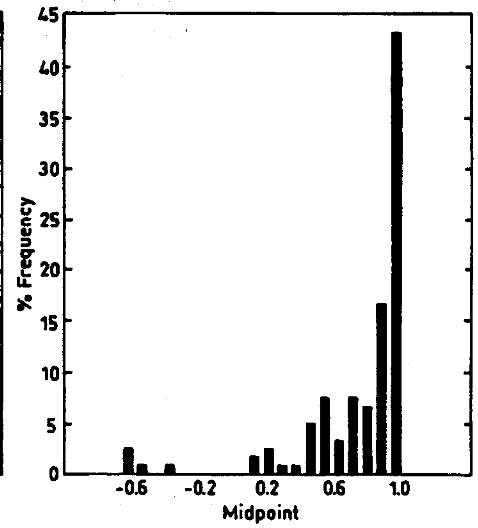

Fig. 28.2

Corr. coeff.: DLR–DM/DLR–SF.

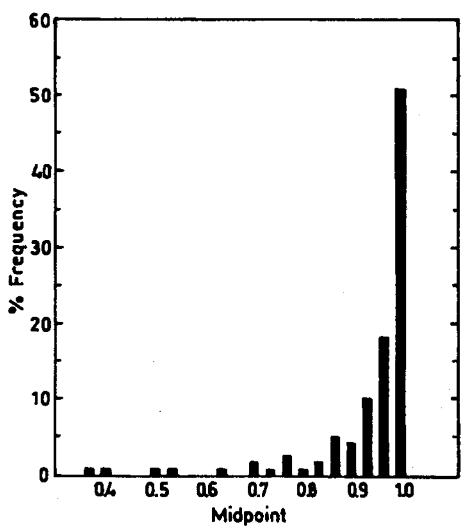

Fig. 28.3

Corr. coeff.: DLR–YEN/DLR–GBP.

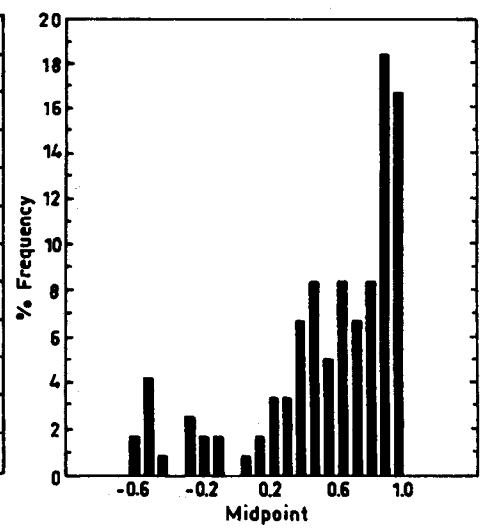

Fig. 28.4

Corr. coeff.: DLR–YEN/DLR–SF.

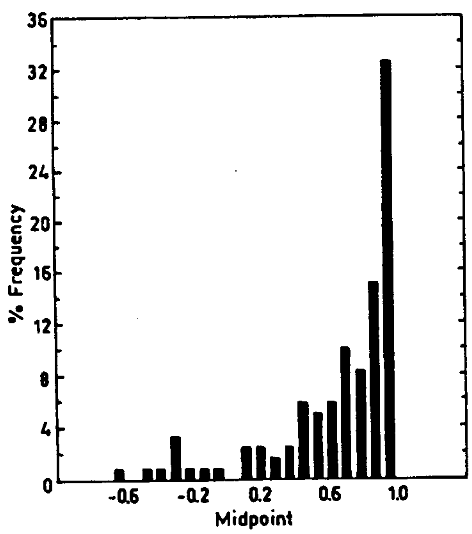

Fig. 28.5

Corr. coeff.: DLR–GBP/DLR–SF.

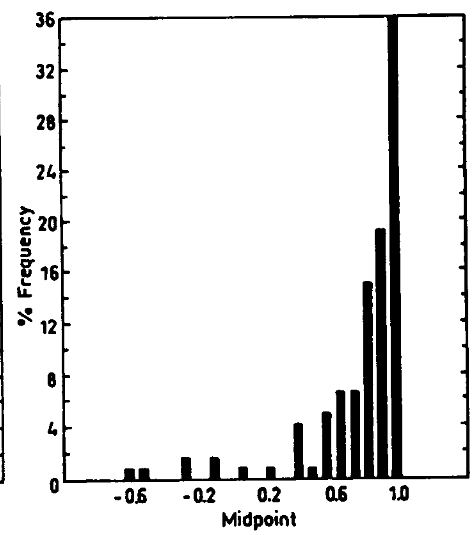

Fig. 28.6

Corr. coeff.: DLR–DM/DLR–YEN
for returns.

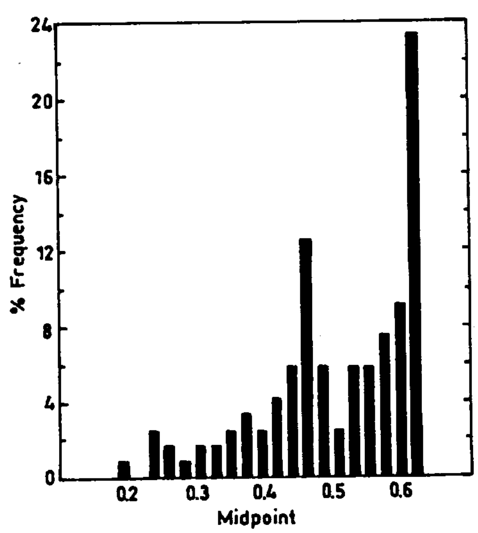

Fig. 28.7

Corr. coeff.: DLR–DM/DLR–GBP
for returns.

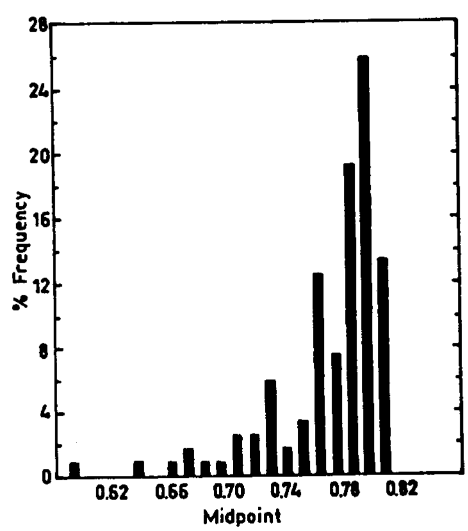

Fig. 28.8

Corr. coeff.: DLR–DM/DLR–SF
for returns.

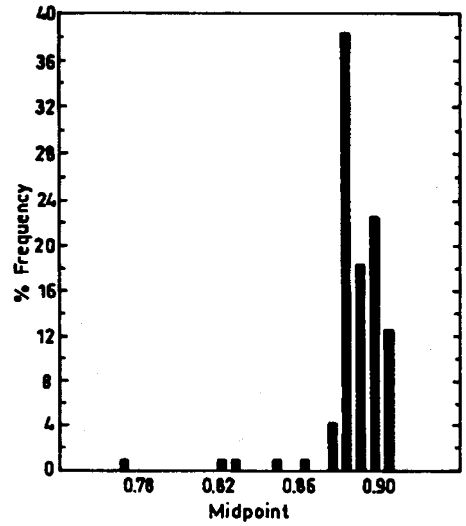

Fig. 28.9

Corr. coeff.: DLR–YEN/DLR–GBP
for returns.

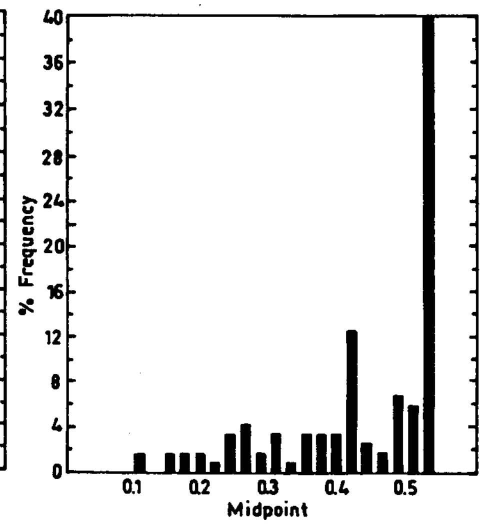

Fig. 28.10

Corr. coeff.: DLR–YEN/DLR–SF
for returns.

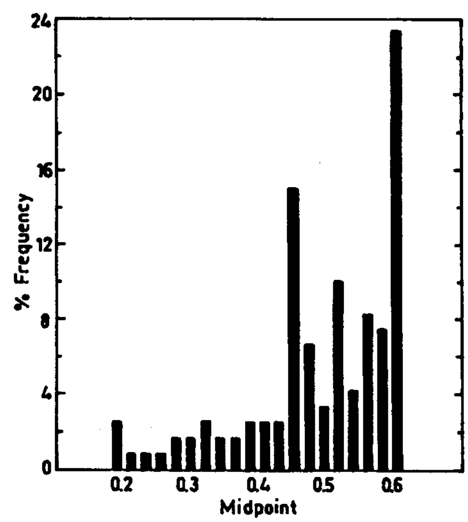

Fig. 28.11

Corr. coeff.: DLR–GBP/DLR–GBP
for returns.

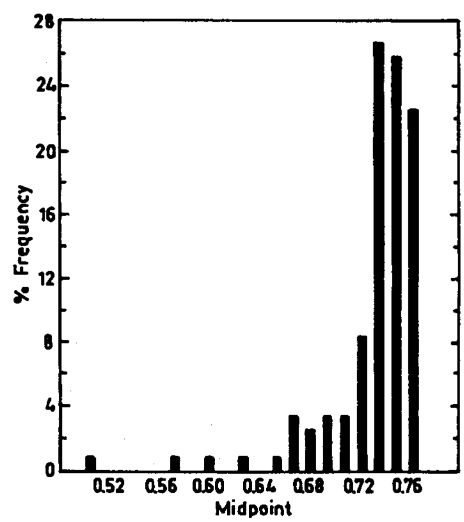

Fig. 28.12

the correlation coefficients are highly unstable and that the greater the mean correlation coefficients, the greater the modes, or the "most likely" correlation coefficients.

Figures 28.1–28.6 are the frequency distributions of the correlation coefficients among the four major exchange rates expressed in US dollar per unit of foreign currency. As all pricing theories for correlation options are based on correlation coefficients of the returns of the underlying assets involved, these correlation coefficients are often more relevant than those of the actual rates or prices in pring correlation options. Using the same daily data, we calculated the monthly correlation coefficients among the returns of these four major exchange rates and found the frequency distributions of these correlation coefficients during the same time period. Figures 28.7–28.12 delineate these frequency distributions. Comparing the mean correlation coefficients of the actual exchange rates and those of the returns, we can easily find that the mean return correlation coefficients are significantly lower than the corresponding mean correlation coefficients of the actual exchange rates. This is because returns always scale down extreme fluctuations of asset prices or exchange rates. Comparing Figures 28.1–28.6 with Figures 28.7–28.12, we can clearly observe that the six return frequency distributions are "more symmetric" around their means than their corresponding actual exchange rate frequency distributions.

28.4. A DISTRIBUTION OF THE CORRELATION COEFFICIENT

In most problems involving two correlated random variables, a bivariate normal distribution is often assumed. The joint density function of a bivariate normal distribution is given in (IV3) with a constant correlation coefficient ρ.

In all the analyses of various kinds of options covered so far in this book, the portfolio analysis in Markowitz (1952 and 1958), the pricing of differential swaps in Turnbull (1993) and Wei (1994), and almost all other studies in finance, the correlation coefficient ρ is assumed to be a constant. The correlation coefficient ρ may be estimated using either one or a combination of the three methods described in Section 28.2. Whereas it is convenient to treat ρ as a constant, it does not capture the stochastic characteristics of the real-world correlation phenomena as evidenced in Section 28.3.

The study of the distribution of the sample correlation coefficient is almost as old as the correlation analysis itself, although interest in the former has dried out in recent decades. "Student" (1908) first studied the distribution of the correlation coefficient for any sample size from a bivariate normal population when the population correlation coefficient is zero. Fisher (1915) obtained a general distribution with any sample size for nonzero population correlation coefficients, also from a bivariate normal population. Fisher's works stimulated studies in this area for more than half a century. In order not to spend too much time on the mathematical aspects of the literature in this area, we will discuss the related studies in Appendix at the end of this book.

Using the joint distribution of the sample variance and covariance of a bivariate normal distribution, Rao (1973) derived a distribution of the correlation coefficient, given the sample size n and the population correlation coefficient ρ. The density function of the correlation coefficient is given by:

$$g(\gamma) = \frac{\Gamma[(n-1)/2]}{\sqrt{\pi}\Gamma[(n-2)/2]}(1-\gamma^2)^{(n-4)/2}, \text{ for } \rho = 0\,, \tag{28.1a}$$

and

$$g(\gamma) = \frac{2^{n-3}}{\pi(n-3)!}(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2} \times \sum_{s=0}^{\infty}\Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma)^s}{s!}, \text{ for } \rho \neq 0\,, \tag{28.1b}$$

where the correlation coefficient $-1 \le \gamma \le 1$, $(n-3!)$ and $s!$ are the factorials of $n-3$ and s, respectively, $\Gamma(p)$ is the standard gamma function which is defined as:

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx\,,$$

and the factorial of any integer n equals $n! = n(n-1)(n-2)\ldots 3\times 2\times 1$.

The density function given in (28.1) is a rather complicated function compared with the density functions of most known theoretical distributions as it has an infinite number of polynomial terms multiplied by the square of the gamma function. In order to see the curvature of the distribution function given in (28.1), we depict $g(\gamma)$ for $n = 10$, $\rho = 0.00$, 0.25, and

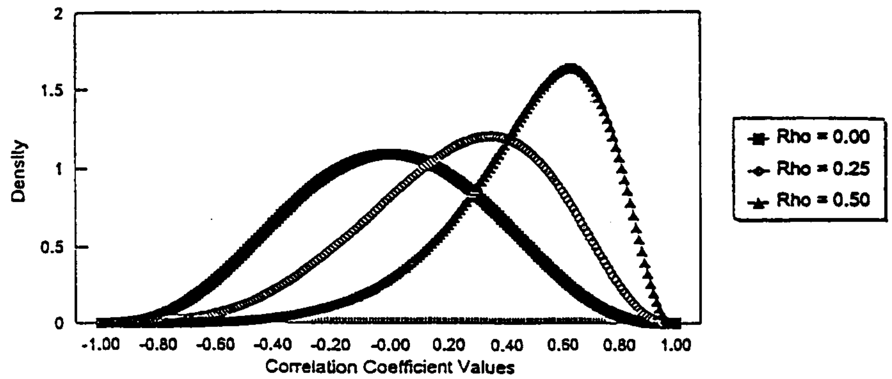

Fig. 28.13.

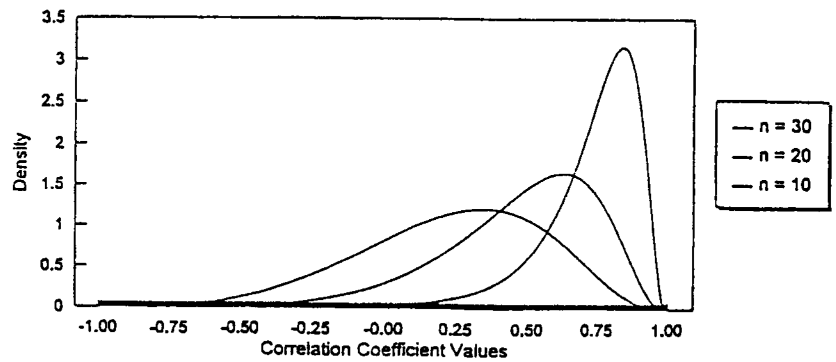

Fig. 28.14.

0.50 in Figure 28.13,[3] which demonstrates that the greater the population correlation coefficient ρ, the more the density function is skewed to the right. Figure 28.14 shows the curvatures of the density function for $n = 10$ and $\rho = -0.50$, 0, and 0.50.

Figure 28.14 shows the curvatures of the density function for $\rho = 0.50$, $n = 10$, 20, and 30. We can observe from Figure 28.15 that the density function becomes thinner for larger n, or the distribution allocates heavier weights for

[3]Considering the population to be specified by the bivariate normal surface, Fisher (1915) obtained the exact sampling distribution of the correlation coefficient. David (1938) provided tables for the distribution of the correlation coefficients based on Fisher's work. Gayen (1951) corrected some errors in their work. For studies on the correlation coefficient in random samples drawn from non-normal universes, see Pearson (1929), Quensel (1938), and Gayen (1951). Although Fisher's (1915) seminal study is the foundation for the aforementioned studies and also for Rao (1973), Fisher's exact distribution function contains higher order $(n-2)$ derivatives of some complicated trigonometric functions with respect to the product of the population correlation coefficient and the random correlation coefficient, and it is not convenient to use.

those correlation coefficients somewhat smaller than the population correlation coefficient ρ for larger n. In fact, the number of observations n captures the length of time back in history. Therefore, the distribution captures the correlation coefficient variations for different number of observations used as evidenced in empirical studies using historical data.

28.5. THE MONOTONICITY OF THE DENSITY FUNCTION

It can be shown that $g(\gamma) = 0$ at $\gamma = -1$ and 1. Due to of the complexity of the density function, its curvature cannot be observed easily. Intuition seems to indicate that the density function reaches the maximum at the population correlation coefficient parameter ρ as in a normal distribution. However, this is not true. We can observe from Figures 28.13– 28.14 that the density function in (28.1) reaches the maximum point somewhere around the population correlation coefficient ρ. In this section, we will find the monotonicity of the density function and the mode of the correlation coefficient, or the correlation coefficient that yields the maximum density value. The first derivative of the density function with respect to γ is

$$\begin{aligned} g'(\gamma) = &-\gamma(n-4)\frac{2^{n-3}}{\pi(n-3)!}(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-6)/2} \\ &\times \sum_{s=0}^{\infty}\Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma)^s}{s!} \\ &+\rho\frac{2^{n-2}}{\pi(n-3)!}(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2} \\ &\times \sum_{s=1}^{\infty}\Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma)^{s-1}}{(s-1)!}, \end{aligned}$$

which can be simplified to

$$\begin{aligned} g'(\gamma) = g(\gamma)\Bigg[&\frac{-\gamma(n-4)}{1-\gamma^2} + 2\rho\sum_{s=1}^{\infty}\Gamma^2\left(\frac{(n+s-1)}{2}\right)\frac{(2\rho\gamma)^{s-1}}{(s-1)!}\Bigg/ \\ &\sum_{s=0}^{\infty}\Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma)^s}{s!}\Bigg]. \end{aligned}$$

It is impossible to obtain a closed-form solution for the critical correlation coefficient γ in terms of the population correlation coefficient ρ and the sample size n, and it is not easy to check the second-order condition directly. We can, however, obtain one relationship that can make our analysis much easier and facilitate significantly the numerical iterations to solve for the critical point. Setting the above first-order derivative to zero and solving the first-order condition for γ yields

$$\gamma_m = \frac{\rho}{\sqrt{\rho^2 + [(n-4)/(4\lambda)]^2} + (n-4)/(4\lambda)}, \tag{28.3a}$$

and
$$\gamma_2 = -\left\{\sqrt{1 + [(n-4)/(4\lambda\rho)]^2} + \frac{n-4}{4\lambda\rho}\right\}, \tag{28.3b}$$

where

$$\lambda = \sum_{s=0}^{\infty} \Gamma^2\left(\frac{n+s}{2}\right)\frac{(2\rho\gamma_m)^s}{s!} \Big/ \sum_{s=0}^{\infty} \Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma_m)^s}{s!} > 1.$$

The two expressions in (28.3a) and (28.3b) are the two roots of the quadratic equation simplified from the first-order condition. When ρ is positive and $n > 4$, the second root γ_2 is clearly smaller than -1, which is beyond the range for possible correlation coefficients. It is obvious that the first root in (28.3a) is always within the reasonable range $(-1, 1)$. Therefore, there is only one critical point at which the first derivative of the density function becomes zero. The critical point can be calculated numerically using (28.3a). Substituting the value of γ_m using (28.3a) into (28.1) yields the mode of the distibution for the sample correlation coefficient.

The second column in Table 28.1 lists the modes of the correlation coefficients for the corresponding population correlation coefficients. We can

Table 28.1. Moments of the correlation coefficient and the correlation coefficient modes for selected estimated correlation coefficient.

Estimated	Modes	Mean	Standard deviation	Skewness	Kurtosis
0.75	0.838	0.7177	0.1954	−2.1698	11.5089
0.5	0.626	0.4787	0.2671	−0.862	3.6774
0.25	0.344	0.2372	0.3174	−0.3968	2.7083
0.1	0.142	0.0946	0.3308	−0.1154	2.4933
0	0	0	0.3333	0	2.4545
−0.1	−0.142	−0.0946	0.3308	0.1554	2.4933
−0.25	−0.344	−0.2372	0.3174	0.3966	2.7083
−0.5	−0.626	−0.4787	0.2671	0.862	3.6774
−0.75	−0.838	−0.7177	0.1954	2.1698	11.5089

observe from Table 28.1 that the absolute values of the modes of the correlation coefficients are moderately larger than those of the corresponding population correlation coefficients. The following proposition better describes the monotonicity of the density function.

Proposition 28.1. The density function in (28.1) increases strictly for $-1 < \gamma < \gamma_m$, decreases strictly for $\gamma_m < \gamma < 1$, and reaches the maximum at $\gamma = \gamma_m$ which is given in (28.3a).

Proof. See Appendix at the end of this chapter. □

Proposition 28.1 states that the density function for any sample size $n > 4$ and population correlation coefficient ρ is always a unimodal function, increasing monotonically from zero to reach a maximum at the mode γ_m, and then decreasing monotonically to zero. The mode γ_m can be understood as the "most likely" correlation coefficient.[4] The monotonicity property is consistent with Kendall and Stuart's (1969, page 388) result for Fisher's (1915) density function.

28.6. THE FIRST FOUR MOMENTS OF THE DISTRIBUTION OF THE CORRELATION COEFFICIENT

Moments of a distribution are often useful measures for us to better understand the distribution, and they are often necessary for specific applications such as approximations. Following Fisher (1921), many researchers have transformed the correlation coefficient γ and studied the moments of the transformation instead of γ. Other researchers such as Hotelling (1953) studied other transformations based on the one by Fisher and their moments. We will analyze in this section the first four moments of the distribution of the correlation coefficient using the density function in (28.1). These first four moments can be obtained as follows after simplifications (see Appendix of this chapter for an outline of the proof):

$$E(\gamma) = \frac{\sqrt{\pi}2^{n-3}}{(n-3)!}(1-\rho^2)^{(n-1)/2}\sum_{s=1}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i} \times \Gamma^2\left(\frac{n}{2}+s-1\right)\frac{(2\rho)^{2s-1}}{(2s-1)!}A(i,n)\,, \tag{28.4a}$$

[4]See page 338 in Soper *et al.* (1917) for a graph with the sample size $n = 25$ and the population correlation coefficient $\rho = 0.6$.

$$E(\gamma^2) = \frac{\sqrt{\pi}2^{n-3}}{(n-3!}(1-\rho^2)^{(n-1)/2}\sum_{s=1}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i}$$
$$\times \Gamma^2\left(\frac{n-1}{2}+s-1\right)\frac{(2\rho)^{2s-2}}{(2s-2)!}\mathrm{A}(i,n), \tag{28.4b}$$

$$E(\gamma^3) = \frac{\sqrt{\pi}2^{n-3}}{(n-3!}(1-\rho^2)^{(n-1)/2}\sum_{s=1}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i}$$
$$\times \Gamma^2\left(\frac{n-1}{2}+s-1\right)\frac{(2\rho)^{2s-2}}{(2s-2)!}\mathrm{A}(i,n), \tag{28.4c}$$

$$E(\gamma^4) = \frac{\sqrt{\pi}2^{n-3}}{(n-3!}(1-\rho^2)^{(n-1)/2}\sum_{s=1}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i}$$
$$\times \Gamma^2\left(\frac{n-1}{2}+s-1\right)\frac{(2\rho)^{2s-2}}{(2s-2)!}\mathrm{A}(i,n), \tag{28.4d}$$

where $\mathrm{A}(i,n) = \frac{\Gamma[i+(n-2)/2]}{\Gamma[i+(n-1)/2]}$ and $\binom{s}{i}$ is the combinatorial sign, $\binom{s}{i} = \frac{s!}{i!(s-i)!}$ which is the number of combinations of choosing i out of s.

The four moments in (28.4a) to (28.4d) look rather complicated, yet they are all functions of only two variables, the population correlation coefficient ρ and the sample size n. The double summations in (28.4a) to (28.4d) can be carried out rather easily for each pair of given n and ρ. In fact, the intermediate number s does not have to be very large as the terms for various s converge very fast. Thus, the summations can be calculated for a limited number of s. Using the first four moments in (28.4a) to (28.4d), we can obtain the mean, standard deviation, skewness, and kurtosis of the correlation coefficient as follows:

$$\bar{\gamma} = \mathrm{E}(\gamma), \tag{28.5a}$$

$$\sigma_\gamma = \sqrt{\mathrm{E}(\gamma^2) - [\mathrm{E}(\gamma)]^2}, \tag{28.5b}$$

$$\eta = \frac{\mathrm{E}(\gamma^3) - 3\bar{\gamma}\mathrm{E}(\gamma^2) + 2\bar{\gamma}^3}{\{\mathrm{E}(\gamma^2) - [\mathrm{E}(\gamma)]^2\}^{3/2}}, \tag{28.5c}$$

and

$$\kappa = \frac{\mathrm{E}(\gamma^4) - 4\bar{\gamma}\mathrm{E}(\gamma^3) + 6\bar{\gamma}^2\mathrm{E}(\gamma^2) - 3\bar{\gamma}^4}{\{\mathrm{E}(\gamma^2) - [\mathrm{E}(\gamma)]^2\}^2}. \tag{28.5d}$$

The mean, standard deviation, skewness, and kurtosis are all functions of both the population correlation coefficient ρ and the sample size n, and Table 28.1 lists their values for a few selected population correlation coefficients and the sample size $n = 10$. We can observe from Table 28.1 that the abosolute value of the mean is, in general, smaller than that of the population correlation coeffient ρ. We can also observe that all the four noncentral moments are symmetric of the population correlation coefficient ρ.

28.7. PRICING CORRELATION OPTIONS WITH UNCERTAIN CORRELATION COEFFICIENTS

Although correlation coefficients are highly unstable, most existing models for correlation options, differential swaps, and other correlation-dependent products are based on the assumption of constant correlation coefficients. In this section, we will combine the uncertain correlation coefficient analysis developed earlier in this chapter with the existing pricing models of correlation options. Although we only consider simple correlation options with two underlying instruments and one correlation coefficient, the method can be extended to models with more than one correlation coefficients.

In pricing all correlation options in a Black-Scholes environment, we assumed that the underlying asset prices follow a lognormal process. We maintain all assumptions made earlier in Part IV with the only extension that the two asset returns are correlated with the correlation coefficient γ whose density function is given in (28.1). The population correlation coefficient ρ is constant as given in (28.1).

Following the standard procedure, correlation option with a given correlation coefficient can be priced using the risk-neutral evaluation relationship. However, with uncertain correlation coefficients, this method has to be modified to capture the fluctuating correlation coefficients. Let $CRP(\gamma)$ stand for the price of a correlation option with the correlation coefficient γ. As γ is between -1 and 1, the actual price of the correlation option with stochastic correlation ($CROP$) can be obtained by integrating CRP(γ) for all possible γ from -1 to 1, or alternatively

$$CROP = \int_{-1}^{1} CRP(\gamma)g(\gamma)d\gamma \,. \tag{28.6}$$

In general, a closed-form solution is impossible for $CROP$ as $CRP(\gamma)$ is often expressed in the form of the cumulative function of the standard bivariate normal distribution. However, numerical integrations can be carried out rather conveniently using most computers so that $CROP$ can be obtained very quickly. Specific formulas of $CRP(\gamma)$ for various kinds of correlation options can be found easily in previous chapters.

Equation (28.6) actually gives the expected price of a correlation option with a stochastic correlation coefficients in terms of the pricing formulas of correlation options with deterministic correlation coefficients and the correlation coefficient density function. The standard deviation of $CROP(\gamma)$ with stochastic correlation coefficients can be obtained similarly as follows:

$$\sigma(CROP) = \sqrt{CROP(2) - CROP^2}\,, \tag{28.7}$$

where

$$CROP(i) = \int_{-1}^{1} [CRP(\gamma)]^i g(\gamma) d\gamma, \;\; i = 2, 3, 4, \ldots \tag{28.8}$$

is the ith noncentral moment of $CRP(\gamma)$ with stochastic correlation coefficients.

The standard deviation of $CRP(\gamma)$ with stochastic correlation coefficients given in (28.7) can be calculated conveniently because the second noncentral moment of $CRP(\gamma)$ with stochastic correlation coefficients given in (28.8) can be calculated similarly as $CROP$ in (28.6). The skewness and kurtosis of $CROP$ with stochastic correlation coefficients can be calculated using (28.5c) and (28.5d) simply by substituting $\bar{\gamma}$ with $CROP$ given in (28.8) and $E(\gamma^i)$ with $CROP(i)$ given in (28.8).

As an example to show how a $CROP$ differs from its corresponding CRP with a given constant correlation coefficient, we use the closed-form solution for an exchange option given in (13.8). Substituting the constant population correlation coefficient ρ with the correlation coefficient γ into (13.8) yields

$$CRP(\gamma) = I_1 e^{-g_1\tau} N(d_{e1}) - I_2 e^{-g_2\tau} N(d_{e2})\,, \tag{28.9}$$

where

$$\begin{aligned}
d_{e2} &= \left[\ln\left(\frac{I_1}{I_2}\right) + (g_2 - g_1 - \frac{1}{2}\sigma_a^2)\tau\right] \Big/ (\sigma_a\sqrt{\tau}),\\
d_{e1} &= d_{e2} + \sigma_a\sqrt{\tau}\,,\\
\sigma_a &= \sqrt{\sigma_1^2 - 2\gamma\sigma_1\sigma_2 + \sigma_2^2}\,,
\end{aligned}$$

and all other parameters are the same as in Chapter 13.

Substituting (28.9) into (28.6), we can find the price of the exchange option to exchange the second asset for the first.

28.8. APPROXIMATING PRICES OF CORRELATION OPTIONS WITH UNCERTAIN CORRELATION COEFFICIENTS

We obtained a pricing formula for correlation options with uncertain correlation coefficients in the previous section. As the first four moments of a correlation coefficient can be calculated easily using (28.4), we can find an approximating formula using them and the Taylor series expansion. Expanding the correlation option price $CRP(\gamma)$ around the expected value of γ, we can readily obtain the following:

$$\begin{aligned} CRP(\gamma) = CRP(\bar{\gamma}) &+ CRP'(\bar{\gamma})(\gamma-\bar{\gamma}) + \frac{1}{2}CRP''(\bar{\gamma})(\gamma-\bar{\gamma})^2 \\ &+ \frac{1}{6}CRP'''(\bar{\gamma})(\gamma-\bar{\gamma})^3 \\ &+ \frac{1}{24}CRP^{(4)}(\bar{\gamma})(\gamma-\bar{\gamma})^4 + \dots, \end{aligned} \tag{28.10}$$

where $\bar{\gamma}$ is the expected value of the correlation coefficient given in (28.4a), $CRP'(\bar{\gamma})$, $CRP''(\bar{\gamma})$, $CRP'''(\bar{\gamma})$, and $CRP^{(4)}(\bar{\gamma})$ are the first-, second-, third-, and fourth-order partial derivatives of the correlation option price with respect to the correlation coefficient γ at $\bar{\gamma}$, respectively.

The first-order partial derivative of the correlation option price with respect to the correlation coefficient γ at $\bar{\gamma}$ is actually the chi of the correlation option at $\bar{\gamma}$. As a matter of fact, the other three partial derivatives can be obtained by taking partial derivative to the pricing formulas for correlation options obtained earlier in this book with respect to the correlation coefficient.

Substituting (28.10) into (28.6) yields:

$$\begin{aligned} CROP = CRP(\bar{\gamma}) &+ \frac{1}{2}CRP''(\bar{\gamma})E[(\gamma-\bar{\gamma})^2] + \frac{1}{6}CRP'''(\bar{\gamma})E[(\gamma-\bar{\gamma})^3] \\ &+ \frac{1}{24}CRP^{(4)}(\bar{\gamma})E[(\gamma-\bar{\gamma})^4] + \dots, \end{aligned} \tag{28.11}$$

where $E[(\gamma-\bar{\gamma})^i]$ is the ith central moment of the correlation coefficient.

The central moment $E[(\gamma-\bar{\gamma})^i]$ can be expressed in terms of the noncentral moments given in (28.4) and the binomial expansion. With the first four noncentral moments in (28.4), we know the first four central moments, and therefore, the correlation option price given in (28.6) can be readily approximated in (28.11).

28.9. THE CERTAINTY EQUIVALENT CORRELATION COEFFICIENT

The previous section shows that the price of a correlation option with an uncertain correlation coefficient is not deterministic, yet we can obtain conveniently the expected price, and its standard deviation, skewness, and kurtosis. However, if we concentrate only on the mean of the price of a correlation option with an uncertain correlation coefficient, we would be able to find, using the same deterministic correlation option pricing formula, a correlation coefficient which yields the same mean value of the correlation option price with an uncertain correlation coefficient. This correlation coefficient may be called the certainty equivalent correlation coefficient (CECC), because the correlation option with an uncertain correlation coefficient can be priced, with a value CECC, using the same deterministic pricing formula as if the correlation coefficient is constant. The CECC is very useful in understanding correlation options with uncertain correlation coefficients and in comparing the uncertain model with the existing deterministic model. However, we must bear in mind that the CECC is "equivalent" only at the first moment of the option price.

28.10. SUMMARY AND CONCLUSIONS

We have provided an uncertain model for the correlation coefficient between any two factors. The model has a great potential in pricing all kinds of correlation options and differential swaps, measuring portfolio risks, and analyzing all kinds of cross-market products. As it better captures the uncertain nature of the correlation coefficient, it should reduce the probability of problems in risk-taking, correlation option pricing, and so on. Our example with exchange options shows that the stochastic model is very different from the existing deterministic model. More tests should be done for each kind of correlation options so that we will be able to conclude how the two models differ for different kinds of correlation options.

An immediate area to explore is how the existing hedging strategies would change when the deterministic model is replaced with the uncertain model, or how correlation options would be hedged when the uncertain model is adopted. The hedging strategies should be very different from the existing ones as the price of the correlation option is no longer deterministic compared to the single option price in the deterministic model. Another application of this model is to analyze portfolio risks and compare the results with those in the deterministic theory of portfolio. The model can also be readily used to price differential swaps. Essentially, any economic and financial theory that has something to do with correlation can be possibly improved with the stochastic correlation coefficient model studied in this chapter. Therefore, this model has a great potential to be used in a variety of areas.

We have reviewed existing distribution functions for correlation coefficients from a bivariate normal population and compared their relative effectiveness in pricing correlation options. Our comparisons show that while the Johnson-Kotz-Rao distribution is more convenient in monotonicity analysis than Hotelling's (1953) hypergeometric distribution, the latter converges much faster than the former for large population sizes and high population correlation coefficients. The first four noncentral moments also converge faster with Hotelling's hypergeometric distribution than with the Johnson-Kotz-Rao distribution. All the other four distribution functions considered are not convenient to use as they involve either high-order derivatives or integrations which always require additional time. Our empirical evidence using daily historical foreign exchange data of the four major exchange rates shows that the theoretical distributions largely capture the randomness of the correlation coefficient.

Correlation risk is a very important aspect of derivatives industry, especially exotic options portfolios. Studies in this area still largely concentrate on how to estimate correlation coefficients. This chapter is the first attempt in the literature to try to capture the uncertain nature of a correlation coefficient with a functional distribution. This is only a first attempt, and it can be extended in many directions. In order not to drag too far away from the major topic of this book, we will leave some related literature and useful results in Appendix of this chapter. Interested readers may refer to it for useful and interesting results.

We have covered almost all popular correlation options trading in the OTC marketplace. We may regard some correlation options described in Part IV as arithmetic options because their payoffs can be considered as arithmetic calculations of two or more than two underlying instruments: addition, subtraction, multiplication, and division. Basket options can be considered as addition options, simple spread options as subtraction options, foreign domestic options as multiplication options, and quotient options as division options, respectively. Multiple spread options can be regarded as options on addition and subtraction together. With the methods developed in Part III for various forms of Asian options and in Part IV for various correlation options, we can essentially price options written on any combination of any one or more of the four arithmetic operations with two or more underlying instruments.

QUESTIONS AND EXERCISES

28.1. Give a plausible interpretation to the uncertainty of a correlation coefficient between two factors.

28.2. What are the major existing methods to estimate correlation coefficients?

28.3. What is the most important shortcoming of a correlation coefficient compared to cointegration discussed at the beginning of Part IV?

28.4. What is GARCH?

28.5. What is the problem with GARCH in estimating correlation coefficients?

28.6. What is an implied correlation coefficient?

28.7. Why can't implied correlation coefficients be used as much as implied volatilities?

28.8. What is a certainty equivalence correlation coefficient?

28.9. What is the advantage of Hotelling's hypergeometric distribution over the Johnson-Kotz-Rao distribution?

28.10. Why does Hotelling's hypergeometric distribution have advantages over the Johnson-Kotz-Rao distribution?

28.11*. Show the first moment of the correlation coefficient in (28.4a).

28.12*. Show the third noncentral moment of the correlation coefficient in (28.4c).

28.13*. Show the fourth moment of the correlation coefficient in (28.4d).

28.14*. Show how to express the third and fourth central moments of the correlation coefficient in terms of their first four noncentral moments.

APPENDIX

A28.1. The Proof of Proposition 1

It is easy to check that $f'(-1) = f'(1) = 0$ using (28.1). From the above construction, we know that γ_m is the only γ between -1 and 1 at which the density function has zero first derivative. Suppose there exists one $-1 < s < \gamma_m$ such that $f'(s) < 0$, $f'(\gamma) < 0$ must be true for all $-1 < \gamma < s$, otherwise there would exist one $-1 < \gamma < s$ at which $f'(\gamma) = 0$ which contradicts the uniqueness of γ_m. As $f'(\gamma) < 0$ for all $-1 < \gamma < s$, $f(s) < f(-1) = 0$, which contradicts the positiveness of the density function. Therefore, for any $-1 < s < \gamma_m$, $f'(s) > 0$. The decreasing part can be similarly proven. As the density function increases strictly before γ_m and decreases after γ_m, $\gamma = \gamma_m$ is certainly a maximum point, and $f"(\gamma_m) < 0$, thus γ_m is the mode.

A28.2. The Outline of the Proof of the Four Moments

Making the substitution $\gamma = \sin(z)$, the integrated functions will become functions of the product of $\sin(z)$ with integer powers and $\cos(z)$ to the power $n-3, z \in [-\pi/2, \pi/2]$. Making the second substitution $z = -v$ for the integrating domain $[-\pi/2, 0]$, the two integrating parts will become identical for some s. After the second substitution, the integration functions will become functions of the product of $\sin(z)$ with even power numbers and $\cos(z)$ to the power $n-3, z \in [0, \pi/2]$. The $\sin(z)$ with even powers can be expanded using $(1-\cos^2 z)^n = \Sigma_{i=0}^{n}(-1)^i \binom{n}{i} [\text{con}(z)]^{2i}$. With this expansion, the integration functions will be sums of $\cos(z)$ with various powers over the integration domain $[0, \pi/2]$. The final results can then be obtained using standard calculus results

$$\int_0^{\pi/2} [\cos(x)]^n dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{1}{2}\frac{\pi}{2}, \text{ if } n \text{ is even},$$

and

$$\int_0^{\pi/2} [\cos(x)]^n dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{2}{3}, \text{ if } n \text{ is odd}.$$

A28.3. The Exact Distribution

"Student" 's (1908) density function is given as:

$$g(\gamma) = \frac{\Gamma[(n-1)/2]}{\sqrt{\pi}\Gamma[(n-2)/2]}(1-\gamma^2)^{(n-4)/2} \tag{A28.1}$$

where Γ is the standard gamma function.

Fisher (1915) obtained the general distribution with any sample size n for nonzero population correlation coefficients from a bivariate normal population. Whereas "Student"'s results were obtained by empirical means, Fisher's results depended on geometrical arguments and analogies, which are more mathematically rigorous. Fisher's exact sampling distribution of the correlation coefficient is given as:

$$g(\gamma) = \frac{1}{\pi\Gamma(n-2)}(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/4}\frac{d^{n-2}}{d(\gamma\rho)^{n-2}}\left[\frac{\cos^{-1}(-\rho\gamma)}{\sqrt{1-\rho^2\gamma^2}}\right], \tag{A28.2}$$

where ρ is the population correlation coefficient, cos(.) is the cosine function, $\cos^{-1}(.)$ is the inverse cosine function, $d^{n-2}(G)/d(\gamma\rho)^{n-2}$ is the $(n-2)$th-order derivative of the function G with respect to $\rho\gamma$, the product of the population and sample correlation coefficients.

Fisher's work depends on a bivariate normal population. For distributions of the sample correlation coefficient in non-normal populations, see Quensel (1938) and Gayen (1951).

Although Fisher's (1915) seminal work has stimulated many other studies, his exact distribution function in (A28.2) contains higher order $(n-2)$ derivatives of some complicated trigonometric functions with respect to the product of the population and sample correlation coefficients, and it is not convenient to use.

A28.4. VARIOUS APPROXIMATIONS

Following Fisher's (1915) seminal work, numerous researchers have tried to approximate the density function in (A28.2) in terms of fundamental mathematical functions. Hotelling (1953) obtained an approximation for the density function using the hypergeometric function. The density function is given as follows:

$$f(\gamma) = \frac{(n-2)(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2}}{\sqrt{2}(n-1)\mathrm{B}\left(\frac{1}{2}, n-\frac{1}{2}\right)(1-\rho^2)^{n-3/2}} \mathrm{F}\left[\frac{1}{2},\frac{1}{2}; n\frac{1}{2}; \frac{1}{2}(1+\rho\gamma)\right], \tag{A28.3}$$

where

$$\mathrm{F}(a,b,c,x) = 1 + \frac{ab}{1\times c}x \frac{a(a+1)b(b+1)}{1\times 2\times c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\times 2\times 3\times c(c+1)(c+2)}x^3 + \dots,$$

is the usual hypergeometric function and

$$\mathrm{B}(p,q) = \int_0^\infty x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

is the beta, or B-function.

Johnson and Kotz (1970) obtained the following approximation

$$g(\gamma) = \frac{(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2}}{\sqrt{\pi}\Gamma[(n-1)/2]\Gamma(n/2-1)} \sum_{s=0}^{\infty} \Gamma^2\left(\frac{n+s-1}{2}\right)\frac{(2\rho\gamma)^s}{s!}, \tag{A28.4}$$

where $s! = s(s-1)(s-2)\dots 3\times 2\times 1$ is the factorial of s.

Johnson and Kotz (1970) also listed three other density functions for the sample correlation coefficient with sample size n:

$$g(\gamma) = \frac{(n-2)(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2}}{\pi} \times \int_0^\infty \frac{dw}{\cos w - \rho r)^{n-1}}, \tag{A28.5a}$$

$$g(\gamma) = \frac{(n-2)(1-\rho^2)^{(n-1)/2}(1-\rho^2)^{(n-4)/2}}{\pi} \times \int_1^\infty \frac{dw}{(w - \rho r)^{n-1}(w^2-1)^{1/2}}, \tag{A28.5b}$$

and

$$g(\gamma) = \frac{(1-\rho^2)^{(n-1)/2}(1-\gamma^2)^{(n-4)/2}}{\pi T(n-2)} \times \left(\frac{\partial}{\sin\theta\partial\theta}\right)^{n-2} \frac{\theta}{\sin\theta}, \text{ with } \theta = \cos^{-1}(\rho\gamma). \tag{A28.5c}$$

Rao's (1973) approximation in (28.1) is exactly the same as the Johnson and Kotz (1970) approximation in (A28.4) if we use the identity

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x),$$

and the well-known gamma function $\Gamma(n-2) = (n-3)!$ Therefore, we consider the Johnson and Kotz distribution in (A28.4) and the Rao distribution in (28.1) the same and call it the Johnson-Kotz-Rao distribution.

Another interesting observation is that the mode (the "most likely") correlation coefficient is always different from the population correlation coefficient and the mean correlation coefficient with the only exception of null population correlation coefficient $\rho = 0$. Scoper *et al.* (1917) showed that the distribution of the correlation coefficient is unimodal for small sample $n > 4$. See Scoper *et al.* (1917) for more details on the determination of the mode.

Garwood (1933) derived cumulative functions for small sample sizes. Hotelling (1953) obtained a cumulative function for the sample correlation coefficient in terms of the hypergeometric functions. These cumulative functions may be useful for other applications, for example, they can be used to express values of options written on the correlation coefficient.

A28.5. COMPARING VARIOUS APPROXIMATIONS

Of the six aforementioned density functions, (A28.2) and (A28.5c) include high-order derivatives, and (A28.5a) and (A28.5b) include integrations which have to be carried out numerically for specific applications. These four density functions are not easy to use because the density values cannot be calculated directly using any given population correlation coefficient and sample size, and additional steps have to be taken. In the case of (A28.2) and (A28.5c), the additional steps involve taking derivatives either analytically or numerically, and in the case of (A28.5a) and (A28.5b), the additional steps involve integrating the appropriate functions numerically. These additional steps certainly take additional time, and what is more, accuracy may not be so high using numerical integrations as using explicit formulas such as (A28.4) or (28.1). We will compare the Johnson-Kotz-Rao distribution given in (28.1) and the Hotelling distribution given in (A28.3).

It is obvious that the terms of the Hotelling (1953) hypergeometric approximation converges faster than the Johnson-Kotz-Rao approximation function because the former is a summation in the power of $(1+\rho\gamma)/2$ and the latter a summation in the power of $2\rho\gamma$. The absolute value of the former is always smaller than one and that of the latter can be greater than one. Thus, the former converges more rapidly than the latter. Kendall and Stuart (1969) noticed that for large sample size n, the first term is often enough for the Hotelling approximation. The error caused by stopping at any stage is considerably less than $2/(1-\rho\gamma)$ multiplied by the last term used.

Corresponding to the first four moments in (28.4) using the Johnson-Kotz-Rao distribution function in (28.1), Ghosh (1951) obtained the first four noncentral moments for the correlation coefficient in terms of the Hotelling hypergeometric functions:

$$\mu_1' = c_n F\left[\frac{1}{2},\frac{1}{2};\frac{1}{2}(n+1);\rho^2\right], \tag{A28.6a}$$

$$\mu_2' = 1 - \frac{(n-2)(1-\rho^2)}{n-1} F\left[1,1;\frac{1}{2}(n+1);\rho^2\right], \tag{A28.6b}$$

$$\begin{aligned}\mu_3' = c_n \Big\{\rho F\left[\frac{1}{2},\frac{1}{2};\frac{1}{2}(n+1);\rho^2(n-1)(n-2)\right] F\left[\frac{1}{2},\frac{1}{2};\frac{1}{2}(n-1);\rho^2\right] \\ - F\left[\frac{1}{2},\frac{1}{2};\frac{1}{2}(n+1);\rho^2\right]\Big\},\end{aligned} \tag{A28.6c}$$

$$\mu_4' = 1 + \frac{(n-2)(n-4)(1-\rho^2)}{2(n-1)} F\left[1, 1; \frac{1}{2}(n+1); \rho^2\right] - \frac{n(n-2)(1-\rho^2)}{4\rho^2} \left\{ F\left[1, 1; \frac{1}{2}(n+1); \rho^2\right] - 1 \right\}, \tag{A28.6d}$$

where $c_n = \frac{2}{n-1}\left\{\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\right\}^2$ and $F(a, b, c, x)$ is the same as in (A28.3).

The first four moments in (A28.6) should converge faster than those in (28.4) because the base in the hypergeometric functions in (A28.6) is ρ^2 which is always smaller than 1 and the base in (28.4) is 2ρ which is greater than ρ^2 for any nonperfect population correlation coefficients.

Several authors have derived moments for transformed or approximated correlation coefficients. Fisher (1921) used a remarkable transformation of the sample correlation coefficient, which is later called the z-transformation in the literature, and derived moments for this transformation. Hotelling (1953) tried to improve the Fisher z-transformation and found a better one. See Ruben (1966) for more in this area. Various transformations may be useful and efficient for specific applications, but they are not the focus of this chapter.

PART V:

OTHER OPTIONS

In Part III and Part IV, we have studied over thirty kinds of exotic options which cover the majority of all exotic options in the OTC marketplace. However, there are quite a few popular exotic options which do not fall into the two categories covered previously. In Part V, we will introduce and analyze these popular exotic options which do not belong to either path-dependent or correlation options.

Part V is organized as follows. Chapter 29 discusses and prices package or hybrid options. Chapter 30 introduces and analyzes nonlinear payoff options which include power options as special cases. Chapter 31 explains and prices compound options or options written on options. Chapter 32 discusses and prices chooser options or "as you like" options. Chapter 33 studies contingent premium options or pay-later options. Chapter 34 discusses briefly other exotic options not covered so far in the book.

Chapter 29

PACKAGE OR HYBRID OPTIONS

29.1. INTRODUCTION

Package options are also called hybrid options. They are baskets of vanilla options or portfolios of vanilla options. In order words, package options can be constructed with vanilla options, their corresponding underlying assets, and cash. Thus, they are not really exotic. Since they can be constructed with vanilla options, they can be priced with the pricing formulas of vanilla options very conveniently. We will introduce a few kinds of popular package options and illustrate how to price them using the pricing formulas of vanilla options in this chapter.

29.2. LADDER OPTIONS

We discussed ladder options briefly in Chapter 1. Ladder options are "more path-dependent" than barrier options. Normally, there are several pre-specified ladders or rungs in a ladder option. Whenever the underlying asset price reaches a pre-specified higher level in a series of pre-specified rungs, the intrinsic value of the option is locked. The payoff of a ladder option (LD) can be given as follows:

$$LD = \max\{\omega \max[S(\tau), L_h] - \omega K, 0], \tag{29.1}$$

where L_h represents the highest ladder level reached the n pre-specified ladders of $(L_1, L_2, \ldots, L_n)$, ω is the option binary operator (set to 1 for a call and -1 for a put), and $S(\tau)$ and K represent the underlying asset price of maturity and strike of the option, respectively.

For simplicity, we first analyze how to price ladder options with one ladder H. When there is only one ladder H, the parameter L_n in (29.1) will be either H or zero, depending whether the ladder is reached or not. Thus, we can price a ladder option with one ladder in two cases: the ladder H is never reached within the life of the option and the ladder is reached.

If the ladder H is never reached, the ladder option is simply an up-out option with barrier H, and strike K. The other case is that the ladder is reached. If the ladder is reached, the payoff in (29.1) can be rewritten as

$$LD1 = \max\{\omega[S(\tau) - K], \omega(H - K)0\}. \tag{29.2}$$

The payoff given in (29.29) can be readily rewritten as

$$LD1 = \omega(H - K) + \max\{\omega[S(\tau) - H], 0\}, \tag{29.3}$$

which indicates that the ladder option payoff can be decomposed into a call-or-nothing (CON) digital option and an op-in option with strike H.

Putting the done analaysis together, we can write the price of a ladder option (PLD) option with one ladder H as follows:

$$\begin{aligned} PLD = {} & UPOUT(S, H, K) + UPIN(S, H, H) \\ & +(H - K)UPIN \text{ Digital } (S, H). \end{aligned} \tag{29.4}$$

where $UPOUT(S, A, B)$ and $UPIN(S, A, B)$ stand for upout and upin option prices with spot price S, trigger A, and strike price B, respectively, and $UPIN$ Digital (S, H) stand for the upin digital option with spot prices and trigger H.

From the above analysis, we know that the payoff of a ladder option with one ladder can be considered as a portfolio of three options: an out option, an in option, and a corresponding digital option. Similar argument can be used to price ladder options with two or more than ladders. We leave this as an exercise.

Example 29.1. Find the price of the ladder option with one ladder H = 90 Japanese yen/US dollar, given the strike price 95 yen/dollar, the spot price 94 yen/dollar, the time to maturity three months, the domestic interest rate 6%, the Japanese interest rate 3%, and the yen/dollar exchange rate volatility 20%.

Substituting $S = 1/94 = \$0.01064$, $K = 1/95 = \$0.01053$, $H = 1/90 = \$0.01111$,$\tau = 3/12 = 0.25$, r = 0.06, g = 0.03, and $\sigma = 0.20$ into (10.45) yields the price of the upout call option:

$$\begin{aligned} UPOUT(S, H, K) &= UPOUT(0.01064, 0.01111, 0.01053) \\ &= \$0.00008, \end{aligned}$$

the price of the upin option

$$UPIN(S,H,H) = UPIN(0.01064, 0.01111, 0.01111) \\ = \$0.000212\,,$$

and the price of the upin digital call option

$$UPIN \text{ Digital } (S,H) = UPIN \text{ Digital } (0.01064, 0.01111) \\ = \$0.661992\,,$$

Therefore, the ladder call option price can be found by substituting the above values into (29.4)

$$PLD = 0.000008 + 0.000212 + (0.01111 - 0.01053) \times 0.661992 \\ = \$0.0006$$

29.3. COLLARS

Collars are getting popular in the market these days, especially in stock compensations. It is reported (Wall Street Journal, Wednesday, Sept. 17, 97, C1) that Ted Turner, Vice-Chairman and the largest single shareholder of Time Warner Inc., put a "collar" on two million shares. The collar essentially protected him against any drop in the stock for the next three years below \$39.63 while requiring him to forfeit any profit above \$60.90. Similar report (Wall Street Journal, Oct. 15, 97, C1) stated that World Com Director David C. McCourt get some portfolio insurance in the form of a "zero cost collar".

The payoff of a collar is given as follows:

$$\min\{\max[S(\tau), K_1], K_2\}\,, \tag{29.5}$$

where $0 < K_1 < K_2$, and $S(\tau)$ is the price of the underlying asset at the option maturity.

The payoff in (29.5) is depicted in Figure 29.1. Figure 29.1 shows that the payoff of a collar is similar to that of the difference between two vanilla call options with different strike prices. As a matter of fact, the payoff given in (29.5) is equivalent to the following

$$\max\,[S(\tau) - K_1, 0] - \max\,[S(\tau) - K_2, 0] + K_1\,. \tag{29.6}$$

Expression (29.6) indicates that a collar can be considered as a portfolio including a long forward contract K_1, a long call with a lower strike price K_1,

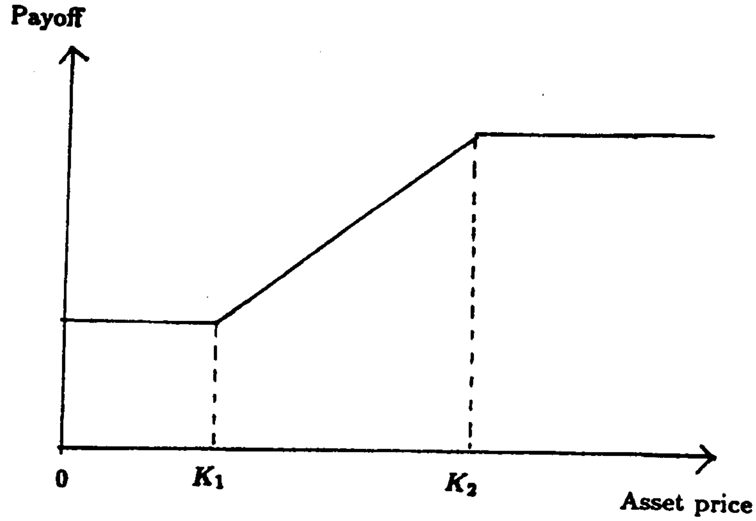

Fig. 29.1. Payoff of a collar.

and a short call with a higher strike price K_2. With the standard method, the price of the collar given in (29.6) can be expressed as

$$COLP = C_{bs}(S, K_1, \sigma, r, g, \tau) - C_{bs}(S, K_2, \sigma, r, g, \tau) + K_1 e^{-r\tau}, \quad (29.7)$$

where $C_{bs}(S, K, \sigma, r, g, \tau)$ stands for the vanilla call option pricing formula given in (3.2) or (10.31) with the spot price S and the strike price K, respectively.

Because the payoff of a collar given in (29.6) is actually the difference between two call options. A collar is actually a call spread plus a forward.

Example 29.2. Find the price of the S&P 500 collarwith two strike parameters $K_1 = \$525$ and $K_2 = \$575$, given the current S&P 500 Index \$555, time to maturity six months, the interest rate 6%, the aggregate dividend 4%, and the volatility 15%.

Substituting $S = \$555$, $K_1 = \$525$, $K_2 = \$575$, $\tau = 6/12 = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (29.7) yields

$$\begin{aligned} COLP &= C_{bs}(555, 525, 0.15, 0.06, 0.04, 0.50) \\ &\quad - C_{bs}(555, 575, 0.15, 0.06, 0.04, 0.50) \\ &\quad + 525e^{-0.06\times 0.50} \\ &= 43.662 - 16.971 + 525 \times 0.9704 = \$536.151\,. \end{aligned}$$

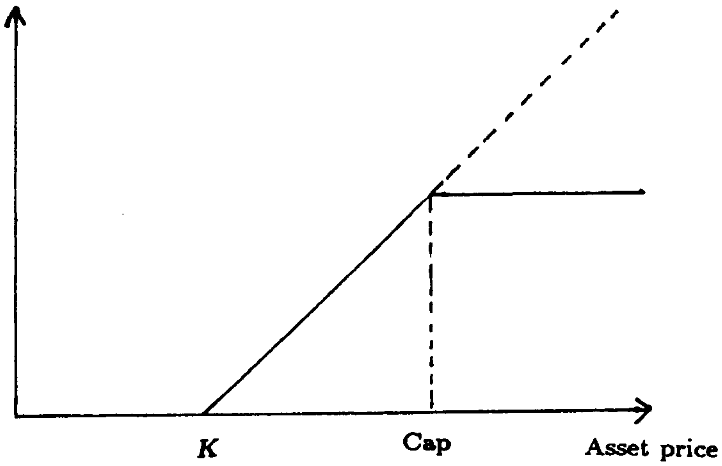

Fig. 29.2. Payoff of a capped call.

29.4. CAPPED CALLS

As we discussed in Chapter 2, vanilla call options have limited liabilities (the premiums) and unlimited upward potential. Because of the unlimited upward potential, vanilla call options are somewhat expensive. If a buyer is willing to sacrifice part of the upward potential which he or she believes to be unlikely, the call option premiums can be reduced. A call option with limited upward payoff is called a capped call. Figure 29.2 depicts the payoff pattern of a capped call. It shows that the payoff of a vanilla call option represented by the broken line is cut off when the payoff of the vanilla call option, or when the underlying asset price reaches the cap at the option maturity. It is obvious that the payoff of the capped call will become the same as that of the corresponding vanilla call option in Figure 2.1 when the cap approaches infinity. The payoff given in Figure 29.2 can be expressed analytically as follows:

$$CAPCALL = \max\{\min[S(\tau), Cap] - K, 0\}, \tag{29.8}$$

where min(. , .), and max (. , .) are functions which give the smaller and larger of two numbers, respectively. K and Cap represent the strike price and the cap of the capped call option, respectively.

The payoff of a capped call option given in (29.8) can be readily obtained by substituting the underlying asset price at the option maturity in the payoff function of a vanilla call option given in (2.1) with the minimum of the underlying asset price at the option maturity and the cap.

The payoff of a capped call given in (29.8) can be expressed alternatively as follows:

$$CAPCALL = \max\left[S(\tau) - K, 0\right] - \max\left[S(\tau) - Cap, 0\right], \qquad (29.9)$$

which indicates that the payoff of a capped call option is the payoff difference between two vanilla call options with strike prices K and Cap, respectively.

Because the payoff of a capped call equals the difference between that of a vanilla call option with strike price K and that of a vanilla call option with strike price Cap, the price of the capped call option must be the same as the difference between the prices of the two corresponding vanilla call options. The price of the capped call (CC) is formally given as follows:

$$CC = C_{bs}(S, K, \sigma, r, g, \tau) - C_{bs}(S, Cap, \sigma, r, g, \tau), \qquad (29.10)$$

where $C_{bs}(S, K, \sigma, r, g, \tau)$ stands for the vanilla call option pricing formula given in (3.2) or (10.31) with spot price S and strike price K, respectively.

It is obvious from (29.10) that the capped call option price approaches the vanilla call option price when the cap approaches infinity because the second term in (29.10) approaches zero as the strike price Cap becomes extremely large.

Example 29.3. Find the price of the S&P 500 capped call option to expire in four months with the strike price K = \$550, given the current S&P 500 Index \$555, the cap \$580, interest rate 6%, the aggregate dividend 4%, and volatility 15%.

Substituting S = \$555, K = \$550, Cap = \$580, $\tau = 4/12 = 0.3333$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (29.10) yields

$$\begin{aligned} CC &= C_{bs}(555, 550, 0.15, 0.06, 0.04, 0.3333) \\ &\quad - C_{bs}(555, 580, 0.15, 0.06, 0.04, 0.3333) = 23.345 - 10.610 \\ &= \$12.735\,. \end{aligned}$$

29.5. FLOORED PUTS

The payoff of a floored put option *(FLPUT)* can be expressed

$$FLPUT = \max\left\{K - \max\left[S(\tau), Floor\right], 0\right\} \qquad (29.11)$$

where *Floor* stands for the floor of the floored put option and other parameters are the same as in (29.8).

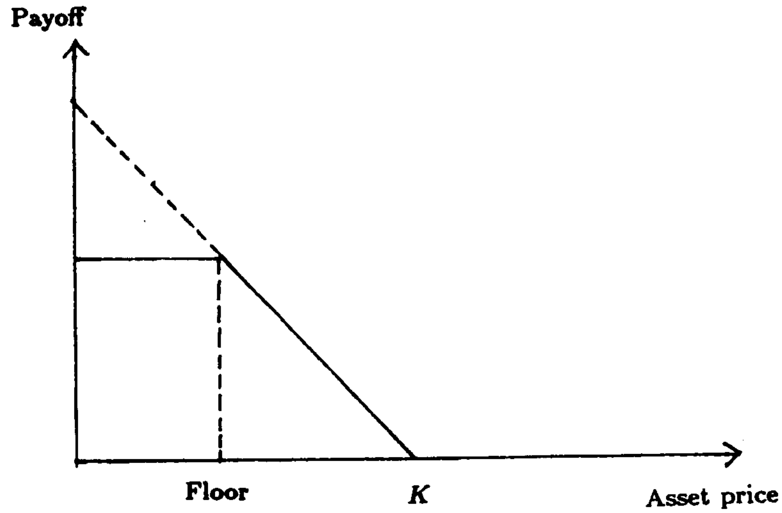

Fig. 29.3. Payoff of a floored put.

Comparing the payoff function of the floored put option with that of the capped call option given in (29.8), we can readily find that whereas the capped call option's payoff is obtained by substituting the underlying asset price at the option maturity in the payoff function of a vanilla call option given in (2.1) with the minimum of the underlying asset price at the option maturity and the cap, the payoff of the floored put option can be obtained by substituting the underlying asset price at the option maturity in the payoff function of a vanilla put option given in (2.2) with the maximum of the underlying asset price and the floor. The payoff of a floored put option is shown in Figure 29.3. It indicates that the payoff of a floored put option equals the difference between the payoff of the corresponding vanilla put option more with the strike price the same as the floor, represented by the broken line, and the payoff of the vanilla put option with strike price K. Following Figure 29.3, the payoff given in (29.11) can be alternatively expressed as

$$FLPUT = \max\left[K - S(\tau), 0\right] - \max\left[Floor - S(\tau), 0\right]. \qquad (29.12)$$

As the payoff of a floored put is the same as the payoff difference between the two vanilla put options with strike prices K and *Floor*, the price of the floored put *(FP)* is the difference between the two corresponding vanilla put options:

$$FP = P_{bs}(S, K, \sigma, r, g, \tau) - P_{bs}(S, Floor, \sigma, r, g, \tau), \qquad (29.13)$$

where $P_{bs}(S, K, \sigma, r, g, \tau)$ stands for the pricing formula of a vanilla put option given in (10.31) with spot price S and strike price K

It is obvious from (29.13) that the floored put option price approaches the vanilla put option price when the floor approaches zero because the second term in (29.13) approaches zero as the strike price *Floor* approaches zero.

Example 29.4. Find the price of the S&P 500 floored put option to expire in four months with strike price K = \$550, given the current S&P 500 Index \$555, the floor \$525, interest rate 6%, the aggregate dividend 4%, and volatility 15%.

Substituting S = \$555, K = \$550, *Floor* = \$525, $\tau = 4/12 = 0.3333$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (29.13) yields

$$\begin{aligned} FP &= P_{bs}(555, 550, 0.15, 0.06, 0.04, 0.3333) \\ &\quad - P_{bs}(555, 525, 0.15, 0.06, 0.04, 0.3333) \\ &= 14.806 - 6.361 \\ &= \$8.445\,. \end{aligned}$$

The capped calls and floored puts studied so for are European-style options. The corresponding American-style options are a little more difficult to price. However, they are not as difficult as standard American options because the density function of the underlying asset within the cap (resp. floor) can be obtained easily if we substitute the upper barrier (resp. lower barrier) in Chapter 10 with the cap (resp. floor). American capped call or floored put options can then be priced with these density functions. Following this in a recent paper, Broadie and Detemple (1995) found closed-form solutions for American capped calls with low dividends.

29.6. BOSTON OPTIONS

Boston options are options with zero initial cost. They are also called break forwards. Since the initial cost for a Boston option is zero, the buyer can buy it without paying any up-front premium. Unlike most other options covered so far in this book which have non-negative payoffs regardless of the underlying asset price at maturity, a Boston option yields negative payoff if the underlying asset price ends up below its strike price. Since there is the probability that the payoff of a Boston option is negative, its expected payoff can be made zero, implying that its price can be zero.

Formally, the payoff of a Boston call option can be given as follows:

$$(F - K) + \max\left[S(\tau) - F, 0\right], \tag{29.14}$$

where $F, K, S(\tau)$ stand for the spot forward price of the underlying asset, the strike price of the option, and the underlying asset price at maturity, respectively.

The payoff function given in (29.14) indicates that a Boston option can be regarded as a portfolio including a simple spread call option with zero strike price on the spread of the underlying asset price at the option maturity and the spot forward price, and forward cash. The second part of (29.14) can also be regarded as a call option with strike price F. Therefore, the price of the Boston option *(BOP)* is the sum of the price of a call option with strike price F and the present value of the forward cash $F - K < 0$:

$$BOP = C_{bs}(S, F, \sigma, r, g, \tau) + e^{-r\tau}(F - K), \tag{29.15}$$

where $C_{bs}(S, F, \sigma, r, g, \tau)$ stands for the extended Black-Scholes formula given in (3.2).

To make the initial cost of the Boston option zero, we can simply solve the equation $BOP = 0$ for strike price K, yielding:

$$K = F - e^{-r\tau}C_{bs}(S, F, \sigma, r, g, \tau). \tag{29.16}$$

29.7. SUMMARY AND CONCLUSIONS

We have introduced a few kinds of package options in this chapter. Package options are actually not new options but baskets of vanilla options, their underlying assets, and cash. To price all these options, we first need to decompose their payoffs into those of vanilla options, their underlying asset prices, and cash, and then express their prices in terms of their corresponding vanilla option prices. Actually, this method can be used to price many other options such as chooser's options. The decomposing method we used in this chapter is not new either. To price a new derivative product, we first need to express its payoff in terms of the payoffs of some products we have already known, and then express the price of the new product in terms of the known products. We will continue to use this method to price other options in the remaining chapters of Part V.

QUESTIONS AND EXERCISES

29.1. What are package options?

29.2. What are the three components of most package options?

29.3. Why may we say that package options are not new options?

29.4. What is a collar?

29.5. How can we decompose the payoff of a collar?

29.6. What is a capped call option?

29.7. Are capped call options always cheaper than their corresponding vanilla call options?

29.8. What is a floored put option?

29.9. What is a Boston option?

29.10. What is unique about the payoff of a Boston option?

29.11. Give an example of another exotic option which may result in negative payoffs.

29.12. Find the price of the S&P 500 collar with two strike parameters K_1 = \$515 and K_2 = \$585, given the current S&P 500 Index \$555, time to maturity four months, interest rate 6%, the aggregate dividend 3.5%, and volatility 18%.

29.13. Find the price of the capped call option if the cap is the same as the upper limit of the collar in Exercise 29.12 and other parameters are the same as in Exercise 29.12.

29.14. Find the price of the floored put option if the floor is the same as the lower limit of the collar in Exercise 29.12 and other parameters are the same as in Exercise 29.12.

29.15. Find the strike price to make the initial cost of a Boston option zero, given the spot price of the S&P 500 Index \$555, time to maturity four months, interest rate 5.8%, the aggregate dividend 3.5%, volatility 18%, and the forward price of the S&P 500 Index \$580.

29.16. Find the price of the ladder option in Example 29.1 if the ladder is changed to 1/85 and other parameters unchanged.

29.17.* Find the pricing formula for a ladder option with two up ladders $H_1 < H_2$.

29.18.* Find the price of the ladder option with the second ladder $H_2 = 1/85 = 0.01176$ and other parameters are the same as in Example 29.1.

29.19* Compare the prices of the ladder options in 29.16 and 29.18.

29.20. Why ladder options become more expensive with more ladders?

Chapter 30

NONLINEAR PAYOFF OPTIONS

30.1. INTRODUCTION

Nonlinear payoff options, as their name implies, are options whose payoffs are nonlinear functions of their underlying asset prices. The nonlinearity can be specified in a number of ways, either as a power, exponential, or other functions of the underlying asset price at the option maturity. Power options are special nonlinear payoff options. The payoff of a power option is a power function of its underlying asset price at maturity. Power options are straightforward extensions of vanilla options. It is well known that the payoff of a vanilla option is simply the difference between the underlying asset price at maturity and its strike price. Graphically, the payoff of a vanilla option is a forty-five degree line starting from the strike price at the horizontal axis representing possible underlying asset price at maturity, as shown in Figure 2.1 for a call option and Figure 2.2 for a put option.

However, there is no reason for the payoff to be always a straight line. Power options have been created to modify the payoff patterns of vanilla options. The name power options comes from the simple algebric term "power", for example, two to the fourth power (2^4) and x to the yth power (x^y). Specifically, the payoff of a power option is either the difference between the underlying asset price at maturity to a nonzero power and the strike price, or the difference between the underlying asset price at maturity and the strike price to a nonzero power.

Since the power is given only to the underlying asset price at maturity in the first kind of power options, we may call them asymmetric power options. Since the power is given to the difference between the underlying asset price at maturity and the strike price in the second kind of power options, we may call them symmetric power options. We will study both asymmetric and symmetric power options in this chapter.

If the power is one in either an asymmetric or symmetric power options, the power option becomes the same as the corresponding vanilla option. If the power is greater (resp. smaller) than one, the power option has a payoff greater (resp. less) than that of the corresponding vanilla option. The purpose of this chapter is to find closed-form solutions for power options and compare them to their corresponding vanilla options. We confine our analysis to a Black-Scholes environment for the purpose of transparency as well as for easy comparisons.

30.2. NONLINEAR PAYOFF OPTIONS

In a typical Black-Scholes environment, the underlying asset return is assumed to follow a lognormal process. Suppose that the underlying asset price is given in (IV1). The payoff of an asymmetric power option ($PAPO$) can be expressed as follows:

$$PAPO = \max\,\{\omega[(S(\tau)]^p - \omega K,\ 0\}, \tag{30.1}$$

where $p \neq 0$ is the power parameter that controls the degree of power in the option; K is the exercise price of the option, max $(.\ ,\ .)$ is a function that gives the larger of two numbers, and ω is the option binary operator (1 for a call option and -1 for a put option).

The payoff function given in (30.1) has one more parameter than that of its corresponding vanilla option. As a matter of fact, p can be either positive or negative. If p is positive, the payoff function given in (30.1) provides the payoff of a power call option when $\omega = 1$ and that of a power put option when $\omega = -1$. However, if p is negative, the payoff function provides the payoff of a power put option when $\omega = 1$ and that of a power call option when $\omega = -1$. Thus, the option operator ω may represent a put option when it is 1 (and $p < 0$) and it may represent a call option when it is -1 (and $p < 0$). Table 30.1 lists all the four possible combinations of p and the option binary operator ω. From it, we know that the power option is a call option when the power parameter and the option operator are of the same sign and it is a put option when they are of opposite signs.

Similarly, the payoff of a symmetric power option is given:

$$PSPO = \max\,\{\omega[S(\tau) - K]^p,\ 0\}\,, \tag{30.2}$$

where all parameters are the same as in (30.1).

Obviously, the payoffs of both asymmetric and symmetric power options given in (30.1) and (30.2) degenerate to the payoff function of a vanilla option when $p = 1$.

Table 30.1. Possible combinations of the power parameter and the option parameter.

Power	Negative	Positive
Call	Put	Call
Put	Call	Put

30.3. PRICING ASYMMETRIC POWER OPTIONS

Using the standard method, we can obtain the expected payoff of the European asymmetric power option given in (30.1) as follows:

$$\begin{aligned} E(PAPO) &= \omega S^p \exp\{p[r - g + \frac{1}{2}(p-1)\sigma^2]\tau\} \\ &\quad \times N[\omega d_p + \omega p\sigma\sqrt{\tau}] - \omega K N(\omega d_p)\,, \end{aligned} \tag{30.3}$$

where

$$d_p = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau + [1 - (1/p)]\ln K}{\sigma\sqrt{\tau}}\,,$$

and all parameters are the same as in (30.1).

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate r. As the risk-neutral valuation relationship guarantees that all assets are expected to appreciate at the same risk-free rate, we can obtain the asymmetric power call option price *(ASPOP)* by discounting the expected payoff given in (30.3) by the risk-free rate:

$$\begin{aligned} ASPOP &= \omega S^p e^{[(p-1)r - pg + p(p-1)\sigma^2/2]\tau} \\ &\quad \times N[\omega d_p + \omega p\sigma\sqrt{\tau}] - \omega K e^{-r\tau} N(\omega d_p)\,, \end{aligned} \tag{30.4}$$

where

$$d_p = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau + [(p-1)]/p]\ln K}{\sigma\sqrt{\tau}}\,.$$

Equation (30.4) is an immediate extension of the Black-Scholes formula for vanilla options. Similar to the extended Black-Scholes formula, it includes two terms: one with the current underlying asset price and the other with the strike price. However, there are obvious differences between them. First of all, the first term includes the current price to the pth power in (30.4) instead of the current price as in the Black-Scholes formula. Secondly, there is one additional factor in the first term, the exponential factor. Thirdly, the argument of the cumulative normal function "d" has one additional part

$[p(p-1)]\ln K$ in the numerator. And lastly, the argument of the cumulative normal distribution function is $p\sigma\sqrt{\tau}$ greater in the first term than in the second term. Substituting $p = 1$ into (30.4) yields:

$$ASPCOP(p = 1) = \omega S e^{-g\tau} N(\omega d + \omega\sigma\sqrt{\tau}) - \sigma K e^{-r\tau} N(\omega d), \quad (30.5)$$

where $d = [\ln(S/K) + (r - g - \sigma^2/2)\tau]/(\sigma\sqrt{\tau})$ is the same as in (10.31).

The pricing formula given in (30.5) is precisely the extended Black-Scholes pricing formula given in (10.31). Thus all the differences between (30.4) and the extended Black-Scholes formula disappear when $p = 1$.

Example 30.1. Find the prices of the power call options to expire in half a year with the power parameter $p = 0.98$ and 1.02, given the spot S&P index \$555, strike price K = \$550, interest rate 6%, the aggregate dividend 4%, and volatility 15%.

Substituting $p = 0.98$ and 1.02, $\sigma = 1$, S = \$555, K = \$550, $\tau = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (30.4) yields

$$\begin{aligned} d_{0.98} &= \frac{\ln(555/550) + (0.06 - 0.04 - 0.15^2/2) \times 0.50 + [(-0.02)/(0.98)] \ln 550}{0.15\sqrt{0.50}} \\ &= -1.088, \\ d_{1.02} &= \frac{\ln(555/550) + (0.06 - 0.04 - 0.15^2/2) \times 0.50 + [(0.1)/(1.1)] \ln 550}{0.15\sqrt{0.50}} \\ &= 1.293, \\ SPC &= 555^{0.98} \exp\{-0.02 \times 0.006 - 0.98 \times 0.04 + 0.98(-0.02) \\ &\quad \times 0.15^2/2] \times 0.5\} N[-1.088 + 0.98 \times 0.15\sqrt{0.5}] \\ &\quad - 550e^{-0.06 \times 0.5} N(-1.088) \\ &= 489.111 \times 0.9805 \times N(-0.984) - 550 \times 0.9704 \times N(-1.088) \\ &= \$4.088\,, \end{aligned}$$

and

$$\begin{aligned} SPC &= 555^{1.02} \exp\{[0.02 \times 0.06 - 1.02 \times +1.02(0.02) \times 0.15^2/2] \times 0.5\} \\ &\quad \times N[1.293 + 1.02 \times 0.15\sqrt{0.5}] - 550e^{-0.06 \times 0.5} N(1.293) \\ &= 629.765 \times 0.9805 \times N(1.401) - 550 \times 0.9704 \times N(1.293) \\ &= \$86.298\,. \end{aligned}$$

Comparing the power call option prices in Example 30.1 and their corresponding vanilla call option price \$28.290, we can readily find that the power call option prices change dramatically with the power parameter. When the power parameter increases (resp. decreases) 2% from 1 to 1.02 (resp. 0.98), the power call option price jumps (resp. drops) from \$28.29 to \$86.298 (resp. \$4.088). Following the same procedure as in Example 30.1, we calculate the power call option prices for various power parameters with other information the same as in Example 30.1. The results are listed in Table 30.2.

Table 30.2. Power call option prices with various power parameters.

p	0.95	0.96	0.97	0.98	0.99	1.00	1.01	1.02	1.03	1.04	1.05
Option price	0.021	0.176	1.010	4.088	12.216	28.29	53.395	86.298	124.817	167.300	213.016

30.4. PRICING SYMMETRIC POWER OPTIONS

For technical reasons, we need to consider whether the power parameter p is an integer or not. We first consider the simple case when the power parameter p is a positive integer. Using the standard method and the binomial expansion,[1] we can obtain the expected payoff of the European symmetric power option (SPO) given in (30.2) with a positive integer power parameter:

$$E(SPO) = \omega \sum_{i=0}^{p} (-1)^i \binom{p}{i} S^{p-i} K^i \exp\{(p-i)\left(r - g\frac{1}{2}\sigma^2\right)\tau + \frac{1}{2}(p-i)^2\sigma^2\tau\} N[\omega d + \omega(n-i)\sigma\sqrt{\tau}], \quad (30.6)$$

where d is the same as in (30.5), $\binom{p}{i}$ is the combinatorial sign which equals $p!/[i!(p-i)!]$ — the number of combinations of choosing i out of p, and $p!$ is the factorial of $p, p! = p(p-1)(p-2)\ldots 3 \times 2 \times 1$.

Using the arbitrage-free arguments, discounting the expected payoff of the European symmetric power option at expiration given in (30.6) by the risk-free interest rate r yields the price of the European symmetric power call option:

[1]For any positive integer p, the following is always true

$$(A-B)^p = \sum_{i=0}^{p} (-1)^i \binom{p}{i} A^{p-i} B^i .$$

$$SPC = \omega \sum_{i=0}^{p} (-1)^i \binom{p}{i} S^{p-i} K^i \exp\{[(p-i)r - (p-i)g + \frac{1}{2}(p-i)(p-i-1)\sigma^2]\tau\} N[\omega d + \omega(p-i)\sigma\sqrt{\tau}] . \quad (30.7)$$

Equation (30.7) appears rather complicated as it includes $p + 1$ terms. We can find its familiarity and generality if we examine a few simple special cases. When $p = 1$, there are $1+1 = 2$ terms in the formula. The symmetric power option with $p = 1$ can be readily simplified to (30.5), the Black-Scholes pricing formula.

When $p = 2$, there are $2+1 = 3$ terms in the formula. The price of the symmetric power option can be simplified to:

$$SPC(p=2) = \omega S^2 \mathrm{e}^{(r-2g+\sigma^2)\tau} N(\omega d + \omega\sigma\sqrt{\tau}) - \omega 2SK \mathrm{e}^{-g\tau} N(\omega d + \omega\sigma\sqrt{\tau}) + \omega K^2 \mathrm{e}^{-r\tau} N(\omega d) . \quad (30.8)$$

We may call the pricing formula given in (30.8) the symmetric square option pricing formula because the payoffs of these options are the squares of the differences between the underlying asset prices at maturity and the strike prices.

When $p = 3$, there are $3+1 = 4$ terms in the formula. The price of the symmetric power option with $p = 3$ can be simplified as follows:

$$SPC(p=3) = \omega S^3 \mathrm{e}^{(2r-3+3\sigma^2)\tau} N(\omega d + \omega 3\sigma\sqrt{\tau}) - 3\omega S^2 K \mathrm{e}^{(r-2g+\sigma^2)\tau} N(\omega d + \omega 2\sigma\sqrt{\tau}) + 3\omega SK^2 \mathrm{e}^{-g\tau} N(\omega d + \omega\sigma\sqrt{\tau}) - \omega K^3 \mathrm{e}^{-r\tau} N(\omega d) . \quad (30.9)$$

We may call the pricing formula given in (30.9) the symmetric cubic option pricing formula because the payoffs of these options are the cubes of the differences between the underlying asset prices at maturity and the strike prices.

We have priced symmetric power options with integer powers. The power parameter p may not, in general, be integer numbers. If the power parameter p is not an integer number, unfortunately, we cannot use the binomial expansion given in this chapter. However, we can use an approximation formula to approach the exact solution. We can obtain the expected payoff of the symmetric power call option given in (30.2) as follows:[2]

$$E(SPO) = \omega \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} S^{p-i} K^i \exp\left\{(p-i)v\tau + \frac{1}{2}(p-i)^2\sigma^2\tau\right\} N[\omega d + \omega(p-i)\sigma\sqrt{\tau}], \qquad (30.10)$$

where $v = r - g - \sigma^2/2$ and d are the same as in the extended Black-Scholes formula given in (30.5).

Discounting the expected payoff of the European symmetric power option at expiration given in (30.10) by the risk-free interest rate r yields the pricing formula of the European symmetric power call option:

$$SPC = \omega \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} S^{p-i} K^i e^{[(p-i-1)r-(p-i)g+\frac{1}{2}(p-i)(p-i-)\sigma^2]\tau} \times N\{\omega[d + (p-i)\sigma\sqrt{\tau}]\}. \qquad (30.11)$$

where $\binom{p}{i}$ is the same combinatorial number as in (30.6) and (30.7) with the exception that the upper limit for the summation p in (30.6) and (30.7) is replaced by infinity.

Equation (30.11) appears rather complicated, yet we can have a better understanding of it with one example. We may call the symmetric power options with $p = 1/2$ symmetric square-root power options because the payoffs of such options are the square-roots of the differences between the underlying asset prices at maturity and the strike prices.

Example 30.2. Find the pricing formula of a symmetric square-root option.

Substituting $p = 1/2$ into (30.11) yields:

$$SPC\left(p = \frac{1}{2}\right) = \omega \sum_{i=0}^{\infty} (-1)^i \binom{0.5}{i} S^{0.5-i} K^i \exp\left\{-\left(\frac{1}{2}+i\right)r - \left(\frac{1}{2}-i\right)g + \frac{1}{2}\left(\frac{1}{2}-i\right)\left(\frac{1}{2}-i-1\right)\sigma^2 \times \left[r - g + \frac{1}{2}(\frac{1}{2}-i)\sigma^2\right]\right\} N\left[\omega d + \omega(\frac{1}{2}-i)\sqrt{\tau}\right]\sigma$$

[2]It can be proven that for any real numbers $|z| < 1$, the following always holds

$$(1-z)^p = 1 - pz + \frac{p(p-1)}{2}z^2 - \frac{p(p-1)(p-2)}{6}z^3 + \ldots = \sum_{i=0}^{\infty}(-1)^i \binom{p}{i} z^i.$$

$$= \sqrt{S}\exp\left[\frac{1}{2}(r-g+\frac{1}{4}\sigma^2)\tau\right] N\left[\omega(d+\frac{1}{2}\sigma\sqrt{\tau})\right] - \frac{1}{2}\frac{K}{\sqrt{S}}$$
$$\times \exp\left[\frac{1}{2}(r-g-\frac{1}{4}\sigma^2)\tau\right] N\left[(d-\frac{1}{2}\sigma\sqrt{\tau})\right]$$
$$-\frac{1}{8}\left(\frac{K^2}{S\sqrt{S}}\right)\exp\left[-\frac{3}{2}\left(-r-g\frac{3}{4}\sigma^2\right)\tau\right] N\left[\omega\left(d-\frac{3}{2}\sigma\sqrt{\tau}\right)\right]$$
$$-\frac{1}{16}\left(\frac{K^3}{S^2\sqrt{S}}\right)\exp\left[-\frac{5}{2}\left(r-g-\frac{5}{2}4\right)\tau\right] N\left[\omega\left(d-\frac{5}{2}\sigma\sqrt{\tau}\right)\right]$$
$$-\frac{5}{16\times 8}\left(\frac{K^4}{S^3\sqrt{S}}\right)\exp\left[-\frac{7}{2}\left(-r-g\frac{5}{4}\sigma^2\right)\tau\right]$$
$$\times N\left[\omega\left(d-\frac{7}{2}\sigma\sqrt{\tau}\right)\right] - \ldots. \tag{30.12}$$

Although there is an infinite number of terms in (30.12) theoretically, the terms approach zero as the number of terms gets larger because both the coefficients and the arguments of the cumulative functions decline monotonically. Therefore, for any specific power parameter p, a limited number of terms is often enough to guarantee an accurate option price.

Example 30.3. Find the price of a symmetric square-root call option to expire in half a year, given other parameters the same as in Example 30.1

Substituting $\omega = 1$, $S = \$555$, $K = \$550$, $\tau = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (30.12) yields

$$d = \frac{\ln(555/550) + (0.06 - 0.04 - 0.15^2/2)\times 0.5}{0.15\sqrt{0.50}} = 0.1314,$$
$$SPC = \sqrt{555}\exp\left[\frac{1}{2}\left(0.06 - 0.04 + \frac{1}{4}\times 0.15^2\right)\times 0.5\right]$$
$$\times N\left[0.1314 + \frac{1}{2}0.15\sqrt{0.50}\right]$$
$$-\frac{1}{2}\frac{550}{\sqrt{555}}\exp\left[-\frac{1}{2}\left(0.06 - 0.04 - \frac{1}{4}\times 0.15^2\right)\times 0.50\right]$$
$$\times N\left[0.1314 - \frac{1}{2}\times 0.15\sqrt{0.50}\right]$$
$$-\frac{1}{8}\left(\frac{550^2}{555\sqrt{555}}\right)\exp\left[-\frac{3}{2}\left(0.06 - 0.04 - \frac{3}{4}\times 0.5\right)\right]$$
$$\times N\left[0.1314 - \frac{3}{2}\times 0.15\sqrt{0.50}\right]$$

$$-\frac{1}{16}\left(\frac{550^3}{555^2\sqrt{555}}\right)\exp\left[-\frac{5}{2}(0.06-0.04-\frac{5}{4}\times 0.15^2)\times 0.5\right]$$

$$\times N\left[0.1314-\frac{5}{2}\times 0.15\sqrt{0.50}\right]-\frac{1}{384}\left(\frac{550^4}{555^3\sqrt{555}}\right)$$

$$\times\exp\left[-\frac{7}{2}\left(0.06-0.04-\frac{7}{4}\times 0.15^2\right)\times 0.50\right]$$

$$\times N\left[0.1314-\frac{7}{2}\times 0.15\sqrt{0.50}\right]-\ldots$$

$$=23.558\times 1.0129\times N(0.1796)-11.673\times 0.9964$$

$$\times N(0.0736)-2.892\times 0.9977\times(-0.0325)-1.433\times 1.010$$

$$\times N(-0.1386)-0.059\times 1.0345\times N(-0.2446)-\ldots.$$

$$=13.108-5.991+1.375-0.634+0.367-\ldots=\$8.082\,.$$

The symmetric square-root call option price \$5.403 in Example 30.3 is significantly lower than its corresponding vanilla call option price \$28.290.

30.5. SENSITIVITIES

It is obvious from our above analysis that the return patterns of power options are flexible with various power parameters. In this section, we try to illustrate that the sensitivities of power options also depend critically on various power parameters compared to that of vanilla options. For simplicity, we only concentrate on the delta and gamma of asymmetric power options. The delta of an asymmetric power option can be readily obtained by taking partial derivative of (30.4) with respect to p. After simplifying:[3]

$$\delta_{asp}=\omega p S^{p-1}e^{[(p-1)r-pg+p(p-1)\sigma^2/2]\tau}N(\omega d_p+\omega p\sigma\sqrt{\tau})\,, \qquad (30.13)$$

where all parameters are the same as in (30.4).

If we substitute $p=1$ into the delta formula given in (30.13), it becomes the delta of a vanilla option, $\omega e^{g\tau}N(\omega d+\omega\sigma\sqrt{\tau})$.

[3]We need to use the following identity to simplify the delta and gamma expressions:

$$S^p e^{[(p-1)r-pg+p(p-1)\sigma^2/2]\tau}f(d_p+p\sigma\sqrt{\tau})=Ke^{-r\tau}f(d_p),$$

which is left as an exercise at the end of this chapter.

The gamma of an asymmetric power can be readily obtained by taking first-order partial derivative of (30.13) with respect to S:

$$\gamma_{asp} = pS^{p-2}e^{[(p-1)r-pg+p(p-1)\sigma^2/2]\tau} \left[\omega(p-1)N(\omega d_p + \omega p\sigma\sqrt{\tau}) + p\frac{f(d_p + p\sigma\sqrt{\tau})}{\sigma\sqrt{\tau}}\right], \quad (30.14)$$

where all parameters are the same as in (30.4).

It can be easily shown that the gamma function given in (30.14) becomes the gamma of a vanila option when $p = 1$.

The sensitivity of the option pricing formula given in (30.4) with respect to the power parameter can be shown to be the following:

$$\frac{\partial}{\partial p}ASPCOP = S^p[\omega(1+\ln S)N(\omega d_p + \omega p\sigma\sqrt{\tau}) + \sigma\sqrt{\tau}] \times e^{[(p-1)r-pg+p(p-1)\sigma^2/2]\tau}, \quad (30.15)$$

which is always positive for $\omega = 1$.

Example 30.4. Find the delta and gamma of the power call option with the power parameter $p = 0.98$ in Example 30.1.

Substituting $p = 0.98$, $\omega = 1$, $S = \$555$, $K = \$550$, $\tau = 6/12 = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (30.13) yields

$$\begin{aligned} \text{delta} &= 0.98 \times 555^{0.98-1} \exp\{[-0.02 \times 0.06 - 0.98 \times 0.04 \\ &\quad + 0.98(-0.02) \times 0.15^2/2] \times 0.5\} \\ &\quad \times N[-1.088 + 0.98 \times 0.15\sqrt{0.5}] \\ &= 0.98 \times 0.8813 \times 0.9805 \times N(-0.984) \\ &= 0.98 \times 0.8813 \times 0.9805 \times 0.1626 \\ &= \$0.1377, \end{aligned}$$

and substituting $p = 0.98$, $\omega = 1$, $S = \$555$, $K = \$550$, $\tau = 6/12 = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.15$ into (30.14) yields

$$\begin{aligned} \gamma_{asp} &= 0.98 \times 555^{0.98-2} \exp\{[-0.02 \times 0.6 - 0.98 \times 0.04 \\ &\quad +0.98(-0.02) \times 0.15^2/2] \times 0.5\} \times \Big[(0.98-1)N(-1.088 \\ &\quad +0.98 \times 0.15\sqrt{0.50}) + 0.98\frac{f(-0.984)}{0.15\sqrt{0.50}}\Big] \end{aligned}$$

$$= 0.98 \times 0.0016 \times 0.9805 \times \left\{-0.02 \times 0.1626 + 0.98 \times \frac{0.2458}{0.15\sqrt{0.50}}\right\}$$
$$= 0.35\%.$$

30.6. SUMMARY AND CONCLUSIONS

Power options are direct extensions of vanilla options so that the payoff curves of power options will not be linear as in the case of vanilla options. Since the power can be set either to the underlying asset price at maturity or to the difference between the underlying asset price at maturity and the strike price, there are two types of power options. We call them asymmetric and symmetric power options, respectively. We have priced both asymmetric and symmetric power options in a Black-Scholes environment and provided Black-Scholes type formulas for these options. Obviously, asymmetric power options have the advantage of simplicity in expression — there are only two terms in the formula for arbitrary power parameters. Although the pricing formulas for symmetric power options are more complicated, they may have particular appeals to some users as their return patterns are different from those of asymmetric power options even with the same power parameters.

Because of the differences between asymmetric and symmetric power options, whether a user chooses to use an asymmetric or a symmetric power option depends on his or her particular need. Due to their flexibility in the payoff patterns, power options have their unique functions many other exotic options do not possess.

QUESTIONS AND EXERCISES

Questions

30.1. What are nonlinear payoff options?

30.2. What are power options?

30.3. What are asymmetric power options?

30.4. What are symmetric power options?

30.5. What is the most important difference between an asymmetric power option and its corresponding symmetric one?

30.6. Can the option binary operator ω alone determine whether an option is a power call option or a power put option? Why?

30.7. Under what conditions can a power option be a power call or put option?

30.8. Are the prices of power call options increasing functions of the power parameter?

Exercises

30.1. Find the prices of the asymmetric power call options to expire in three months with the power parameter $p = 0.99$ and 1.01, given the spot S&P index \$555, strike price $K =$ \$550, interest rate 6%, the aggregate dividend 4%, and volatility 20%.

30.2. Find the prices of the asymmetric power options to expire in four months with the power parameter $p = 0.5$, given the spot S&P index \$555, strike price $K =$ \$24, interest rate 6%, the aggregate dividend 4%, and volatility 20%.

30.3. Find the prices of the asymmetric power options to expire in five months with the power parameter $p = 1.2$, given the spot S&P index \$555, strike price $K =$ \$1960, interest rate 6%, the aggregate dividend 4%, and volatility 20%.

30.4. Find the deltas of the two power call options in Exercise 30.1.

30.5. Find the gammas of the two power call options in Exercise 30.1.

30.6. Find the corresponding symmetric power call option prices in Exercise 30.1.

30.7.* Show the identity given in Footnote 3 of this chapter.

30.8.* Show the sensitivity given in (30.15).

30.9*. Find the first four terms of the pricing formula for symmetric call options with the power parameter $p = 1/4$.

30.10. Find the sixth term in formula (30.12).

30.11. Find the price of the square-root options in Example 30.3 if the time to maturity is three months.

30.12. Find the price of the square-root options in Example 30.3 if the volatility is 25% and other parameters remain the same as in Example 30.3.

30.13. What can you find from Example 30.3 and Exercises 30.11 and 30.12?

Chapter 31

COMPOUND OPTIONS

31.1. INTRODUCTION

Compound options are options written on options. Geske (1977a) demonstrated that risky securities with sequential payouts can be valued as compound options. He priced compound options in terms of multivariate normal distribution functions. Geske and Johnson (1984b) corrected some errors in Geske (1977a). Geske (1979a) first priced compound options in a Black-Scholes environment and found their sensitivity measures. Compound options have been used extensively in option pricing literature to price American options. Roll (1977), Geske (1979b, 1981), and Whaley (1981) developed an appropriate valuation formula based on compound options to price American call options.

As indicated in Chapter 4, there is a positive probability of early exercise with American call options provided that the dividend is high enough. Unlike American call options, there is always the probability of early exercise with American put options even when the underlying stock pays no dividends. Considering the valuation of an American put option a compound option, Geske and Johnson (1984a) developed an elegant analytic formula that requires the explicit summation of two infinite series containing multinormal cumulative functions. Selby (1987) provided a new general identity relating sums of nested multi-normal distributions, and used this identity to improve the computational speed and accuracy of both the Roll-Geske-Whaley American call, and the Geske-Johnson American put option pricing formulas by reducing significantly the number of integrals to be evaluated.

The purpose of this chapter is to introduce and price all the basic types of compound options. As there are two types of vanilla options, call options and put options, there are four types of compound options: a call option written on a call option, a call option written on a put option, a put option written on a call option, and a put option written on a put option. Our objective in this chapter is to price compound options in a Black-Scholes environment. With the integration method introduced in Chapter 11 for exotic barrier options, we can price compound options very conveniently.

31.2. COMPOUND OPTIONS

The payoff of a European compound option with strike price k and time to maturity τ_1 written on a vanilla option ($CMPD$) with strike price K and time to maturity $\tau > \tau_1$ can be expressed as follows:

$$CMPDOC = \max\{\omega C[S(\tau_1),\ K,\ \sigma,\ r,\ g,\ \tau - \tau_1,\ \omega'] - \omega k,\ 0\}, \quad (31.1)$$

where $C[S(\tau_1), K, \sigma, r, g, \tau - \tau_1, \omega']$ stands for the price of the underlying option with the underlying asset price $S(\tau_1)$ at τ_1, strike price K, volatility σ interest rate r, the payout rate of its underlying asset g, and the time to maturity of the option from τ_1 to the maturity of the option $\tau : \tau - \tau_1$, respectively.

There are two option binary operators in (31.1): ω is the binary option operator for the compound option and ω' for the underlying option. Because each of the two option binary operators has two values, there are four combinations for the two option binary operators representing the four types of compound options: $\omega = \omega'$ for a compound call option written on a vanilla call option, $\omega = -1$ and $\omega' = 1$ for a compound put option written on a vanilla call option, $\omega = 1$ and $\omega' = -1$ for a compound call option written on a vanilla put option, and $\omega = \omega' = -1$ for a compound put option written on a vanilla put option, respectively.

31.3. PRICING COMPOUND OPTIONS

Assuming that the underlying asset price follows the standard geometric Brownian motion specified in (IV1) as in all previous chapters, we can express the price of a vanilla option in closed form. Assume that the underlying asset price at τ_1 is known, the vanilla call option price can be given as follows from (10.31):

$$\begin{aligned} C[S(\tau_1), K, \sigma, r, g, \tau - \tau_1, \omega'] &= \omega' S(\tau_1) e^{-g(\tau - \tau_1)} N\{\omega' d_{1bs}[S(\tau_1)]\} \\ &\quad - \omega' K e^{-r(\tau - \tau_1)} N\{\omega' d_{bs}[S(\tau_1)]\}, \end{aligned} \quad (31.2)$$

where

$$d_{bs}[S(\tau_1)] = \frac{\ln[S(\tau_1)/K + v(\tau - \tau_1)}{\sigma\sqrt{\tau - \tau_1}}, d_{1bs}[S(\tau_1)] = d_{bs}[S(\tau_1)] + \sigma\sqrt{\tau - \tau_1},$$

and

$$v = r - g - \sigma^2/2\,.$$

For convenience, we repeat the price solution of the standard geometric Brownian motion with yield g at τ_1:

$$S(\tau_1) = S\exp[v\tau_1 + \sigma z(\tau_1)], \tag{31.3}$$

where τ_1 is the time to maturity of the compound option, $S = S(t)$ is the spot underlying asset price, and $z(\tau_1)$ is a standard Gauss-Wiener process.

The underlying asset price $S(\tau_1)$ is obviously lognormally distributed. Let

$$x = \ln[S(\tau_1)/S]\,,$$

where x is normally distributed with mean $v\tau_1$ and variance $\sigma^2\tau_1$. Let

$$u = (x - v\tau_1)/\sigma\sqrt{\tau_1}$$

stands for the standardized normal variable for the log-return of the underlying asset at τ_1.

Intuitively, the larger the standardized variable u is, the greater the underlying asset price at τ_1, the larger the logarithm return x, the larger the price $S(\tau_1)$, and in turn the higher (resp. lower) the underlying call (resp. put) option price at τ_1 from (31.2), thus, the more (resp. less) likely the compound call (resp. put) option will be in-the-money. Thus, there must exist one critical value of u such that the underlying call option price given in (31.2) equals the strike price of the compound option. At this critical point, the compound option is at-the-money. The critical point can be obtained by solving the following equation:

$$C[Se^{v\tau_1 + u\sigma\sqrt{\tau_1}},\ K,\ \sigma,\ r,\ g,\ \tau - \tau_1,\ \omega'] = k\,, \tag{31.4}$$

where the left-hand side of the equation represents the call option price given in (31.2) and the k at the right-hand side is the strike price of the compound option.

Example 31.1. Find the critical value given in (31.4) with the compound option to expire in three months, the underlying S&P 500 call option to expire in six months, the spot S&P 500 Index \$555, the strike price of the S&P 500 call option \$550, the strike price of the compound option \$3, interest rate 6%, the aggregate dividend of S&P 500 4%, and the volatility of the S&P 500 Index 18%.

We use the trial and error method to find the critical u value. Substituting $\omega' = 1, u = -2, S = \$555, K = \$550, \tau_1 = 0.25, \tau = 3/12 = 0.25, r = 0.06, g = 0.04$, and $\sigma = 0.18$ into the left-hand side of (31.4) using (31.2) yields the underlying call option price

$$C[555 \times e^{(0.06-0.04-0.18^2/2)\times 0.25-2\times 0.18\sqrt{0.25}}, 550, 0.18, 0.06, 0.04, 0.5-0.25, 1]$$
$$= \$3.12$$

which is greater than the strike price of the compound option \$3. Since the call option price is an increasing function of the standardized variable u, we need to try a smaller value than -2. Substituting $u = -2.2$ and other parameters into the left-hand side of (31.4) using (31.2) yields the underlying call option price

$$C[555 \times e^{(0.06-0.04-0.18^2/2)\times 0.25-2\times 0.18\sqrt{0.25}}, 550, 0.18, 0.06, 0.04, 0.5-0.25, 1]$$
$$= \$2.28$$

which is smaller than the strike price of the compound option \$3. As the strike price of the compound option \$3 is between the two option prices \$2.28 and \$3.12, the critical value must be between the two trial values of u: -2.2 and -2. Continuing this procedure, we find the critical value of the compound option $u = -2.0259$.

The above example shows how to solve the equation given in (31.4). The trial and error method illustrated in Example 31.1 is intuitive and can be programmed in any computer conveniently. Other computer algorithms can be used to improve the calculation speed, yet they are not the focus of this book. Let

$$\begin{aligned} y &= y(S,\ K,\ k,\ \sigma,\ r,\ g,\ \tau_1,\ \tau,\ \omega') \\ &= u(S,\ K,\ k,\ \sigma,\ r,\ g,\ \tau_1,\ \tau,\ \omega') \end{aligned} \tag{31.5}$$

stand for the solution of (31.4) multiplied by -1. With this solution, we can obtain the pricing formula of a compound option written on a vanilla option (*CMPD*) by discounting the expected payoff of the compound option given in (31.1) using the integration method illustrated in Appendix of Chapter 11:

$$\begin{aligned} CMPD = {} & \omega' S e^{-g\tau} N_2[\omega\omega'(y + \sigma\sqrt{\tau_1}), \omega' d_1(\tau), \omega\sqrt{\frac{\tau_1}{\tau}}] \\ & - \omega\omega' K e^{-r\tau} N_2\left[\omega\omega' y, \omega' d(\tau), \omega\sqrt{\frac{\tau_1}{\tau}}\right] \\ & - \omega k e^{-r\tau_1} N(\omega y)\,, \end{aligned} \tag{31.6}$$

where

$$d(\tau) = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, d_1(\tau) = d(\tau) + \sigma\sqrt{\tau},$$

and $N_2(a, b, c)$ is the cumulative function of a standard bivaiate normal distribution with upper limits a and b and the correlation coefficient c, and y is given in (31.5).

The pricing formula given in (31.6) has three terms, the first two terms look similar to the two terms in the pricing formula of a correlation digital option given in (15.15) and the third term results from the strike price of the compound option. The univariate cumulative function in the third term can be interpreted as the probability of the compound option to end up in-the-money as in vanilla options.

Example 31.2. Find the compound call and put option prices in Example 31.1.

Substituting the critical value $y = 2.0259$, $\omega = \omega' = 1$, and other parameters into (31.5) yields the compound call option price:

$$\begin{aligned} &555e^{-0.04\times0.5}N_2[2.059 + 0.18\sqrt{0.25}, 0.213, \sqrt{0.5}] - 550e^{-006x0.5} \\ &\quad \times N_2[2.0259, 0.086, \sqrt{0.5}] - 3 \times e^{0.04\times0.25}N(2.0259) \\ &\quad = 555 \times 0.9802 \times 0.584 - 550 \times 0.9704 \times 0.534 - 3 \times 0.99 \times 0.9789 \\ &\quad = \$29.789, \end{aligned}$$

and the corresponding compound put option price can be obtained by substituting the critical value $y = 2.0259$, $\omega = -1$, $\omega' = 1$, and other parameters into (31.6)

$$\begin{aligned} &-555e^{-0.04\times0.05}N_2[-2.0259 - 0.18\sqrt{0.25}, -0.213, \sqrt{0.5}] + 550e^{-0.06\times0.5} \\ &\quad \times N_2[-2.0259, -0.086, \sqrt{0.5}] + 3 \times e^{-004\times0.25}N(-2.0259) \\ &\quad = -555 \times 0.9802 \times 0.017 + 550 \times 0.9704 \times 0.021 + 3 \times 0.99 \times 0.0214 \\ &\quad = \$2.024. \end{aligned}$$

Example 31.3. Find the critical value given in (31.4) for a compound option written on a put option and other parameters are the same as in Example 31.1.

Substituting $\omega' = -1, u = 2, S = \$555, K = \$550, \tau_1 = 0.25, \tau = 6/12 = 0.50, r = 0.06, g = 0.04$, and $\sigma = 0.18$ into the left-hand side of (31.4) using

(31.2) yields the underlying put option price

$$C[555 \times e^{(0.06-0.04-0.18^2/2)\times 0.25+2\times 0.18\sqrt{0.25}}, 550, 0.18, 0.06, 0.04, 0.5-0.25, -1]$$
$$= \$1.86$$

which is lower than the strike price of the compound option $3. Because the call option price is an increasing function of the standardized variable u, we need to try a larger value than -2. Substituting $u = -1$ and other parameters into the left-hand side of (31.4) using (31.2) yields the underlying call option price

$$C[555 \times e^{(0.06-0.04-0.18^2/2)\times 0.25+1\times 0.18\sqrt{0.25}}, 550, 0.18, 0.06, 0.04, 0.5-0.25, 1]$$
$$= \$7.68$$

which is greater than the strike price of the compound option $3. As the compound option strike price $3 is between the two option prices $1.86 and $7.68, the critical value must be between the two trial values of u: -2 and -1. Continuing this procedure, we find the critical value of the compound option $y = -1.6922$.

31.4. PUT-CALL PARITY FOR COMPOUND OPTIONS

Using the same argument as in Chapter 3 to derive the put-call parity for vanilla options, we can readily obtain the put-call parity for compound options

$$CMPD(\omega = 1) - CMPD(\omega = -1) = C(S, k, \omega', \tau) - k e^{-r\tau_1}, \quad (31.7)$$

where $CMPD(\omega = 1)$ and $CMPD(\omega = -1)$ stand for the compound call and put option prices, respectively, strike price $k, C(S, k, \tau, \omega')$ stands for the underlying vanilla option price with spot price S, time to maturity τ, and option binary operator ω', respectively, and other parameters are the same as in (31.6). It is worth noting that the first part on the right-hand side of (31.7) is the value of BS with time to maturity τ but not $\tau - \tau_1$!

The intuition behind the put-call parity given in (31.7) is that the difference of the payoffs of the compound call option and its corresponding compound put option written on the same underlying vanilla option is always $C(S, \tau_1, \omega') - k$, regardless of whether the underlying vanilla option price $C(S, \tau_1, \omega')$ ends up above or below the strike price k. The parity is thus obtained following the same argument as the put-call parity for vanilla options in Chapter 3. We leave the proof as an exercise.

Example 31.4. Find the compound call and put option prices written on the underlying put option using the critical value in Example 31.3.

Substituting the critical value $y = -1.6922$, $\omega = 1$, $\omega' = -1$, and other parameters into (31.6) yields the price of the compound call option on the underlying put option:

$$\begin{aligned}
&-555e^{-0.04\times 0.5}N_2[-1.6922 + 0.18\sqrt{0.25}, -0.213, -\sqrt{0.5}]\\
&\qquad + 550e^{-0.06\times 0.5}N_2[-1.6922, -0.086, -\sqrt{0.5}]\\
&\qquad -3 \times e^{-0.04\times 0.25}N(-1.6922)\\
&\quad = -\ 555 \times 0.9802 \times N_2[-1.6022, -0.213, -0.7071]\\
&\qquad + 550 \times 0.970 \times N_2[-1.6922, -0.086, -0.7071]\\
&\qquad -3 \times 0.99 \times (1 - 0.0453)\\
&\quad = -\ 555 \times 0.9802 \times 0.37967 + 550 \times 0.9704 \times 0.42178 - 2.835\\
&\quad = -\ 206.545 + 225.112 - 2.835\\
&\quad = \$15.732\,.
\end{aligned}$$

and the corresponding compound put option price can be obtained by substituting the critical value $y = 2.0259$, ω= -1, and other parameters into (31.6):

$$\begin{aligned}
&555e^{-0.04\times 0.5}N_2[-2.0259 - 0.18\sqrt{0.25}, -0.213, \sqrt{0.5}]\\
&\qquad + 550e^{-006\times 0.5}N_2[-2.0259, -0.086, \sqrt{0.5}]\\
&\qquad + 3 \times e^{0.04\times 0.25}N(-2.0259)\\
&\quad = -\ 555 \times 0.9802 \times 0.017 + 550 \times 0.9704 \times 0.021 + 3 \times 0.99 \times 0.0214\\
&\quad = \$2.024\,.
\end{aligned}$$

31.5. PRICING AMERICAN OPTIONS USING COMPOUND OPTION PRICING FORMULAS

The compound option pricing formulas have been used extensively in option pricing literature to price American options. Roll (1977), Geske (1979b, 1981), and Whaley (1981) developed an appropriate valuation formula to price American call options. Geske and Johnson (1984a) developed an elegant analytic formula to price American put options that requires the explicit summation of two infinite series containing multi-normal cumulative functions. These analytic formulas for American options contain an infinite number of terms. Since these formulas are exact in the limit, arbitrary accuracy can be obtained by extrapolating from a sequence of limited terms to the actual solutions containing an infinite series. The serious drawback of these analytic formulas is that the cumulative function values of multivariate normal distributions have to be estimated and it takes a lot of time to estimate these values when the number of dimension is large. It is beyond the scope of this book to discuss estimations of the cumulative function values. Interested readers may check Selby (1987) for more detailed references in this area.

31.6. TRIGGER COMPOUND OPTIONS

Compound options are often used in project finance. The combination of trigger options and compound options can certainly increase the flexibility of compound options described earlier in this chapter. The purpose of this section is to introduce and price all basic types of trigger compound options. As there are four types of vanilla compound options and four basic types of barrier options (down-in, down-out, up-in, and up-out), there are altogether sixteen basic types of trigger compound options. Our objective in this section is to price trigger compound options is a Black-Scholes environment. With the integration method used in Chapter 11 and those in Chapter 31 for vanilla compound options, we can price barrier compound options conveniently.

There are eight types of out compound options, because there are two types of underlying options ($\omega' = 1$ and -1), two types of compound options ($\omega = 1$ and -1), and two types of out (down $\theta = 1$ and up $\theta = -1$) options. Symmetrically, there are eight types of knock-in compound options. We first start to price out-compound options and then in-compound options.

Out-Compound Options

With the compound option payoff function given in (31.1) instead of that for a vanilla option following a similar procedure as in pricing standard knockout options as in Chapters 10 and 11 and as in pricing compound options earlier in Chapter 31, we can price down-out compound option using the same restricted density function given in (10.24) or (11.56).

There is some difference in pricing trigger compound options and vanilla compound options. For a vanilla compound option, we can perform standard integration once we obtain the critical point as given in (31.5). Yet, we have to consider the barrier level which divided the density function for a trigger compound option. In our particular example of a down out call on a call, the density is zero for all points below the standardized barrier.

$$-d_{bs}(S, H, \tau_1) = [\ln(H/S) - v\tau_1]/(\sigma\sqrt{\tau_1}) .$$

Thus, the lower limit for the integration should be the minimum of this standardized barrier and the critical point $-u(S, K, k, \sigma, r, g, \tau_1, \tau, \omega')$ given by (31.5). With this in mind, we can find prices of all trigger compound options readily as follows.

The price of a down out call on a call $[(\theta\omega\omega') = (111)]$, a down out put (1-1-1), an up out on put (-11-1), or an up out put on call (-1-11) Out $(\theta, \omega, \omega')$ can be given can be obtained as follows

$$\begin{aligned}
&\text{Out}(\theta, \omega, \omega') \\
&\quad = \theta\omega\left(\frac{\omega + \omega'}{2}\right)\Bigg(CMPD\{S, \omega, \omega', \min[y, d(S, H, \tau_1)]\} \\
&\quad - \left(\frac{H}{S}\right)^{2v/\sigma^2} CMPD\left\{\frac{H^2}{S}, \omega, \omega', \min[y, d(S, H, \tau_1)] + \frac{2a}{\sigma\sqrt{\tau_1}}\right\}\Bigg) \\
&\quad - \theta\omega\left(\frac{\omega - \omega'}{2}\right)\Bigg(CMPD\{S, \omega, \omega', \theta\min[\theta y, \theta d(S, H, \tau_1)]\} \\
&\quad - \left(\frac{H}{S}\right)^{2v/\sigma^2} CMPD\left\{\frac{H^2}{S}, \omega, \omega', \theta\min[\theta y, \theta d(S, H, \tau_1)] + \frac{2a}{\sigma\sqrt{\tau_1}}\right\}\Bigg),
\end{aligned} \tag{31.8}$$

where

$$d_{bs}(S, K, \tau_1) = \frac{\ln(S/K) + v\tau_1}{\sigma\sqrt{\tau_1}},\ a = \ln\left(\frac{H}{S}\right),$$

and $CMPD(S, \omega, \omega', d)$ is given in (31.6).

We can find the intuition in deriving (11.8) through the specific example of a down call on a call $(\theta\omega\omega' = 111)$. The payoff a down call on a call is given in (31.1) if the low barrier is not touched throughout the life of the option, otherwise the options is knocked out with some rebate or nothing. For simplicity, we neglect rebate for the time being. Without the low barrier, we can simply find the solution of (31.4) and find the price of vanilla compound option using (31.6) through integrating the underlying asset price at compound option maturity treating the solution of (31.4) as the lower bound of the integration. Yet with the a low barrier, the density of the underlying asset price at compound option maturity is simply zero for prices below the barrier, therefore, we have to use the larger of the solution of (31.4) and the standardized barrier in our integration in order to obtain the payoff of the barrier compound option. Carrying out the derivation as outlined above following similar steps as to derive (31.6) for vanilla compound options, we can find the pricing formula for a down call on a call (DCC):

$$DCC = CMPD\{S, 1, 1, \min[y, d(S, H, \tau_1)]\}$$
$$- \left(\frac{H}{S}\right)^{2v/\sigma^2} CMPD\left\{\frac{H^2}{S}, 1, 1, \min[y, d(S, H, \tau_1)] + \frac{2a}{\sigma\sqrt{\tau_1}}\right\}\Bigg)$$

which is exactly the same as given in (31.8) if we substituting ($\theta\omega\omega' = 111$) into (31.8). Other three cases can be similarly shown.

The formula given in (9) provides pricing the expression for four of the eight types of trigger compound options. The price of a down out call on a put $[\theta\omega\omega') = (11-1)]$, a down out put on a call (1-11), an up out call on a call (-111), or an up out put on a put (-1-1-1) Out $(\theta, \omega, \omega')$ can be given can be obtained as follows

$$\text{Out}(\theta, \omega, \omega')$$
$$= 0 \text{ if } \theta d(S, H, \tau_1) < \theta y \text{ and}$$
$$= CMPD\{S, -\omega, \omega', \theta\omega \min[\theta\omega y, \theta\omega d(S, H, \tau_1)]\}$$
$$- CMPD\{S, -\omega, \omega', \omega \min[\omega' y, \omega' d(S, H, \tau_1)]\}$$
$$- \left(\frac{H}{S}\right)^{2v/\sigma^2} \left(CMPD\left\{\frac{H^2}{S}, -\omega, \omega', \theta\omega \min[\theta\omega y, \theta\omega d(S, H, \tau_1)] + \frac{2a}{\sigma\sqrt{\tau_1}}\right\}\right.$$
$$- CMPD\left\{\frac{H^2}{S}, -\omega, \omega', \omega' \min[\omega' y, \omega' d(S, H, \tau_1)] + \frac{2a}{\sigma\sqrt{\tau_1}}\right\} \tag{31.9}$$

otherwise, where all parameters and expressions are the same as in (31.8).

In-Compound Options

The two formulas given in (31.8) and (31.9) provide prices for all eight types of out compound options. Because the sum of the price of an in compound option $\ln(\theta, \omega, \omega')$ and that of its corresponding out option $\text{Out}(\theta, \omega, \omega')$ always equals the price of their corresponding underlying compound option, or

$$\ln(\theta, \omega, \omega') + \text{Out}(\theta, \omega, \omega') = CMPD(\omega, \omega'), \tag{31.10}$$

we can find the price of a knock-in compound option readily simply by subtracting the out compound price from their corresponding vanilla compound option price, or

$$\ln(\theta, \omega, \omega') = CMPD(\omega, \omega') - \mathrm{Out}(\theta, \omega, \omega').$$

31.7. SUMMARY AND CONCLUSIONS

Compound options are options written on options. Since there are two types of vanilla options, there are four types of compound options: a call on a call, and a call on a put, a put on a call, and a put on a put. We provided pricing formulas for all the four types of compound options in this chapter. The compound option pricing formulas are often used to price American options but this is beyond the scope of this book.

QUESTIONS AND EXERCISES

31.1. What are compound options?

31.2. How many types of compound options are there?

31.3. Find the critical value given in (31.4) with the compound option to expire in two months, the underlying S&P 500 call option to expire in five months, the spot S&P 500 Index $555, the strike price of the S&P 500 call option $550, the strike price of the compound option $2.50, interest rate 6%, the aggregate dividend of S&P 500 4%, and the volatility of the S&P 500 Index 25%.

31.4. Find the compound call option price using the critical value in Exercise 31.3.

31.5. Show the put-call parity given in (31.7).

31.6. Find the corresponding compound put option price using the compound call option price in (31.4) and the put-call parity given in (31.7).

31.7. Find the critical value given in Equation (31.4) with all the information the same as in Exercise 31.3 with the only exception that the underlying option is a put instead.

31.8. Find the price of the compound call option written on a put option using the critical value in Exercise 31.7.

31.9. Find the corresponding compound put option price using the compound call option price in Exercise 31.8 and the put-call parity given in (31.7).

Chapter 32

CHOOSER OPTIONS

32.1. INTRODUCTION

Chooser options are also called "you-choose" options or "as-you-like" options. A chooser option gives its holder the right to decide at a prespecified time in the future before the maturity of the option whether he or she would like the option to be a call or a put option. This prespecified date is normally called the choice date. The holder decides whether the option is a call or a put depending on his or her expectation of whether the expected payoff of a call will exceed that of a put or otherwise, based on the price movement from the present to the choice date. If the call and put are specified with the same price and maturity time, the chooser option is called a standard, simple, or regular chooser option. Otherwise, it is called a complex chooser option.

Chooser options can be used to hedge exposures that may or may not realize. They are also useful for speculating on changes in the volatility of the underlying asset. Chooser options were particularly popular on the run-up to the 1993 US Congressional vote on the North American Free Trade Agreement (NAFTA). They attracted investors who believed that the passage of NAFTA would substantially boost Mexican equities, but were uncertain whether the bill would be passed. Last October, structures that allowed investors to choose between calls and puts on the US dollar/Deutsche mark cross received a lot of attention when the health of the German Chancellor Helmut Hohl's administration was uncertain.

Chooser options came into existence in the late eighties. Rubinstein (1991, 1992) provided good explanations to chooser options. In Rubinstein's formula, a nonlinear equation has to be solved numerically using the iteration method. Nelken (1993) pointed out the limitation of Rubinstein's method as the iteration method may be slow for convergence in solving the nonlinear equation. He proposed to use numerical integration to find the values of chooser options directly.

Given the time to maturity of a standard chooser option, the further the choice date is away from present, the more flexibility and advantages the holder will have to choose between a call and a put as there exists less time before maturity and hence less uncertainty, thus the more expansive the standard chooser option should be. At one extreme when the choice date is present, the chooser option buyer has to pay the higher of the call price and the put price. At the other extreme when the choice date is at the option maturity, the chooser option price must equal the sum of the call price and the put price.

We concentrate on European-style chooser options in a Black-Scholes environment in this chapter for transparency and easy comparisons with vanilla options.

32.2. CHOOSER OPTIONS

In a Black-Scholes environment the underlying asset return is assumed to follow a lognormal process. Assume that the underlying asset price follows the geometric Brownian motion given in (IV1). Let τ_1 stand for the choice date of a chooser option. The holder of a chooser option (*CHS*) has to make the following decision to obtain the higher value of the call and put options under scrutiny:

$$CHS = \max\{C[S(\tau_1), K_c, \tau_c], P[S(\tau_1), K_p, \tau_p]\}, \tag{32.1}$$

where $C[S(\tau_1), K_c, \tau_c]$ stands for a price of a vanilla call option with the price of the underlying asset $S(\tau_1)$ at the choice time τ_1, the strike price K_c, and the time to maturity $\tau_c, \tau_1 < \tau_c \leq \tau$, and $P[S(\tau_1), K_p, \tau_p]$ stands for the price of a vanilla put option with the price of the underlying asset $S(\tau_1)$ at the choice time τ_1, the strike price K_p, and the time to maturity $\tau_p, \tau_1 < \tau_p \leq \tau$.

The pricing formulas $C[S(\tau_1), K_c, \tau_c]$ and $P[S(\tau_1), K_p, \tau_p]$ are given in (31.2). As $S(\tau_1)$ is not known at the current time, the two prices $C[S(\tau_1), K_c, \tau_c]$ and $P[S(\tau_1), K_p, \tau_p]$ are not known either. However, we know the distribution of $S(\tau_1)$ given in (31.3). With it, we could obtain the expected value for the payoff of the chooser option given in (32.1) using the integration method developed in Appendix of Chapter 11.

32.3. PRICING SIMPLE CHOOSER OPTIONS

In the general situation, both the strike prices and the time to maturity of the call and put options in (32.1) may be different from each other. In other words, not only may the strike prices be different from each other, but also the time to maturity of the two options may be different. However, the call and put options in most chooser options have the same time to maturity as that of the chooser options themselves. Chooser options are normally classified into two groups: simple chooser options and complex chooser options. When the strike prices of the call and put options are the same, the chooser option is called a simple chooser option, and when the strike prices of the call and put options are not the same, the chooser option is called a complex chooser option. Simple chooser options are easier to price than their corresponding complex counterparts. We will price simple chooser options first and then price complex chooser options in the next section.

For a simple chooser option, the call option and its corresponding put option have the same strike price. Let K stand for the common strike price, we can express the price of the put option in terms of the corresponding call option price using the put-call parity given in (3.25):

$$P[S(\tau_1), K, \tau] = C[S(\tau_1), K, \tau] - S(\tau_1)e^{-g(\tau-\tau_1)} + Ke^{-r(-\tau-\tau_1)}. \quad (32.2)$$

Substituting the put option price given in (32.2) into (32.1) and simplifying the result yields the following:

$$CHS = C[S(\tau_1), K, \tau] + \max[Ke^{-r(\tau-\tau_1)} - S(\tau_1), 0]. \quad (32.3)$$

Using the distribution of $S(\tau_1)$ given in (31.3) and the identity given in (13.6), we can find the expected value of a simple chooser option (*EXCHS*) at the choice time as follows:

$$\begin{aligned} EXCHS = {} & Se^{-g\tau+r\tau_1}\{N[d_{1bs}(\tau)] - N[-d(\tau,\tau_1) - \sigma\sqrt{\tau_1}]\} \\ & - Ke^{-r\tau+r\tau_1}\{N[d_{bs}(\tau)] - N[-d(\tau,\tau_1)]\}, \end{aligned} \quad (32.4)$$

where $d(\tau,\tau_1) = d(\tau_1) + \frac{(r-g)(\tau-\tau_1)}{\sigma\sqrt{\tau_1}}$, and $d(s) = d(S, K, s)$ is the argument in the standard extended Black-Scholes formula with the spot and strike prices S and K and the time to maturity s, respectively.

We can obtain the price of the simple chooser option (*PCHS*) by discounting the expected payoff in (32.4) at the risk-free rate of return r from the choice time back to the present:

$$\begin{aligned} PCHS = {} & Se^{-g\tau}\{N[d_{1sb}(\tau)] - N[-d(\tau,\tau_1) - \sigma\sqrt{\tau_1}]\} \\ & - Ke^{-r\tau}\{N[d_{bs}(\tau)] - d(\tau,\tau_1)]\}, \text{ or} \end{aligned} \quad (32.5)$$

or

$$\begin{aligned} PCHS = {} & C(S,K,\tau) - Se^{g\tau}N[-d(\tau,\tau_1) - \sigma\sqrt{\tau_1}] \\ & + Ke^{-r\tau}N[-d(\tau,\tau_1)], \end{aligned} \quad (32.6)$$

where $C(S, K, \tau)$ stands for the extended Black-Scholes call option pricing formula with the spot and strike prices S and K and the time to maturity τ, and all other parameters are the same as in (32.4).

The pricing formula in (32.5) appears a little bit different from other pricing formulas. We can analyze some special cases of (32.6) to have a better understanding of the intuition behind it. The following corollaries list the special results of the pricing formula in (32.6).

Corollary 32.1. The pricing formula of a simple chooser option in (32.6) degenerates to that of a vanilla call option if the choice time is zero and $S > Ke^{-(r-g)\tau}$, and to that of a put option if the choice time is zero and $S < Ke^{-(r-g)\tau}$.

Proof. We can readily show that the argument $d(\tau, \tau_1)$ in (32.6) approaches positive (resp. negative) infinity when the choice time is zero and $S >$ (resp. $<$)$Ke^{-(r-g)\tau}$. The results in Corollary 32.1 are immediate from these values of $d(\tau, \tau_1)$ and the identity $N(x) + N(-x) = 1$ for any real number x. □

Corollary 32.2. The vanilla call (resp. put) option is more expansive than the vanilla put (resp. call) option if the choice time is zero and $S >$ (res. $<$)$Ke^{-r(r-g)\tau}$.

Proof. Immediate from the pricing formula of vanilla options and the identity $N(x) + N(-x) = 1$ for any real number x. □

Combining the results in Corollaries 32.1 and 32.2, we can readily conclude that the simple chooser option price equals the larger of the corresponding vanilla call and put options in the special case when the choice time is the same as present.

Corollary 32.3. The simple chooser option pricing formula in (32.6) degenerates to the sum of the corresponding vanilla call and put option prices when the choice time approaches the time to maturity of the option.

Proof. Immediate from the pricing formula of vanilla options because the argument $d(\tau, \tau_1)$ in (32.6) approaches $d(S, K)$ when the choice time approaches the time to maturity of the option. □

The result given in Corollary 32.3 imply that buying a simple chooser option with the choice time the same as the time to maturity of the option involved costs the buyer the same as buying both the corresponding call and put options.

where $C(S, K, \tau)$ stands for the extended Black-Scholes call option pricing formula with the spot and strike prices S and K and the time to maturity τ, and all other parameters are the same as in (32.4).

The pricing formula in (32.5) appears a little bit different from other pricing formulas. We can analyze some special cases of (32.6) to have a better understanding of the intuition behind it. The following corollaries list the special results of the pricing formula in (32.6).

Corollary 32.1. The pricing formula of a simple chooser option in (32.6) degenerates to that of a vanilla call option if the choice time is zero and $S > Ke^{-(r-g)\tau}$, and to that of a put option if the choice time is zero and $S < Ke^{-(r-g)\tau}$.

Proof. We can readily show that the argument $d(\tau, \tau_1)$ in (32.6) approaches positive (resp. negative) infinity when the choice time is zero and $S >$ (resp. $<$)$Ke^{-(r-g)\tau}$. The results in Corollary 32.1 are immediate from these values of $d(\tau, \tau_1)$ and the identity $N(x) + N(-x) = 1$ for any real number x. □

Corollary 32.2. The vanilla call (resp. put) option is more expansive than the vanilla put (resp. call) option if the choice time is zero and $S >$ (res. $<$)$Ke^{-r(r-g)\tau}$.

Proof. Immediate from the pricing formula of vanilla options and the identity $N(x) + N(-x) = 1$ for any real number x. □

Combining the results in Corollaries 32.1 and 32.2, we can readily conclude that the simple chooser option price equals the larger of the corresponding vanilla call and put options in the special case when the choice time is the same as present.

Corollary 32.3. The simple chooser option pricing formula in (32.6) degenerates to the sum of the corresponding vanilla call and put option prices when the choice time approaches the time to maturity of the option.

Proof. Immediate from the pricing formula of vanilla options because the argument $d(\tau, \tau_1)$ in (32.6) approaches $d(S, K)$ when the choice time approaches the time to maturity of the option. □

The result given in Corollary 32.3 imply that buying a simple chooser option with the choice time the same as the time to maturity of the option involved costs the buyer the same as buying both the corresponding call and put options.

where $C(S, K, \tau)$ stands for the extended Black-Scholes call option pricing formula with the spot and strike prices S and K and the time to maturity τ, and all other parameters are the same as in (32.4).

The pricing formula in (32.5) appears a little bit different from other pricing formulas. We can analyze some special cases of (32.6) to have a better understanding of the intuition behind it. The following corollaries list the special results of the pricing formula in (32.6).

Corollary 32.1. The pricing formula of a simple chooser option in (32.6) degenerates to that of a vanilla call option if the choice time is zero and $S > Ke^{-(r-g)\tau}$, and to that of a put option if the choice time is zero and $S < Ke^{-(r-g)\tau}$.

Proof. We can readily show that the argument $d(\tau, \tau_1)$ in (32.6) approaches positive (resp. negative) infinity when the choice time is zero and $S >$ (resp. $<$)$Ke^{-(r-g)\tau}$. The results in Corollary 32.1 are immediate from these values of $d(\tau, \tau_1)$ and the identity $N(x) + N(-x) = 1$ for any real number x. □

Corollary 32.2. The vanilla call (resp. put) option is more expansive than the vanilla put (resp. call) option if the choice time is zero and $S >$ (res. $<$)$Ke^{-r(r-g)\tau}$.

Proof. Immediate from the pricing formula of vanilla options and the identity $N(x) + N(-x) = 1$ for any real number x. □

Combining the results in Corollaries 32.1 and 32.2, we can readily conclude that the simple chooser option price equals the larger of the corresponding vanilla call and put options in the special case when the choice time is the same as present.

Corollary 32.3. The simple chooser option pricing formula in (32.6) degenerates to the sum of the corresponding vanilla call and put option prices when the choice time approaches the time to maturity of the option.

Proof. Immediate from the pricing formula of vanilla options because the argument $d(\tau, \tau_1)$ in (32.6) approaches $d(S, K)$ when the choice time approaches the time to maturity of the option. □

The result given in Corollary 32.3 imply that buying a simple chooser option with the choice time the same as the time to maturity of the option involved costs the buyer the same as buying both the corresponding call and put options.

Example 32.1. Find the price of a simple chooser options to expire in half a year with the choice time in three and five months, given the current S&P 500 Index \$555, the strike price \$540, the interest rate 6%, the aggregate dividend 4%, and the volatility 20%.

Substituting S = \$555, K = \$540, $r = 0.06$, $g = 0.04$, $\sigma = 0.20$, $\tau_1 = 3/12 = 0.25$, $\tau = 0.50$ into (32.6) yields

$$d(0.50, 0.25) = \left[\ln\frac{555}{540} + \left(0.06 - 0.04 - \frac{1}{2} \times 0.02\right)\right] \Big/ \left(0.20\sqrt{0.25}\right) = 0.324,$$

$$d(0.50, 0.25) = d(0.25) + \frac{(0.06 - 0.04)(0.50 - 0.25)}{0.02\sqrt{0.25}} = 0.374,$$

$$\begin{aligned} PCHS &= C(555, 540, 0.50) - 555 \times e^{0.04\times 0.05} N[-0.374 - 0.20\sqrt{0.25}] \\ &\quad + 540 \times e^{-0.06\times 0.50} N[-0.374] \\ &= \$53.908, \end{aligned}$$

and substituting $\tau_1 = 5/12 = 0.4167$ and other parameters into (32.6) yields

$$\begin{aligned} d(0.4167) &= \left[\ln\left(\frac{555}{540}\right) + \left(0.06 - 0.04 - \frac{1}{2} \times 0.02^2\right)\right. \\ &\quad \left. \times\, 0.4167\right] \Big/ \left(0.02\sqrt{0.4167}\right) = 0.2768, \end{aligned}$$

$$\begin{aligned} d(0.50, 0.4167) &= d(0.4167) + \frac{(0.06 - 0.04)(0.50 - 0.4167)}{0.20\sqrt{0.4167}} \\ &= 0.2897, \end{aligned}$$

$$\begin{aligned} PCHS &= C(555, 540, 0.50) - 555 \times e^{0.04x0.50} N[-0.2897 - 0.20\sqrt{0.4167}] \\ &\quad + 540 \times e^{-0.06x0.50} N[-0.2897] \\ &= \$59.691. \end{aligned}$$

The results in Example 32.1 show that the simple chooser option price with the choice time in five months (\$59.691) is \$5.783 more expensive than that in three months (\$53.908). These results are consistent with the intuition that a simple chooser option is more to the advantage to the holder with longer choice time, or less time before the maturity of the option, because he or she has more information to forecast the underlying asset price at maturity. Therefore, the holder has to pay more. We have the general sensitivity result with respect to the choice time.

Corollary 32.4. The sensitivity of the simple chooser option price with respect to the choice time is:

$$\frac{\partial PCHS}{\partial \tau_1} = \frac{S\sigma e^{-g\tau}}{2\sqrt{\tau_1}} f[d(\tau, \tau_1) + \sigma\sqrt{\tau_1}] > 0, \tag{32.7}$$

where $f(.)$ is the density function of a standard normal distribution and other parameters are the same as in (32.5) and (32.6).

Proof. Taking partial derivative of (32.6) with respect to the choice time τ_1 yields (32.7) after simplifying using the identity (see Exercise 32.16):

$$Se^{-g\tau} f[d(\tau, tau_1) + \sigma\sqrt{\tau_1}] = Ke^{-r\tau} f[d(\tau, \tau_1)]\,. \tag{32.8}$$

□

Example 32.2. Find the sensitivities of the simple chooser option prices with respect to the choice time in Example 31.1.

Substituting $\tau_1 = 0.25$, $d(\tau, \tau_1) = 0.374$, and other parameters into (32.7) yields

$$\begin{aligned}\frac{\partial PCHS(\tau_1 = 0.25)}{\partial \tau_1} &= \frac{555 \times 0.20 \times e^{00.04 \times 0.05}}{2\sqrt{0.25}} f[0.374 + 0.20\sqrt{0.25}] \\ &= \$38.799,\end{aligned}$$

and substituting $\tau_1 = 0.4167$, $d(\tau, \tau_1) = 0.2897$, and other parameters into (32.7) yields

$$\begin{aligned}\frac{\partial PCHS(\tau_1 = 0.4167)}{\partial \tau_1} &= \frac{555 \times 0.20 \times e^{-0.04 \times 0.50}}{2\sqrt{0.25}} f[0.2897 + 0.20\sqrt{0.4167}] \\ &= \$39.756.\end{aligned}$$

The results in Example 32.2 indicate that the simple chooser option price becomes more sensitive to the choice time as the choice time gets closer to the time to maturity of the option. Using the sensitivity results in Example 32.2, we can readily find that the simple chooser option in Example 32.1 with the choice time in five months will be approximately 11 ($39.756 \times 1/365 = \$0.109$) cents and 76 ($39.756 \times 7/365 = \$0.762$) cents more expensive if the choice time is postponed for one day and one week, respectively.

32.4. PRICING COMPLEX CHOOSER OPTIONS

We provided a closed-form solution for simple chooser options and obtained a simple comparative static analysis in the previous section. Complex chooser options are more complicated than simple chooser options because we cannot use the put-call parity to simplify the expression in (32.1). The most critical decision for a chooser option's holder is to judge the relative values of the call and put options under scrutiny. Following the method to find a critical value which divides whether a compound option is in-the-money or out-of-the-money in the previous chapter, we can simply find a critical value which determines the relative magnitudes of the call and put options involved by solving a nonlinear equation. The nonlinear equation of u is given:

$$C[Se^{v\tau_1+u\sigma\sqrt{\tau_1}}, K_c, \sigma, r, g, \tau-\tau_c] = P[Se^{v\tau_1+u\sigma\sqrt{\tau}}, K_p, \sigma, r, g, \tau-\tau_p],$$

or

$$\begin{aligned} S(\tau_1)e^{-g(\tau-\tau_c)}N\{d_{1bs}[S(\tau_1), K_c, \tau-\tau_c]\} + S(\tau_1)e^{-g(\tau-\tau_p)} \\ \times N\{-d_{1bs}[S(\tau_1), K_c, \tau-\tau_p]\} - K_c e^{-\tau r}N\{d_{bs}[bs(\tau_1), K_c, \tau-\tau_c]\} \\ -K_p e^{-r(\tau-\tau_p)}N\{-d_{bs}[S(\tau_1), K_p, \tau-\tau_p]\} = 0, \end{aligned} \quad (32.9)$$

where

$$\begin{aligned} d_{bs}[S(\tau_1), K, s] &= \frac{\ln[S(\tau_1)/K] + (r-g-\sigma^2/2)s}{\sigma\sqrt{s}}, \\ d_{1bs}[S(\tau_1), K, s] &= d_{bs}[S(\tau_1), K, s] + \sigma\sqrt{s}, \\ S(\tau_1) &= Sexp[v\tau_1 + \sigma z(\tau_1)] = Se^{v\tau_1+u\sigma\sqrt{\tau_1}}, \\ v &= r-g-\sigma^2/2, \end{aligned}$$

and K_c, K_p, τ_c and τ_p are the strike prices and the time to maturity of the call and put options, respectively, $\tau_1 < \tau_c, \tau_p, \leq \tau$.

Example 32.3. Find the critical value in Equation (32.9) for a complex chooser option to expire in nine months, the strike prices of the call and put options are \$550 and \$560, both the call and put options are to expire in nine months, the choice time is three months, the spot S&P 500 Index \$555, the interest rate 6%, the aggregate dividend 4%, and the volatility of the S&P 500 Index 20%.

We will use the trial and error method to find the critical u value. Substituting $u = 0$, $S = \$555$, $K_c = \$550$, $K_p = \$560$, $\tau_1 = 3/12 = 0.25$, $\tau = \tau_c = \tau_p = 6/12 = 0.50$, $r = 0.06$, $g = 0.04$, and $\sigma = 0.20$ into the left hand side of (32.9) using (31.2) yields \$5.41 which is greater than zero. Substituting $u = 0.10$ and other parameters into the left hand side of (32.9) using (31.2) yields −\$22.46 which is smaller than zero. The two trial values indicate that the solution must be between 0 and 0.10. Following the same procedures, we can obtain the solution $u = 0.01904$.

The above example shows how to solve equation (32.9). The trial and error method illustrated in Example 32.3 is intuitive and can be programmed in any computer to solve Equation (32.9) conveniently. Other computer algorithms can be used to improve the calculation speed, yet they are not the focus of this book. Let

$$\begin{aligned} y &= y(S, K_c, K_p, \sigma, r, g, \tau_1, \tau_c, \tau_p, \tau) \\ &= -u(S, K_c, K_p, \sigma, r, g, \tau_1, \tau_c, \tau_p, \tau) \end{aligned} \tag{32.10}$$

stand for the solution of (32.9) multiplied by −1. With this solution, we can obtain the pricing formula of a complex chooser option (*CXCS*) by discounting the expected payoff of the complex chooser option given in (32.1) and using the integration method illustrated in Appendix of Chapter 11:

$$\begin{aligned} CXCS = {} & Se^{-g\tau_c} N_2\left[y + \sigma\sqrt{\tau_1}, d_1(K_c, \tau_c), \sqrt{\frac{\tau_1}{\tau_c}}\right] \\ & - K_c e^{-r\tau_c} N_2\left[y, d(K_c, \tau_c), \sqrt{\frac{\tau_1}{\tau_c}}\right] \\ & - Se^{-g\tau_p} N_2\left[-y - \sigma\sqrt{\tau_1}, -d_1(K_p, \tau_p), \sqrt{\frac{\tau_1}{\tau_p}}\right] \\ & - K_p e^{-r\tau_p} N_2\left[-y, -d(K_p, \tau_p), \sqrt{\frac{\tau_1}{\tau_p}}\right], \end{aligned} \tag{32.11}$$

where y is the solution of (32.9) multiplied by −1 given in (32.10),

$$d(K, s) = \left[\ln\left(\frac{S}{K}\right) + \left(r - g + \frac{1}{2}\sigma^2\right)s\right] \Big/ \left(\sigma\sqrt{s}\right), d_1(s) = d(s) + \sigma\sqrt{s},$$

and $N_2(a, b, c)$ is the cumulative function of the standard bivariate normal distribution with the upper limits a and b and the correlation coefficient c.

Example 32.4. Find the price of the complex chooser option in Example 32.3.

Substituting $S = \$555$, $K_c = \$550$, $K_p = \$560$, $\tau_1 = 3/12 = 0.25$, $\tau = \tau_c = \tau_p = 6/12 = 0.50$, $r = 0.06$, $g = 0.04$, $\sigma = 0.20$, and the critical solution $y = -0.01904$ into (32.11) yields

$$
\begin{aligned}
d(K_c, \tau_c) &= \left[\ln\left(\frac{555}{550}\right) + \left(0.06 - 0.04 + \frac{1}{4} + \frac{1}{2} \times 0.20^2\right)0.50\right] \Big/ \left(0.20\sqrt{0.5}\right) \\
&= 0.064, \\
d_1(K_c, \tau_c) &= 0.064 + 0.20\sqrt{0.5} = 0.205, \\
d(K_p, \tau_p) &= \left[\ln\left(\frac{555}{550}\right) + \left(0.06 - 0.04 + \frac{1}{2} \times 0.20^2\right)0.50\right] \Big/ \left(0.20\sqrt{0.5}\right) \\
&= -0.063, \\
d_1(K_p, \tau_p) &= -0.063 + 0.20\sqrt{0.50} = 0.078,
\end{aligned}
$$

and

$$
\begin{aligned}
CXCS &= 555e^{-0.04\times 0.5}N_2[-0.01904 + 0.20\sqrt{0.25}, 0.205, 0.7071] \\
&\quad - 550e^{-0.06\times 0.5}N_2[-0.01904, 0.064, 0.7071] \\
&\quad - 550e^{-0.04\times 0.5}N_2[0.01904 - 0.20\sqrt{0.25}, -0.078, 0.7071] \\
&\quad + 560e^{-0.06\times 0.5}N_2[0.01904, 0.063, 0.7071] \\
&= 555 \times 0.9802 \times 0.4316 - 550 \times 0.9704 \times 0.3833 - 555 \\
&\quad \times 0.9802 \times 0.3437 + 560 \times 0.9704 \times 0.3913 \\
&= \$55.885\,.
\end{aligned}
$$

32.5. SUMMARY AND CONCLUSIONS

A chooser option can reduce its holder's regret in the sense that he or she will not regret making a wrong decision to have bought a vanilla call or put. We have discussed chooser options and provided a closed-form pricing formula for simple chooser options in this chapter. A closed-form pricing formula for complex chooser options is obtained in terms of bivariate normal cumulative functions using the critical values which divide the relative magnitudes of the constituent call option and its corresponding put option in a complex chooser option.

A chooser option is an option whose holder has the choice to specify at a predetermined time before the maturity of the option whether it will be a call or a put option. Since the holder of a chooser option has the right to decide the nature of the option, the chooser option is more to the advantage of the holder, and hence the holder should pay a higher price than buying either the corresponding call or put option. Thus a chooser option is more expensive than either the corresponding call or put option. Chooser options are generally classified into simple chooser options and complex chooser options. When the strike prices of the call and the put options are the same and the two options have the same time to maturity in a chooser option, the chooser option is called a simple chooser option. Otherwise, the chooser option is called a complex option.

QUESTIONS AND EXERISES

32.1. What are chooser options?

32.2. How many types of chooser options are there?

32.3. What is a simple chooser option?

32.4. What is a complex chooser option?

32.5. Why can chooser options reduce their holders' regret?

32.6. Find the prices of simple chooser options to expire in half a year with the choice time in two and four months, given the current S&P 500 Index \$555, the strike price \$545, the interest rate 6%, the aggregate dividend 4%, and the volatility 20%.

32.7. Comparing the results in Exercise 32.6 with those in Example 32.1, what can you conclude?

32.8. Find the sensitivities of the simple chooser option prices in Exercise 32.6 with respect to the choice time.

32.9. Show the results in Corollary 32.4.

32.10. Show the identity in (32.8).

Chapter 33

CONTINGENT PREMIUM OPTIONS

33.1. INTRODUCTION

Since the word contingent stands for something conditional, a contingent premium stands for a conditional premium and a contingent premium option is an option with a conditional premium, or premium under certain conditions. There are quite a few types of contingent premium options, the most popular ones being pay-later options and money-back options. We will first concentrate on pay-later options and then illustrate general contingent premium options in this chapter.

Option buyers often come across the problem of letting options go expired worthless after paying the premiums up-front. The regret of getting nothing after paying up-front premiums can be diminished with pay-later options. The holder of a pay-later option, as the phrase "pay-later" implies, does not pay the writer any up-front premium. Actually, the holder of a pay- later option does not pay any money up-front at all if the option expires out-of-the-money. The holder needs, nevertheless, to pay the writer a prespecified premium only when the option turns out to be in-the-money. These options are sometimes called "collect-on-delivery" options because the writer of such options can collect premiums only when there is something to deliver (the options are in-the-money). Clearly, pay-later options capture investors' desires to avoid unnecessary payment for out-of-the-money options. Nevertheless, they are not riskless, because the holder of a pay-later option has to pay a prespecified premium which is very often more expensive than the otherwise equivalent vanilla option even when the option is slightly in-the-money. Royal Bank of Canada recently pitched another kind of pay-later options called reverse contingent premium options. Clients have to pay premiums if the options are not exercised.

Gastineau (1994b) discussed the concept of contingent premium options. Kat (1994) studied both path-independent and path-dependent contingent premium options in a Black-Scholes environment. We will confine our analysis to a Black-Scholes environment as in all other analyses in this book for transparency and easy comparisons with vanilla options.

33.2. PAY-LATER AND REVERSE PAY-LATER OPTIONS

In a typical Black-Scholes environment, the underlying asset price is assumed to follow a lognormal process. Suppose that the underlying asset price follows the standard geometric process given in (IV1). The payoff of a pay-later option can be expressed formally as follows:

$$\begin{aligned} PPL &= [\omega S(\tau) - \omega K] - Q \text{ if } \omega S(\tau) > \omega K\,, \\ &= 0, \text{ if } \omega S(\tau) \leq \omega K\,, \end{aligned} \tag{33.1}$$

where Q is the prespecified price that the pay-later option buyer has to pay the writer when the option turns out to be in-the-money, K is the exercise price of the option, and ω is the option binary operator (1 for a call option and -1 for a put option).

And the payoff of a reverse contingent premium or reverse pay-later option can be expressed formally as follows:

$$\begin{aligned} PPL &= [\omega S(\tau) - \omega K] \text{ if } \omega S(\tau) > \omega K\,, \\ &= -Q',\ \omega S(\tau) \leq \omega K\,, \end{aligned} \tag{33.2}$$

where Q' is the prespecified price that the reverse contingent premium option buyer has to pay the writer when the option turns out to be out-of-the-money and other parameters are the same as in (33.1).

Using the standard method in obtaining the expected payoff of a vanilla option as in Chapter 2, we can obtain the expected payoff of the European pay-later option given in (33.1) as follows:

$$E(PPL) = \omega S e^{(r-g)\tau} N[\omega d + \omega\sigma\sqrt{\tau}] - (\omega K + Q)N(\omega d)\,, \tag{33.3}$$

where

$$d = \frac{\ln(S/K) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\,.$$

Arbitrage arguments permit us to use the risk-neutral evaluation approach by discounting the expected payoff of an option at expiration by the risk-free interest rate r. We can obtain the pay-later option price (PLP) by discounting the expected payoff given in (33.3) by the risk-free rate τ,

$$PPL = \omega S e^{-g\tau} N[\omega d + \omega\sigma\sqrt{\tau}] - (\omega K + Q)e^{-r\tau} N(\omega d)\,, \tag{33.4}$$

where all parameters are the same as in (33.3).

Alternatively, Formula (33.4) can be expressed as follows:

$$PPL = C_{bs}(S, K, \omega) - Qe^{-r\tau} N(\omega d) , \tag{33.5}$$

where $C_{bs}(S, K, \omega)$ is the price of the otherwise equivalent Black-Scholes option given in (10.31) with the same strike price if the premium is paid up-front.

There are "full pay-later" options and "partial pay-later" options. Whereas the holder of a "full pay-later" option does not pay any up-front premium, one pays a percentage of an amount smaller than the price of the otherwise equivalent vanilla option for a "partial pay-later" option. Assume that the holder pays ψ percent of the price of the otherwise equivalent vanilla option for a "partial pay-later" option, $0 \leq \psi \leq 1$. At one extreme, when $\psi = 0$, the holder pays nothing at all, the "partial pay-later" option becomes a "full pay-later" option. At the other extreme when $\psi = 1$, the holder pays the same premium as the otherwise equivalent vanilla option, the "partial pay-later" option degenerates to the otherwise equivalent vanilla option. Thus, "partial pay-later" options include both "full pay-later" and vanilla options as special cases. Therefore, we will simply analyze "partial pay-later" options in the rest of this section.

As the holder of a "partial pay-later" option pays $\psi C_{bs}(S, K, \omega)$ up-front, arbitrage condition implies that the present value of its expected payoff should be the same as what he or she pays, or $\psi C_{bs}(S, K, \omega)$.

Solving $PPL = \psi C_{bs}(S, K, \omega)$ for Q in (33.5) yields

$$Q = \frac{(1 - \psi) C_{bs}(S,\ K,\ \omega) e^{r\tau}}{N(\omega d)} , \tag{33.6}$$

and its equivalent present value is

$$PV(Q) = \frac{(1 - \psi) C_{bs}(S,\ K,\ \omega)}{N(\omega d)} . \tag{33.7}$$

The pricing formula given in (33.7) for "partial pay-later" options includes pricing formulas for both "full pay-later" options and vanilla options as special cases when the partial pay-later parameter $\psi = 0$ and 1, respectively. The formula given in (33.6) is actually the future value of the Black-Scholes option price (the unpaid portion) compounding continuously at the risk-free rate r divided by the risk-neutral probability that the option turns out to be in-the-money. It can also be understood as that the future value of the Black-Scholes call option premium compounding continuously at the risk-free rate r should be the same as the expected payment of the option in the risk-neutral world. Alternatively, the option premium should be the same whether it is paid up-front $C_{bs}(S, K, \omega)$ or later on Q in the risk-neutral world.

Example 33.1. After appreciating against the US dollar for about 20% from ¥100 per dollar since earlier 1995, the Japanese yen has begun to depreciate against the dollar, the dollar-yen rate being 97 yen per dollar now. Find the present values of the predetermined premiums of the "partial pay-later" options with partial pay-later parameters $\psi = 0$ and 0.50 to expire in three months, given the strike price 100 yen per dollar, the dollar-yen exchange rate volatility 20%, and the US and Japanese interest rates 5.9% and 3.2%, respectively.

Substituting $\omega = 1, \psi = 0, S = 1/97 = 0.0103, K = 1/100 = 0.01, \tau = 3/12 = 0.25, \sigma = 0.20, r = 0.059$, and $g = r_f = 0.032$ into (33.7) yields

$$d = \frac{\ln(0.0103/0.01) + (0.059 - 0.032 - 0.20^2/2) \times 0.25}{0.20\sqrt{0.25}} = 0.322\,,$$

the "full pay-later" call option price:

$$C_{bs}(0.0103,\ 0.01,\ 1)/N(0.322) = 0.0008/0.6095 = \$0.0025,$$

and the "full pay-later" put option price:

$$P_{bs}(0.0103,\ 0.01,\ 1)/N(-0.322) = 0.00043/(1 - 0.6095) = \$0.0011.$$

Substituting $\psi = 0.50$ into (33.7) using the results above for $\psi = 0$ yields the prices for the "partial pay-later" call and put options as

$$(1 - 0.50) \times 0.0025 = \$0.00125, \quad \text{and}$$
$$(1 - 0.50) \times 0.0011 = \$0.00055, \quad \text{respectively.}$$

The "full pay-later" call and put option prices \$0.0025 and \$0.0011 in Example 33.1 are significantly higher than their corresponding vanilla option prices 0.0008 and 0.00043, respectively.

Following the same steps as in pricing "partial pay-later" options above, we can readily find the price of a reverse "partial pay-later" option (RPPL):

$$RPPL = C_{bs}(S,\ K,\ \omega) - Q' e^{-r\tau} N(-\omega d)\,, \tag{33.8}$$

where $C_{bs}(S,\ K,\ \omega)$ is the price of the otherwise equivalent Black-Scholes option given in (10.31) with the same strike price if the premium is paid up-front, and Q' is the prespecified cash payment if the option ends up out-of-the-money.

Solving $RPPL = \psi C_{bs}(S,\ K,\ \omega)$ for Q' for Q' in (33.8) yields

$$Q' = \frac{(1-\psi)C_{bs}(S,\ K,\ \omega)e^{r\tau}}{N(-\omega d)}, \tag{33.9}$$

and its present equivalent value is

$$PV(Q') = \frac{(1-\psi)C_{bs}(S,\ K,\ \omega)}{N(-\omega d)}, \tag{33.10}$$

where ψ is the same percentage of the corresponding vanilla option price paid up-front as in (33.6), $0 \le \psi \le 1$.

The present value of the predetermined premium for a pay-later option given in (33.7) and that for the corresponding reverse pay-later option given in (33.10) satisfy the following condition:

$$PV(Q)N(\omega d) = PV(Q')N(-\omega d)\,. \tag{33.11}$$

Example 33.2. Find the prices of the corresponding reverse "full pay-later" options in Example 33.1.

Using the vanilla call option price $C_{bs}(0.0103,\ 0.01,\ 1)=$ \$0.0008 obtained in Example 33.1, $\psi = 0$, we can obtain the price of the corresponding reverse "full pay-later" call option using (33.1):

$$C_{bs}(0.0103,\ 0.01,\ 1)/N(-0.322) = 0.0008/(1-0.6095) = \$0.002\,,$$

and the reverse "full pay-later" put option price:

$$P_{bs}(0.0103,\ 0.01,\ 1)/N(0.322) = 0.00043/0.6095 = \$0.00071\,.$$

33.3. CONTINGENT PREMIUM OPTIONS (CPOs)

We studied "partial pay-later" options in the previous section. "Partial pay-later" options are one special type of contingent premium options. Kat (1994) studied both path-independent and path-dependent contingent premium options in a Black-Scholes environment. We will describe some of Kat's results briefly in this section.

Kat defined the payoff of a contingent premium option (CPO) as the sum of the payoff of its corresponding vanilla option and a series of prespecified "supershares". A supershare is a composite asset-or-nothing digital option which yields a prespecified cash provided the underlying asset price ends up within a certain range. See Chapter 15 for a more detailed study of supershares. Formally, the payoff of a CPO (PCPO) is given

$$PCPO = \max[\omega S(\tau) - \omega K,\ 0] + \sum_{i=1}^{n} X_i \text{Prob}[a_i \le S(\tau) < a_i + 1]\,, \tag{33.12}$$

where $a_1, a_2, \ldots, a_n$ and a_{n+1} represent boundaries of segments in ascending order with the upper boundary a_{i+1} not included in the ith segment, and X_i is the prespecified cash payment if the underlying asset price ends up in the ith segment.

Since the contingent premium (CP) feature must span the complete outcome space, $a_1 = 0$ and $a_{n+1} = +\infty$ must hold true for all CPOs. We can readily show that the payoff given in (33.12) becomes the payoff of a pay-later option given in (33.1) if we specify $X_1 = 0, X_2 = -Q, a_1 = 0, a_2 = K$, and $a_3 = +\infty$.

The probability $\text{Prob}[a_i \le S(\tau) < a_{i+1}]$ in (33.12) is actually the risk-neutral probability that the underlying asset price finishes in the segment (a_i, a_{i+1}) in a risk-neutral world. Using the results given in Chapter 15 for supershares, we can readily write the price of a CPO as follows

$$CPO = C_{bs}(S, K, \omega) + e^{-r\tau} \sum_{i=1}^{n} X_i \{N[d(a_i)] - N[d(a_{i+1})]\}\,, \tag{33.13}$$

where

$$d(a_i) = \frac{\ln(S/a_i) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = \frac{\ln(S/a_i) + v\tau}{\sigma\sqrt{\tau}}, i = 1, 2, \ldots, n+1\,,$$

$C_{bs}(S,\ K,\ \omega)$ is the price of the corresponding vanilla option with strike price K, and ω is the option binary operator (1 for a call and -1 for a put).

The price of the CPO given in (33.13) is actually a linear function of the prespecified payment with the corresponding coefficients as the discounted risk-neutral probability that the underlying asset price ends up in the corresponding segment, and the price of the corresponding vanilla option as the intercept. We can see the linearity clearly from an example.

Example 33.3. Find the pricing formulas of the CPOs to expire in half a year, given the spot price of the underlying asset \$100, interest rate 5%, volatility of the underlying asset 20%, payout of the underlying asset zero, the strike price of the option \$105, $a_1 = 0, a_2 = \$105$, and $a_3 = +\infty$.

Using the vanilla option pricing formula given in (10.31), we can find the vanilla call and put option prices $C(100, 105, 1) = \$4.524$ and $C(100, 105, -1) = 6.932$.

Substituting $S = \$100$, $K = \$105$, $r = 0.05$, $g = 0$, $\sigma = 0.20$, $\tau = 0.50$, $a_1 = 0$, $a_2 = \$105$, and $a_3 = +\infty$ into (33.13) yields the discounted risk-neutral probabilities

$$e^{-r\tau}\text{Prob}[a_1 \leq S(\tau) < a_2] = e^{-0.05\times 0.50}\{N[d(0)] - N[d(105)]\} = 0.580\,,$$

and

$$e^{-r\tau}\text{Prob}[a_2 \leq S(\tau) < a_3] = e^{-0.05\times 0.50}\{N[d(105)] - N[d(\infty)]\} = 0.395\,.$$

Substituting the vanilla option prices and the above discounted risk-neutral probabilities into (33.13) yields the CP call and put option prices

$$\text{CP call option price} = 4.524 + 0.580X_1 + 0.395X_2\,, \tag{33.14a}$$

and

$$\text{CP put option price} = 6.932 + 0.580X_1 + 0.395X_2\,, \tag{33.14b}$$

where the constants in the above two linear equations represent the prices of the vanilla call and put options.

Example 33.4. Find the amount of cash the holder of the "full pay-later" call and put options has to pay in Example 33.3.

For a "full pay-later" call option, $X_1 = 0$ and $X_2 = -Q$. Substituting $X_1 = 0$ and $X_2 = -Q$ into the linear equation CP call price $= 0$ given in (33.14) and solving for Q yields

$$Q = 4.524/0.395 = \$11.453\,.$$

For a "full pay-later" put option, $X_1 = -Q$ and $X_2 = 0$. Substituting $X_1 = -Q$ and $X_2 = 0$ into the linear equation CP put price $= 0$ given in (33.14b) and solving for Q yields

$$Q = 6.932/0.580 = \$11.952\,.$$

which yields a prespecified cash provided the underlying asset price ends up within a certain range. See Chapter 15 for a more detailed study of supershares. Formally, the payoff of a CPO (PCPO) is given

$$PCPO = \max[\omega S(\tau) - \omega K,\ 0] + \sum_{i=1}^{n} X_i \text{Prob}[a_i \le S(\tau) < a_i + 1]\,, \tag{33.12}$$

where $a_1, a_2, \ldots, a_n$ and a_{n+1} represent boundaries of segments in ascending order with the upper boundary a_{i+1} not included in the ith segment, and X_i is the prespecified cash payment if the underlying asset price ends up in the ith segment.

Since the contingent premium (CP) feature must span the complete outcome space, $a_1 = 0$ and $a_{n+1} = +\infty$ must hold true for all CPOs. We can readily show that the payoff given in (33.12) becomes the payoff of a pay-later option given in (33.1) if we specify $X_1 = 0, X_2 = -Q, a_1 = 0, a_2 = K$, and $a_3 = +\infty$.

The probability $\text{Prob}[a_i \le S(\tau) < a_{i+1}]$ in (33.12) is actually the risk-neutral probability that the underlying asset price finishes in the segment (a_i, a_{i+1}) in a risk-neutral world. Using the results given in Chapter 15 for supershares, we can readily write the price of a CPO as follows

$$CPO = C_{bs}(S, K, \omega) + e^{-r\tau} \sum_{i=1}^{n} X_i \{N[d(a_i)] - N[d(a_{i+1})]\}\,, \tag{33.13}$$

where

$$\begin{aligned} d(a_i) &= \frac{\ln(S/a_i) + (r - g - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln(S/a_i) + \upsilon\tau}{\sigma\sqrt{\tau}}, i = 1, 2, \ldots, n+1\,, \end{aligned}$$

$C_{bs}(S,\ K,\ \omega)$ is the price of the corresponding vanilla option with strike price K, and ω is the option binary operator (1 for a call and -1 for a put).

The price of the CPO given in (33.13) is actually a linear function of the prespecified payment with the corresponding coefficients as the discounted risk-neutral probability that the underlying asset price ends up in the corresponding segment, and the price of the corresponding vanilla option as the intercept. We can see the linearity clearly from an example.

Example 33.3. Find the pricing formulas of the CPOs to expire in half a year, given the spot price of the underlying asset \$100, interest rate 5%, volatility of the underlying asset 20%, payout of the underlying asset zero, the strike price of the option \$105, $a_1 = 0, a_2 = \$105$, and $a_3 = +\infty$.

Using the vanilla option pricing formula given in (10.31), we can find the vanilla call and put option prices $C(100, 105, 1) = \$4.524$ and $C(100, 105, -1) = 6.932$.

Substituting $S = \$100$, $K = \$105$, $r = 0.05$, $g = 0$, $\sigma = 0.20$, $\tau = 0.50$, $a_1 = 0$, $a_2 = \$105$, and $a_3 = +\infty$ into (33.13) yields the discounted risk-neutral probabilities

$$e^{-r\tau}\text{Prob}[a_1 \le S(\tau) < a_2] = e^{-0.05\times0.50}\{N[d(0)] - N[d(105)]\} = 0.580\,,$$

and

$$e^{-r\tau}\text{Prob}[a_2 \le S(\tau) < a_3] = e^{-0.05\times0.50}\{N[d(105)] - N[d(\infty)]\} = 0.395\,.$$

Substituting the vanilla option prices and the above discounted risk-neutral probabilities into (33.13) yields the CP call and put option prices

$$\text{CP call option price} = 4.524 + 0.580X_1 + 0.395X_2\,, \tag{33.14a}$$

and

$$\text{CP put option price} = 6.932 + 0.580X_1 + 0.395X_2\,, \tag{33.14b}$$

where the constants in the above two linear equations represent the prices of the vanilla call and put options.

Example 33.4. Find the amount of cash the holder of the "full pay-later" call and put options has to pay in Example 33.3.

For a "full pay-later" call option, $X_1 = 0$ and $X_2 = -Q$. Substituting $X_1 = 0$ and $X_2 = -Q$ into the linear equation CP call price = 0 given in (33.14) and solving for Q yields

$$Q = 4.524/0.395 = \$11.453\,.$$

For a "full pay-later" put option, $X_1 = -Q$ and $X_2 = 0$. Substituting $X_1 = -Q$ and $X_2 = 0$ into the linear equation CP put price = 0 given in (33.14b) and solving for Q yields

$$Q = 6.932/0.580 = \$11.952\,.$$

33.4. MONEY-BACK OPTIONS

Money-back options are another type of CPOs. A money-back option pays back when the option finishes in-the-money. A money-back call (resp. put) option might be a very attractive instrument for an investor with a strong view on a rise (resp. fall) of the underlying price. Before giving the pricing formula of money-back options, let's take an example.

Example 33.5. Find the prices of money-back options in Example 33.3.

For a money-back call (MBC) option, we can specify $X_1 = 0$ and $X_2 = MBC$. Substituting $X_1 = 0$, $X_2 = MBC$, and CP call price $= MBC$ into the linear equation (33.14a) and solving for MBC yields

$$MBC = 4.524/(1 - 0.395) = \$7.478.$$

For a money-back put (MBP) option, we can specify $X_1 = MBP$ and $X_2 = 0$. Substituting $X_1 = MBP, X_2 = 0$, and CP put price $= MBP$ into the linear equation (33.14b) and solving for MBP yields

$$Q = 6.932/(1 - 0.580) = \$16.505.$$

Substituting $a_1 = 0, a_2 = K$, and $a_3 = +\infty$ into (33.13) and using the method illustrated in Example 33.5, we can readily find the price for a money-back (MB) option as follows

$$MB(S, K, \omega) = \frac{C_{bs}(S, K, \omega)}{1 - e^{-r\tau} N(\omega d)}, \tag{3.15}$$

where all parameters are the same as in (33.13).

33.5. PATH-DEPENDENT CPOs

The CPOs studied so far in this chapter are path-independent. Their prices depend only on the prices of the underlying asset at the option maturity. Besides path-independent CPOs, there are path-dependent CPOs with receivables dependent on the paths taken by the underlying asset prices to reach their values at expiration. There are different ways to make the payoff of a CPO dependent on the path. Barrier-driven CPOs are combinations of barrier options studied in Chapters 10 and 11 and path-independent CPOs covered earlier in this chapter. As a matter of fact, we can provide pricing formulas for barrier-driven CPOs readily using the results in Chapters 10, 11 and 15, and the method to price path-independent CPOs earlier in this chapter. We leave some as exercises at the end of this chapter for interested readers.

33.6. SUMMARY AND CONCLUSION

We have analyzed the payment of a pay-later option in a risk-neutral environment. Our results show that the amount the buyer has to pay when the option turns out to be in-the-money is actually the future value of the corresponding Black-Scholes call option premium compounding continuously at the risk-free rate r divided by the probability that the option turns out to be in-the-money. This is not surprising because the expected value of a pay-later option should be zero in the risk-neutral world.

A pay-later option can somewhat reduce option buyers' regret of letting options expire worthless after paying up-front premiums. However, they are not riskless because the payment when the option turns out to be in-the-money (no matter how little in-the-money) may far dominate the payoff. Still, their attractiveness can dominate the riskiness if the holder has a strong perspective that the option will turn out to be deep-in-the-money.

We have only discussed European-style pay-later options. They can be extended to American-style ones. These American-style pay-later options can be structured in connection with barrier options such that no premium is paid if a prespecified trigger is never touched. Otherwise, a premium needs to be paid either at the option maturity or as soon as the barrier is touched.

QUESTIONS AND EXERCISES

Questions

33.1. What are contingent premium options?

33.2. What are pay-later options?

33.3. What are "partial pay-later" options?

33.4. What are reverse pay-later options?

33.5. What are money-back options?

33.6. What are path-dependent CPOs?

33.7. What are path-independent CPOs?

33.8. Is there anything in common between a money-back option and its corresponding pay-later option?

33.9. Under what conditions will the payoff of a general CPO given in (33.12) become that of a reverse pay-later option?

33.10. Is a money-back option always more expensive or cheaper than its corresponding vanilla option?

33.11. Why is a money-back option always more expensive or cheaper than its corresponding vanilla option?

33.12. What is the difference between a money-back option and its corresponding pay-later option?

Exercises

33.1. Find the present values of the predetermined premiums for the "partial pay-later" options with partial pay-later parameters $\psi =$ 0.25 and 0.75 in Example 33.1.

33.2. Find the prices of the corresponding reverse "partial pay-later" options in Exercise 33.1.

33.3.* Show (33.13).

33.4. Find the pricing linear expressions for the corresponding CP at-the-money options in Example 33.3.

33.5. Find the amount of cash the holder of the "full pay-later" call and put options has to pay in Exercise 33.4.

33.6. Find the prices of money-back options in Exercise 33.4.

33.7. Find the prices of the corresponding money-back options in Exercise 33.1 with $\psi = 0$.

33.8.* Show the delta of a "partial pay–later" call option is as follows

$$\delta_{plc} = (1-\psi)\left\{\frac{N(d+\sigma\sqrt{\tau})}{N(d)} + \frac{f(d+\sigma\sqrt{})N(d) - f(d)N(d+\sigma\sqrt{\tau})}{\sigma\sqrt{\tau}N^2(d)}\right\}$$

where $f(z)$ is the density function of the standard normal distribution.

33.9. Find the delta of the "partial pay-later" options in Exercise 33.1 using the formula in Exercise 33.8.

33.10.* Find the formula of the theta of a "partial pay-later" option.

33.11. Show the expected payoff function given in (33.3).

33.12. Show (33.8).

33.13.* Find the formula of the present value of an amount X_i to be received when the underlying price hits the barrier H_i (Hints: use the conditional density functions for barrier options given in Chapter 10).

33.14.* Find the pricing formula of a barrier-driven CP call option with one barrier.

33.15.* Find the pricing formula of a barrier-driven CP put option with one barrier.

33.16.* Find the pricing formula of a barrier-driven CP call option with n barriers.

Chapter 34

OTHER EXOTIC OPTIONS

34.1. INTRODUCTION

It is not easy to classify existing exotic options into a small number of groups according to their characteristics. Besides the two major groups of exotic options described in Part III and Part IV and the five types of exotic options covered earlier in Part V, there are quite a few other kinds of exotic options. We will introduce them briefly in this chapter.

34.2. MID-ATLANTIC/BERMUDA/MODIFIED AMERICAN OPTIONS

Mid-Atlantic options are also known as Bermuda options, limited exercise options, or modified American options. As the phrase mid-Atlantic implies, a mid-Atlantic option is a hybrid of American and European options. Instead of being exercised any time before maturity as a standard American option, a mid-Atlantic option can be exercised only at prespecified discretely-spaced time points before the maturity of the option. Thus, Bermuda options are quasi-American options. They are sometimes called modified American options because of this quasi-American property. At the inception of a Bermuda option, besides the regular specifications of a vanilla option, the discrete dates of exercise must also be specified. As Bermuda options possess properties of both American and European options, their premiums are between those of their corresponding American and European options.

There are no closed-form solutions for Bermuda options in a Black-Scholes environment. However, they can be priced very conveniently using the binomial method described in Chapter 4. We can illustrate this in the following example.

Example 34.1. Find the prices of the Bermuda put options with exercise frequency daily, weekly, and monthly, respectively, given the spot and strike prices \$100, interest rate 10%, payout rate of the underlying asset zero, time to maturity half a year, and volatility of the underlying asset 20%.

We can readily find the number of periods in each situation by dividing the time to maturity by the exercise frequency. The number of periods with monthly, weekly, and daily exercise frequency is 6, 26, and 182, respectively with half a year of time to maturity. Substituting $S = K = \$100$, $\sigma = 0.20$, $\tau = 0.5$, $r = 0.10$, $g = 0$, $n = 6$, 26, and 182 into the binomial tree model in Chapter 4 yields the prices of the Bermuda put options \$3.835, \$3.896, and \$3.916, respectively.

From the results of Example 34.1, we can readily observe that the price of a Bermuda put option increases with the exercise frequency. This is consistent with the intuition that the higher the exercise frequency, the closer the Bermuda option is to the corresponding American option, and therefore the higher its price will be.

Most options we have studied so far in this book are options with continuous observations. As a matter of fact, it costs a great deal to monitor the underlying asset price continuously. Since Bermuda options are options with given observation frequency, they provide reasonable solutions at a lower cost. Essentially, all kinds of exotic options can be modified to incorporate the discrete observation characteristic to form Bermuda exotic options. For instance, we may have Bermuda barrier options, Bermuda lookback options, Bermuda correlation options with hourly, daily, or even weekly observations.

34.3. INSTALLMENT OPTIONS

Installment options, as their name implies, allow investors to pay their premiums in installments, thereby cutting the up-front premiums and offering the flexibility of canceling the options early if needed. Pension fund managers are among those who use installment options most. They buy installment options to protect their portfolios at a lower cost. Installment options have also been used in equity, interest rate, currency, and commodity markets.

An installment option is a form of compound options. It can be considered as a series of compound options or a string of extendable options. After paying a minimum up-front premium, the investor has a choice to continue the option by paying the installment payments or let the option expire. A typical installment option calls for a buyer to make four equal payments on a quarterly basis. If a payment is not made, then the option expires worthless automatically. Therefore, installment options make it possible for investors to protect their underlying assets or portfolios at a lower cost.

There is always a tradeoff between the up-front premium and the total premium. Whereas an installment option requires less up-front premium than the corresponding vanilla option, the total premium of the installment payment is more than that of the corresponding vanilla option if the investor chooses to make all the installment payments. Since whether the investor chooses to continue to pay the installment payments is uncertain, installment options have some characteristics of American options. Thus they cannot be priced easily in closed-form. Although we can illustrate how to price installment options using the methods to price American options and compound options, it is not our objective here to show how to price them precisely. Besides the known factors which affect the prices of vanilla options, the number of payments and the time interval between two consecutive payments are also important in affecting installment option prices.

34.4. EXPLODING OPTIONS

Exploding options are American-style capped call options or floored put options compared to the European-style ones studied in Chapter 29. Thus, the primary difference between an exploding option and its corresponding capped call or floored put option is that the exploding option expires at parity when the underlying asset trades at or through a trigger price, normally the cap or floor level. Gastineau (1993b) discussed how an exploding option works and how to apply it in practice. Specific contracts may have slightly different terms on which the exploding feature is activated, but we will only describe the major characteristics of exploding options in this section.

There always exists an earlier exercise trigger range in which the option "explodes" or matures when the price of the underlying instrument touches this range. The trigger range is normally from the cap level to infinity for an exploding call option, and from zero to the floor for an exploding put option. The payoff of an exploding option is similar to that of a capped call (resp. floored put) option with an up (resp. down) knockout property. The payoff can be formally given as follows:

$$\max[\omega S(\tau) - \omega K,\ 0] \quad \text{if } \omega S(T) < \omega COF(\omega) \text{ for all } T \in (0, \tau), \text{ and} \qquad (34.1a)$$

$$\omega COF(\omega) - \omega K \quad \text{otherwise}\,, \qquad (34.1b)$$

where

$$COF(1) = Cap,\ COF(-1) = Floor\,,$$

K, *Cap*, and *Floor* represent the strike price, the cap of the capped call option, and the floor of the floored put option, respectively, ω is the binary option operator (1 for a call option and -1 for a put option).

The payoff given in (34.1) is almost the same as that for a knockout option analyzed in Chapter 10, with the only exception that the rebate for the knockout option is replaced with the expression $\omega(COF)(\omega) - \omega K$ given in (34.1b). The first part of the payoff of the exploding option given in (34.1a) is actually the payoff of a capped call or a floored put option under the condition that the trigger range is never touched within the life of the option, and the second part can be regarded as the rebate of an up (resp. down) knockout digital call (resp. put) option. Because we have shown the pricing formulas for knockout barrier options in both Chapters 10 and 11, we do not need to repeat them here. Interested readers can simply apply the results in either Chapter 10 or Chapter 11 to price an exploding option.

34.5. LADDER OPTIONS

We discussed ladder options with one ladder in Chapter 29. They are the simplest kind of ladder options. In general, there are two or more than two ladders or rungs in a ladder option. Ladder options work essentially in the same manner as clique options, and similar in some ways to lookback options. Whenever the underlying asset price reaches a prespecified higher level in a series of predetermined ladders for the underlying asset, the intrinsic value of the option is locked in and a new strike price is established at that level, whereas the strike price of a clique option can only be reset on certain dates regardless of the underlying asset level. Thus, the strike price of a clique option is set at prespecified time, while the strike price of a ladder option is set at prespecified levels of the underlying asset price.

Ladder options are also similar to lookback options. Whereas every higher or lower price of the underlying asset implies a reset strike price for a lookback option, an increase or decrease of the underlying asset price within two consecutive prespecified ladder levels does not affect the payoff of the ladder option. Only when changes of the underlying asset price are beyond some prespecified ladder levels is the strike price of the ladder option reset. The relationship between a ladder option and its corresponding lookback option can be better understood in the limiting case of an infinite number of ladders. When there is an infinite number of evenly-spaced ladders, the distance between two consecutive ladders approaches zero and every increase or decrease of the underlying asset price implies a reset strike price for the ladder option, the same as for the corresponding lookback option. Thus, a ladder option becomes a lookback option when there is an infinite number of evenly-spaced ladders. Since ladder options are similar to lookback options and lookback options are normally more expensive than their corresponding vanilla options, ladder options usually cost more than their corresponding vanilla options, especially with more ladders.

Street (1992) discussed and priced ladder options. Interested readers can read Street (1992) for more detailed description for ladder options. Heynen and Kat (1994b) illustrated how a ladder option can be packaged with vanilla options and barrier options. We leave this as an exercise at the end of this chapter.

34.6. SHOUT/DEFERRED-STRIKE OPTIONS

Shout options are also called deferred-strike options. As the phrase "deferred-strike" implies, a shout option is an option whose strike price can be specified as the underlying asset price at any time before the maturity of the option. Thomas (1993) and Gastineau (1994a) discussed shout options. The level of the strike is ultimately set at a specific relationship to the spot, for example, 5% or 3% below the spot, or 3% or 5% above, during a period of time normally starting on the trade date and ending on a date agreed upon at the trade time. After the strike is specified according to the terms in the contract or after the shouting time, the shout option becomes a vanilla option until the maturity of the option.

Shout options possess characteristics of American options. Since optimal timing or the "shouting" time is uncertain, there is no straightforward way to price shout options. However, they can be priced using either the binomial tree method described in Chapter 4 or some analytical approximations.

34.7. LOCK-IN OPTIONS

A lock-in option is an option which allows its holder to settle the option payoff at a time before the contracted option maturity, but transactions take place only at the expiration date. There are European lock-in options and American lock-in options. Whereas the lock-in time is prespecified in a European lock-in option, it is not contracted *ex ante* but can be chosen by the option holder at any time until the payment date in an American lock-in option. American lock-in options might be considered as "deferred-transaction" American options compared to "deferred-rebate" options studied in Chapter 10. Modeling the optimal decision on when to "shout" in pricing shout options presents considerable additional complexity beyond what is present in a lock-in option contact.

While European lock-in options are less costly than vanilla options because of smaller time values and delayed payment of option payoffs, American lock-in options permit an investor to fix the option payoff at a more favorable time than merely waiting until the option expires. Although American lock-in options share the early exercise flexibility of standard American options and shout options, they are cheaper than both of them. Yu (1994) priced both European and American lock-in options and found approximated pricing formulas for American lock-in options. The price of a European lock-in option (ELKIN) can be readily expressed as follows:

$$ELKIN = e^{-r(\tau-T)} C_{bs}(S, K, T, \omega), \qquad (34.2)$$

where $C_{bs}(S, K, T, \omega)$ is the extended Black-Scholes pricing formula with spot price S, strike price K, time to maturity T, the binary option operator ω (1 for a call and -1 for a put option), and T is the prespecified lock-in time, $T < \tau$.

It is obvious that the price of any European lock-in option is always lower than its corresponding vanilla option because the discounting factor is always smaller than one and the lock-in time T is always smaller than the time to maturity τ. Interested readers may refer to Yu (1994) for the approximated pricing formulas for American lock-in options.

34.8. RESET OPTIONS

A reset option, as its name implies, allows the buyer to lower the strike price of a call option or raise the strike price of a put option before its expiration, if the option is out-of-the-money on specified reset dates during a reset time period, or if the spot reaches a prespecified level. Reset options can be used to protect the investors from betting wrong. There are a number of ways to specify a reset option because there are many different ways to reset the strike price and the resetting conditions can be very different. If the strike price is set when the option is out-of-the-money on prespecified reset dates, the reset option is symmetric to a clique option in the sense that both the underlying asset price in the clique option and the strike price in the reset option are allowed to reset on prespecified reset dates. If the strike price is set according to whether the spot reaches a prespecified level, the reset option is similar to a barrier option in the sense that the payments for both reset options and barrier options are determined by whether the spot reaches a prespecified level.

34.9. CONVEX OPTIONS

Convex options can modify the return spectrum of vanilla options. For example, if investors believe that the Japanese Nikkei 225 index will increase 15% in the coming six months because of the appreciation of the US dollar against yen, they could buy a convex option that offers a larger return for the first 15% rise in the index than a vanilla option, but pays less as the market goes higher. Convex options are often used in emerging markets although they can be used in any markets, and can better monetarize investors' views on the underlying market moves. They can actually be constructed easily with power options studied in Chapter 30 using diferent power values.

34.10. "ROLL UP PUTS" AND "ROLL DOWN CALLS"

"Roll up puts" and "roll down calls" are actually variations of vanilla barrier options. Gastineau (1994b) discussed such options. These options contain the provision that if the price of the underlying asset moves against a prespecified trigger level — up for a put option and down for a call option — the strike price of the option is revised and the original option turns into a barrier option. More specifically, when the trigger price is hit, the strike price of the "roll up put" option is raised, and the option becomes an up-and-out put option with an outstrike price placed at an even higher price. Symmetrically, a "roll down call" option becomes a down-out-call with an outstrike price placed at an even lower price after the trigger is hit, the "roll down call" option's strike price being lowered.

From the above description, we may understand that if we consider vanilla barrier options as conditional (whether the triggers are hit or not within the lives of the options) options with payments in terms of vanilla options, we can consider "roll up puts" and "roll down calls" as conditional options with payments in terms of vanilla barrier options. Thus, these "roll" options are "higher generation" options than most other exotic options because their payments are in terms of vanilla barrier options.

"Roll up puts" and "roll down calls" give an investor the opportunity of a favorable reset of the strike price and provide an obvious penalty. They are attractive to investors who feel that they may be early implementing a bullish or bearish position in a specific market. These options can be priced with the formulas of vanilla barrier options because we can duplicate their payoffs with portfolios of vanilla barrier options and vanilla options.

34.11. SUMMARY

We have described nine kinds of exotic options in this chapter. All of them, with the exception of ladder options which are path-dependent, are exotic options with various degrees of earlier-exercise property. Thus they can be largely considered as American-style exotic options. Shout options and lock-in options can also be considered as path-dependent because their values depend on the underlying asset price at the shouting time and the lock-in time, respectively.

So far in this book we have introduced and priced most of the exotic options in the OTC marketplace. The list is nevertheless not complete. Since the innovation process is still continuing, it makes no sense to list all exotic options in one book. However, we can analyze essentially all other kinds of exotic options conveniently with the descriptions and techniques developed in this book.

QUESTIONS

34.1. What are Bermuda options?

34.2. Why is the price of a Bermuda option always between the prices of their corresponding European-style and American-style options?

34.3. Does the price of a Bermuda option increase or decrease with the exercise frequency? Why?

34.4. What is an installment option? What is the advantage of using installment options?

34.5. What are exploding options?

34.6. What are ladder options?

34.7. What is the relationship between ladder options and lookback options?

34.8. Is there any similarity between ladder options and clique options?

34.9. What are shout options?

34.10. Why are shout options also called deferred-strike options?

34.11. What are lock-in options?

34.12. Why are European lock-in options always cheaper than their corresponding vanilla options?

34.13. What are reset options? Why can reset options be similar to clique options?

34.14. What are convex options?

34.15. Are ladder options similar to barrier options? Why?

34.16. Show how ladder options can be packaged with vanilla options and barrier options.

34.17. What is a "roll up put" option?

34.18. What is a "roll down call" option?

34.19. Why can we say that "roll up puts" and "roll down calls" are "higher generation" options than most other exotic options covered in this book?

34.20. Can you think of one kind of exotic option not covered in this book?

34.21.* Show how the the payoff of a ladder option with two up ladders can be duplicated with barrier options and digital options?

34.22.* Show how the the payoff of a ladder option with two up ladders can be duplicated with n upper ladders?

PART VI:

HEDGING EXOTIC OPTIONS AND FURTHER DEVELOPMENT OF EXOTIC OPTIONS

Chapter 35

HEDGING EXOTIC OPTIONS

35.1. INTRODUCTION

To structure and price exotic options is one thing, to hedge them is another, and often a more challenging and trickier one.

We have mainly concentrated on introducing and pricing various kinds of exotic options in this book. Hedging exotic options may be as important as, if not more important than, pricing them, because outcomes with any exotic options can be very uncertain without hedging them. It is normally more complicated to hedge exotic options than to hedge their corresponding vanilla options, because it generally requires an extensive knowledge of advanced derivatives products and mathematics, and the hedging results are normally less intuitive than those of vanilla options. Applying the "no-free-lunch" argument, we may consider an additional difficulty in hedging exotic options — the "tradeoff" of the additional convenience in using exotic options to achieve specific objectives.

Hedging is closely related to pricing, and to some degrees, the way an option is hedged determines how it can be priced. Since it is complicated to hedge exotic options in general, it would take as long a book as this simply to discuss how to hedge various kinds of exotic options. Since the objective of this book is to introduce and price various kinds of exotic options, we will discuss only some general issues in hedging exotic options in this chapter.

35.2. DYNAMIC HEDGING

From our description of the Black-Scholes model in Chapter 2, we know that a stock call (resp. put) option at any time can be replicated with a weighted portfolio of a risky stock and riskless zero-coupon bonds or cash. We know how to calculate their weights in the portfolio: long (resp. short) delta amount [given in (3.32)] of the underlying asset and short (resp. long)

$KN(\omega d)$ amount of zero-coupon bonds. Instead of owning an option, the investor can in principle own a portfolio of a stock and riskless bonds with weights specified above. These weights depend on all the parameters which affect the Black-Scholes option value: the spot underlying asset price, the strike price of the option, the volatility of the underlying asset, the interest rate, the payout rate of the underlying asset, and the time to maturity of the option. As time passes by, the time to maturity of each option declines, and these weights change continuously with it and/or the underlying asset price, regardless of changes in other parameters in the model.

The above portfolio is called the dynamic replicating portfolio because the weights in the portfolio change dynamically with time. A call can be hedged by going short the dynamic delta amount of the underlying asset against a long position in the option to eliminate all the risks related to stock price moves. Although the method works perfectly in principle, there are a few difficult problems associated with it. First of all, continuous weight adjustment is impossible because continuous trading is impossible. Weights are actually adjusted at discrete time intervals. Secondly, transaction costs are always involved with any weight adjustment and they cannot be neglected in practice. Transaction costs grow with the frequency of weight adjustment. Thus, traders have to compromise between accuracy of hedging and the costs associated with it.

35.3. STATIC REPLICATION

To some degrees, static replication was motivated by the limitations of dynamic hedging described above. Static replication stands for replicating exotic options statically with standard options together with the underlying assets. Specifically, it is a portfolio of vanilla options with varying strikes and maturities, and fixed weights that will not require any further adjustment. This portfolio will precisely replicate the value of a target option for a chosen range of future time and market levels. This method is called static replication because the weights in the portfolio are fixed and no adjustment is necessary as in dynamic hedging with un-negligible transaction costs, and it was developed by Derman, Ergener, and Kani (1994, 1995).

This method is quite intuitive, because as long as the payoff of any target option can be replicated with vanilla options and the underlying assets, the price of the target option must be the same as the price of the structured portfolio of vanilla options. Otherwise arbitrage would occur. Since we know how to hedge vanilla options, we can hedge the target option by hedging the vanilla options within the structured portfolio.

The static replication method has obvious advantages over dynamic hedging since there is no need to adjust the weights in the structured portfolio, and there is no additional transaction cost for the weight adjustment. However, this method has its serious problems. Only in some rare cases can a target option be replicated with a limited number of vanilla options. In other words, a target option, in general, can only be replicated perfectly using the static replication method with an infinite number of vanilla options with various strike prices and time to maturity. Although a portfolio including a limited number of vanilla options may approximate the corresponding portfolio including an infinite number of vanilla options, the approximation may result in inaccuracy somewhat similar to that resulting from infrequent adjustment of weights in dynamic hedging. What is more, the replicating portfolio is not unique for a particular target option.

Another more serious problem with the static replication method is that there is one implied assumption behind the method — perfect liquidity for the constituent vanilla options — because neither the two articles mentioned the availability of constituent vanilla options with various strike prices and time to maturity. An infinite number of vanilla options with arbitrary strike prices and/or time to maturity can be a serious problem, because such options normally do not exist in organized exchanges. It would require a significant maintenance cost and/or capital simply to maintain this infinite number of vanilla options in order to duplicate a given exotic option.

35.4. REPLICATING AND HEDGING DIGITALS

Digital options, as we argued in Chapter 15, are the simplest of all options including vanilla options. Despite this, they possess some most difficult problem in option theory, and particularly in option risk management. And this problem exists in many other kinds of exotic options such as barrier options, ladder options, lookback options, and so on. In this section, we will try to illustrate this problem and show how it can be partially solved.

It is well known that the simplest digital option, a cash-or-nothing option or simply a CON option, exhibits zero delta almost for any possible underlying asset prices. However, its delta behaves extraordinarily around its strike price: it jumps from zero to positive (resp. negative) infinity for a CON call (resp. put) option from the lower neighborhood of the strike price toward its strike price, and then returns to zero again after the strike price. Such jumps create tremendous, undesirable shifts in the values of portfolios including such options. This problem has been attacked by all risk managers since digital options and other options with digitality came into existence.

A very popular method developed by Bowie and Carr (1994) shed some lights on this problem. The method that Bowie and Carr used is portfolio replication. Specifically, they considered a portfolio which includes long positions of an infinite number of vanilla put options with strike prices slightly higher than the strike price of the targeted CON put option, and short positions of an equal number of vanilla put options with strike prices slightly lower than the strike price of the targeted CON put option. Algebraically, they consider the following portfolio:

$$PF(n) = \frac{n}{2}\left[P\left(K + \frac{1}{n}\right) - P\left(K - \frac{1}{n}\right)\right], \tag{35.1}$$

where K stands for the strike price of the targeted CON put option, $P(KK)$ is the price of the vanilla put option with strike price KK given in (10.31), and $n/2$ is the number of vanilla put options in the portfolio.

It can be readily demonstrated that the value of the portfolio given in (35.1) approaches the value of the corresponding CON put option when the number of vanilla put options approaches infinity, or

$$\underset{n \to +\infty}{\text{Limit}}\, PF(n) = \mathrm{e}^{-r\tau} N[-d(K)], \tag{35.2}$$

where $PF(n)$ is given in (35.1) and $d(K)$ is the stand argument in the Black-Scholes formula with spot price S, strike price K, interest rate r, and time to maturity τ. See Exercise 35.10 for a proof.

The right hand side of (35.2) is exactly the pricing formula for a CON put option given in (35.1) with $\omega = -1$. The result given in (35.2) can be directly interpreted as follows: a portfolio including long positions of an infinite number of vanilla put options with strike prices slightly higher than the strike price of the targeted CON put option, and short positions of an equal number of vanilla put options with strike prices slightly lower than the strike price of the targeted CON put option replicate the targeted CON put option perfectly.

The problem of hedging digital options seems to have been solved with the replicating method shown in (35.2). Yet the problem is not so simple. If we ponder over (35.2), we would find that it is no more than a limiting identity in calculus. In order for it to make some financial sense, two implied assumptions have to be made: one is that vanilla put options with strike prices slightly above and below the strike price of the corresponding CON put option are available simultaneously. The other is that an infinite number of such vanilla put options are also available. Obviously these two assumptions are equivalent to assuming perfect liquidity of the vanilla put options with arbitrary strike prices. Certainly, such assumptions cannot be satisfied in practice, and therefore, the replicating portfolio in (35.2) cannot contain much financial meaning beyond its mathematical equivalence. Thus, liquidity is the common limitation to both the Bowie and Carr's replicating method to replicate digital options and the Derman-Ergener-Kani static replication method to replicate and hedge exotic options in general.

35.5. SUMMARY

Nothing in this world comes free. Any additional return above riskless return is accompanied with some risk. The difficulties in hedging exotic options may be understood as a tradeoff of the convenience and flexibility in using exotic options. In order to enjoy the convenience and flexibility of exotic options, we have to bear the inconvenience and difficulties in hedging them. The objective of this chapter is not to find out ways to hedge exotic options, but rather to point out the difficulties in hedging them. As we said at the beginning of this chapter, it would take another book to treat how to hedge the popular exotic options covered in this book. We leave this topic to another volume.

QUESTIONS

35.1. What is dynamic hedging?

35.2. What is the major problem with dynamic hedging?

35.3. What is static replication?

35.4. What is the advantage of static replication over dynamic hedging?

35.5. What is the problem with static replication?

35.6. Why is it difficult to hedge a CON digital option?

35.7. What are the two implied assumptions behind the replicating portfolio in (35.2)?

35.8. Why is it said that the replicating portfolio in (35.2) is no more than a limiting identity in calculus?

35.9. What is the common problem with the Bowie-Carr replicating method and the Derman-Ergener-Kani static replication method?

35.10.* Show the limit result given in (35.2).

Chapter 36

FURTHER DEVELOPMENT

36.1. CURRENT STATUS OF EXOTIC OPTIONS DEVELOPMENT

The innovation process in creating new exotic options is still continuing although the growth rate has somewhat slowed down. The trend seems to be in two directions: one is to combine two or more basic kinds of exotic options to form another kind with more flexibility, and the other is to create options on "exotic underlying instruments" such as inflation indicators, environmental measures, insurance contracts, etc. We will discuss the first trend in this section and the second in the following one.

Actually, we have already illustrated quite a few exotic options which combine two or more basic kinds of exotic options. For example, the combination of an option with binary characteristics and a barrier option forms a binary-barrier option [Reiner and Rubinstein (1991b)], the combination of an Asian option and a barrier option results in an Asian barrier option with average barriers as studied in Chapter 11, the combination of options with binary characteristics and correlation characteristics forms a correlation-binary option, the combination of a barrier option and a pay-later option forms a contingent premium option with American characteristics, the combination of trigger options and compound options to form trigger compound options as illustrated in Chapter 31, and so on. As a matter of fact, most exotic barrier options in Chapter 11 are combinations of more basic kinds of exotic options. These new exotic options have greater flexibility and should better serve clients' hedging and speculating purposes.

As a matter of fact, ladder options can also be considered as combinations of barrier options. Roll up puts and roll down calls are combinations of barrier options and vanilla options. Most options we have studied in this book are options with continuous observations. The combination of Bermuda options with any kind of exotic options can form exotic options with discrete observations. We may call them Bermuda exotic options. Combinations of the basic exotic options covered in this book will form a significant number of composite exotic options.

36.2. EXOTIC OPTIONS WRITTEN ON EXOTIC UNDERLYING INSTRUMENTS

The other group of exotic options is options written on exotic underlying instruments. Such options include credit options written on credit measurement, insurance options written on insurance contracts, real estate options written on real estate indexes, inflation options written on inflation indexes, environment options written on environment measurement, options on economic cycles, etc.

Credit risk options (CROs), or simply credit options, are probably the most popular, recent, and rapidly growing innovations among the above mentioned options. Das (1995) priced CROs with multi-factor models and stochastic interest rates. He also provided a general discrete time approach, and illustrated how to hedge the CROs. His stochastic interest-rate models appear to allow all plausible shapes of default spread curves.

Whittaker and Kumar (1995) gave a very good description of most popular credit derivatives. In general, credit derivatives permit investors to manage credit exposures by separating their views on credit from other market variables. The underlying markets for credit derivatives can be bank loans, corporate debts, trade receivables, emerging markets, municipal debts, as well as the credit exposures generated from other derivative-linked activities. Particularly, a credit option is a privately negotiated OTC option between two counter-parties to meet the specific credit-related hedging or investment objectives of the customer. A credit call (resp. put) option gives its buyer the right, but not the obligation, to buy (resp. to sell) an underlying credit-sensitive asset or credit spread at a predetermined price for a predetermined period of time. These basic structures can be understood as vanilla credit options with which exotic credit options can be structured. Besides vanilla credit options, there are also exotic credit options. The most popular exotic credit options are barrier and digital credit options.

36.3. POSSIBLE REASONS FOR THE GROWTH OF EXOTIC OPTIONS IN THE PAST

There are many reasons which could explain the rapid growth of exotic options in the past decade or so. Discussing these major reasons will help us perceive the future development of exotic options.

First of all, cost reduction has probably been the most important factor propelling the rapid growth of exotic options. Most exotic options covered in this book have lower premiums than their corresponding vanilla options. The low premiums of exotic options make them more attractive to investors, because they provide cheaper means for hedging and/or investment.

Off-balance sheet transactions have been another factor for the rapid growth of exotic options. Most exotic options are relatively new, and their transactions are largely off-balance sheet. Most institutions and individual investors possess more flexibility using these off-balance sheet products.

Thirdly, demands for more sophisticated instruments in risk management have also contributed to the rapid growth of exotic options. As illustrated in this book, most exotic options can satisfy investors with some special features. With the development in market globalization, market integration, technology, and others, the demands for specific instruments have increased to meet specific objectives in risk management. Very often, the specific objectives may be achieved with vanilla options, yet the vanilla products are not convenient. Exotic options can provide more specific and efficient ways to achieve the required objectives.

Fourthly, exotic options could be used to achieve some tax benefits for investors. Fifthly, technological advancement in data storage and computing power makes it possible to process a larger amount of data within a much shorter time than ever before. Manufacturers of semiconductors continue to reduce the cost of computing, and the number of arithmetic operations per second continues to grow exponentially with new chips. The increased computing power has made it possible to obtain reasonable results for more complex products within a reasonable period of time.

Processing a larger amount of data within a shorter period of time also makes it possible for institutions and individual investors to know various aspects of their portfolios in greater details than before. Knowing their risks better, they can decide what risks they have to tolerate in their business and what risks they want to get rid of. Thus, more specific needs in risk management require more specific instruments to achieve them.

Last but not least, the low interest rate and the poor economic environment of the early 1990s have sent investors hunting for "yield pickup" had helped exotic options to boom. Exotic options have been used in many creative ways to achieve enhanced yields and other goals.

36.4. OTHER FACTORS AFFECTING FURTHER DEVELOPMENT

In a recent article, Gastineau and Margolis (1995) discussed the future development of equity derivatives. Although most arguments there are for equity derivatives, they are also applicable to exotic options in general. Gastineau and Margolis discussed both the supply and demand sides of equity derivatives business. They pointed out that although the demand side has played a significant role in pushing the business, the supply side has also been important. Besides the factors we discussed in the previous section, they also discussed a few more.

One is the demand for derivatives in asset allocation and liability management. Growing emphasis on systematic asset allocation and application of quantitative tools to portfolio management is progressively combined with comparable techniques for liability management. Improvements in technology permit the creation of instruments that respond to changing market conditions quickly.

Another factor they discussed is regulation, and its effects on both the supply and demand sides. Whereas there are no reliable ways to predict the regulatory response to derivatives development, there has been a clear pattern of regulation in the past. Regulations may delay the introduction of a new product, but they are unlikely to be able to stop a close competitor from satisfying a market need for a particular product. Whereas in rare cases regulations may stimulate demands for a product when a favorable market environment, such as favorable accounting treatment, is established, they normally create impediments to free markets, restricting demand and/or supply, and in turn the total quantity of goods or services is less than that it would be in a free market environment.

36.5. WHAT LIES AHEAD?

There still remains one very important factor: the current general market perception of, or attitude to, derivatives, especially exotic derivatives, resulting from the chain of events since 1993 mentioned at the beginning of this book. This general attitude is a very negative factor to the demand side of the derivatives industry. However, with better understanding of derivatives products, the general public, and corporates in particular, will change the current attitude towards derivatives as time passes by. This is because, as argued by Zhang (1995e), it is not the products that cause all the losses, but how you use them. As in the case of automobile accidents, it is not the vehicles that cause accidents, but how we drive them.

Besides the last factor discussed in Section 36.3 which is absent in the current market environment, and the negative market sentiments towards derivatives discussed above, all other positive factors have been in function. Therefore, we may conclude that the derivatives industry, especially the exotic derivatives products, will continue to grow, albeit at a more relaxed rate.

Let's conclude this book with John Hull's remarks on the future of derivatives:

> *New derivative securities are being developed at an exciting pace. There can be little doubt that important new ideas and new results will continue to emerge.*
>
> *John Hull (1993)*

QUESTIONS

36.1. What are the current trends in the exotic options market?

36.2. What are exotic underlying instruments?

36.3. What are credit options?

36.4. Give the names of two popular credit exotic options.

36.5. Give a few examples of exotic options which may be cheaper or more expensive than their corresponding vanilla options.

36.6. How many kinds of composite (with two different kinds of basic exotic options) exotic options can be formed with n basic kinds of exotic options?

36.7. What are the possible factors which have contributed to the rapid growth of exotic options in the past decade or so?

36.8. What was the particular reason for the growth of exotic derivatives in the early 1990s?

36.9. What is the current market attitude towards derivatives, especially exotic derivatives? Why?

36.10. Can you give one additional factor not covered in this chapter which could have contributed to the growth of exotic options?

Appendix I

PAYOFF FUNCTIONS FOR VARIOUS OPTIONS

ω is the option binary operator (1 for a call option and -1 for a put option), θ is the direction binary operator (1 for down and -1 for up)

Absolute Options

$$\max[\omega|I_1(\tau) - I_2(\tau)| - \omega k, 0]$$

Alternative Options (calls), either-or options, or best of the two options

$$\max\{\max[\mathrm{I}_1(\tau)/\mathrm{I}_1 - \mathrm{k}_1, 0],\ \max[\mathrm{I}_2(\tau)/\mathrm{I}_2 - \mathrm{K}_2, 0]\} \text{ or}$$
$$\max\{\max[\ln(\mathrm{I}_1(\tau)/\mathrm{I}_1) - \mathrm{k}_1, 0],\ \max[\ln(\mathrm{I}_2(\tau)/\mathrm{I}_2) - \mathrm{k}_2, o]\}$$

American Options

$$\max\ [\omega \mathrm{S}(T) - \omega K, 0], T \in (0, \tau]$$

Asian Options

Geometric Asian Options

$$\max[\omega \mathrm{GA}(\tau) - \omega \mathrm{K}, 0]$$

Asian Options with Geometric Averages as Strike Prices

$$\max[\omega \mathrm{S}(\tau) - \omega GA(\tau), 0]$$

Arithmetic Asian Options

$$\max[\omega \mathrm{AA}(\tau) - \omega \mathrm{K}, 0]$$

Asian Options with Arithmetic Averages as Strike Prices

$$\max[\omega \mathrm{S}(\tau) - \omega \mathrm{AA}(\tau), 0]$$

Flexible Geometric Asian Options

$$\max[\omega \mathrm{FGA}(\tau) - \omega \mathrm{K}, 0]$$

Asian Options with Flexible Geometric Averages as Strike Prices

$$\max[\omega \mathrm{S}(\tau) - \omega \mathrm{FGA}(\tau), 0]$$

Flexible Arithmetic Asian Options

$$\max[\omega \mathrm{FAA}(\tau) - \omega \mathrm{K}, 0]$$

Asian Options with Flexible Arithmetic Averages as Strike Prices

$$\max[\omega \mathrm{S}(\tau) - \omega \mathrm{FAA}(\tau), 0]$$

Asset-Or-Nothing (AON) Options:

$$\mathrm{S}(\tau) \ \text{ if } \ \omega \mathrm{S}(\tau) - \omega \mathrm{K} > 0$$

Barrier Options

Knock-in vanilla barrier options

$$\max[\omega \mathrm{S}(\tau) - \omega \mathrm{K}, 0] \text{ if } \theta \mathrm{S(t)} > \theta \mathrm{H} \ \& \ \theta \mathrm{S(T)} \le \theta \ \mathrm{H} \text{ for some } 0 < \mathrm{T} \le \tau, \text{ and}$$
$$\text{Rebate } \mathrm{Rm}(\tau) \text{ if } \theta \mathrm{S(t)} > \mathrm{H} \ \& \ \theta \mathrm{S(T)} > \theta \ \mathrm{H} \text{ for all } < \mathrm{T} \le \tau$$

Knockout vanilla barrier options

$$\max[\omega \mathrm{S}(\tau) - \omega \mathrm{K}, 0] \text{ if } \theta \mathrm{S(t)} > \theta \ \mathrm{H} \ \& \ \mathrm{S(T)} > \theta \ \mathrm{H} \text{ for all } 0 < \mathrm{T} \le \tau, \text{ and}$$
$$\text{Rebate } \mathrm{R}(\tau) \text{ if } \theta \mathrm{S(t)} > \theta \ \mathrm{H} \ \& \ \theta \mathrm{S(T)} \le \theta \ \mathrm{H} \text{ for some } 0 < \mathrm{T} \le \tau$$

Floating barrier options: substitute the constant barrier H in vanilla barrier options with $He^{\gamma T}$

Asian Barrier options: substitute the constant barrier H in vanilla barrier options with

FAA for flexible arithmetic Asian barrier options and with

FGA for flexible geometric Asian barrier options

Basket Options

$$\max\left[\omega \sum_{i=1}^{n} \mathrm{w}_i \mathrm{I}_i(\tau) - \omega \mathrm{K}, \ 0\right]$$

Boston Options

$$F - K + \max[S(\tau) - F,\ 0]$$

Capped Call Options

$$\max\ \{\min[S(\tau),\ \text{Cap}] - K,\ 0\}$$

Cash-Or-Nothing (CON) Options:

$$\text{Cash if } \omega S(\tau) - \omega K\ > 0\ \text{ and 0 otherwise}$$

Chooser Options/As-You-Like Options/You-Choose-Options

Simple Chooser Options

$$\max\{C[S(\tau_1),\ K],\ P\ [S(\tau_1),\ K]\}, 0 < \tau_1 < \tau$$

Complex Chooser Options

$$\max\{C[S(\tau_1 0,\ K_1],\ P[S(\tau_2),\ K_2]\}, K_1 \neq\ K_2, 0 < \tau_1, \tau_2 < \tau$$

Clique Options

$$\max[\omega S(\tau) - \omega K, \omega S(\tau) - \omega S(\tau_i), 0],\ i = 1, 2, \ldots,\ n, \tau_i < \tau \text{ is prespecified}$$

Compound options

$$\max\{C[S(\tau),\ K] - k,\ 0\},\ \text{a call option on a vanilla a call option}$$

$$\max\{k - C[S(\tau),\ K], 0\},\ \text{a put option on a vanilla a call option}$$

$$\max\{P[S(\tau),\ K] - k, 0\},\ \text{a call option on a vanilla a put option}$$

$$\max\{k - P[S(\tau),\ K], 0\},\ \text{a put option on a vanilla a put option}$$

Contingent Premium Options/Pay-Later Options

$$\omega S(\tau) - \omega K - Q \text{ if } \omega S(\tau) > \omega K \text{ and 0 if otherwise}$$

Correlation Digital Options

$$\omega[S(\tau) - K] \text{ if } \omega M(\tau) > \omega K \text{ and 0 if otherwise}$$

Dual-Strike Options

$$\max\{\omega[I_1(\tau) - K_1], \omega_1[I_2(\tau) - K_2], 0\}$$

Equity-Linked Foreign Exchange Options

$$I_1(\tau)\ \max\ [\omega I_2(\tau) - \omega K_e, 0]$$

European Options

$$\max\ [\omega S(\tau) - \omega K, 0]$$

Exchange Options

$$\max\ [I_1(\tau) - I_2(\tau), 0]$$

Exploding Options

$\max\ [\omega S_{(\tau)} - \omega K, 0]$ if $\omega S(T) < \omega COF(\omega)$ for all $T \in 0, \tau)$ and $\omega COF(\omega) - \omega K$ otherwise, where COF(1) = Cap, COF(-1) = Floor

Floored Puts

$$\max\ \{K - \max[S(\tau),\ Floor], 0\}$$

Foreign Equity Options

$$\max\ [\omega I_1(\tau) - \omega K_f, 0]$$

Foreign Domestic Options

$$\max\ [\omega I_1(\tau) * I_2(\tau) - \omega K, 0]$$

Forward-Start Options

$$\max[\omega S(\tau) - \omega S(\tau_1), 0], 0 < \tau_1 < \tau$$

Gap options

$$\omega S(\tau) - \omega X, 0 \text{ if } \omega S(\tau) - \omega K > 0$$

LookBack Options

Floating Strike LookBack Call Options

$$\max\ \{S(\tau) - \min[S(t)]\},\ t \in [0, \tau]$$

Floating Strike LookBack Put Options

$$\max\ \{\max[S(t)] - S(\tau)\},\ t \in [0, \tau]$$

Fixed Strike LookBack Call Options

$$\max\ \{\max[S(t)] - K\},\ t \in [0, \tau]$$

Fixed Strike LookBack Put Options

$$\max\ \{K - \min[S(t)]\},\ t \in [0, \tau]$$

Partial LookBack Call Options

$$\max\ \{S - \lambda \min[S(t)]\},\ t \in [0, \tau], \lambda > 1$$

Partial LookBack Put Options

$$\max\ \{\lambda \max[S(t)] - S(\tau)\},\ t \in [0, \tau], 0 < \lambda < 1$$

Nonlinear Payoff Options

$$\max\ \{A + B[S(\tau) - K]^p, 0\},\ B\ \neq 0 \text{ and } p\ \neq 0$$

Options On the Best of Several Assets

$$\max\ \{\omega \max[I_1(\tau),\ I_2(\tau),\ I_3(\tau), \ldots, I_n(\tau)] - \omega K, 0\}$$

Options On the Worst of Several Assets

$$\max\ \{\omega \min[I_1(\tau),\ I_2(\tau), \ldots, I_n(\tau)] - \omega K, 0\}$$

Options Paying the Best and Cach

$$\max[I_1(\tau),\ I_2(\tau),\ I_3(\tau), \ldots,\ I_n(\tau),\ K]$$

Options Paying the Worst and Cach

$$\min\ [I_1(\tau),\ I_2(\tau),\ I_3(\tau), \ldots,\ I_n(\tau)], K]$$

Out-Performance Options

$$\max\{\omega[\ln(I_1(\tau 0/I_1) - \ln(I_2(\tau)/I_2)] - \omega K, 0\}$$

Power Options

Symmetric Power Options

$$\max\{[S(\tau) - \{K]^p, 0\},\ p\ \neq -1, 0$$

Asymmetric Power Options

$$\max[S^p(\tau) - K, 0],\ p \neq 0$$

Product Options/Foreign Domestic Options

$$\max\,[\mathrm{I}_1(\tau) * \mathrm{I}_2(\tau) - \mathrm{K}, 0]$$

Quanto Options

$$\bar{\mathrm{I}}_2 \max\,[\omega \mathrm{I}_1(\tau) - \omega \mathrm{K}_f, o] \text{ in domestic currency, and}$$
$$\bar{\mathrm{I}}_2 \mathrm{I}_2 \max\,[\omega \mathrm{I}_1(\tau) - \omega \mathrm{K}_f, 0] \text{ in foreign currency}$$

Quotient Options

$$\max\,[\mathrm{I}_1(\tau)/\mathrm{I}_2(\tau) - \mathrm{K}, 0] \text{ or } \max\,[\mathrm{I}_2(\tau)/\mathrm{I}_1(\tau) - \mathrm{K}, 0]$$

Receive of the Worst and Cash Options

$$\min\,[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau),\ \mathrm{K}],\ \mathrm{K} > 0$$

Receive of the Worst Options

$$\min\,[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau)]$$

Shout Options or Deferred Strike Options

$$\max[\mathrm{S}(\tau) - \mathrm{K},\ \mathrm{S}(\tau) - \mathrm{S}(\tau_1), 0], 0 < \tau_1 < \tau$$

Spread Options

Simple Spread Options:

$$\max\,[\mathrm{a}\ \mathrm{I}_1(\tau) + \mathrm{b}\ \mathrm{I}_2(\tau) - \mathrm{K}, 0],\ \mathrm{a} > 0,\ \mathrm{b} < 0$$

Spread over the Rainbow:

$$\max\,\{\mathrm{a}\max[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau)] + \mathrm{b}\min[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau)] - \mathrm{K}\}$$

Multiple Spread Options:

$$\max\left[\mathrm{a}\sum_{i=1}^{n} \mathrm{w}_i \mathrm{I}_i(\tau) + \mathrm{b}\sum_{j=1}^{m} \mathrm{w}_{n+j}\mathrm{I}_{n+j}(\tau) - \mathrm{K}, 0\right],\ \mathrm{a} > 0,\ \mathrm{b} < 0$$

Two-Color Rainbow Options

$$\max\{\max\,[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau)] - \mathrm{K}, 0\} \text{ or}$$
$$\max\{\min\,[\mathrm{I}_1(\tau),\ \mathrm{I}_2(\tau)] - \mathrm{K}, 0\}\ \max\{\mathrm{C}[\mathrm{S}(\tau),\ \mathrm{K}],\ \mathrm{P}[\mathrm{S}(\tau),\ \mathrm{K}]\}$$

Appendix II

TABLE OF THE CUMULATIVE FUNCTION VALUES OF THE STANDARD NORMAL DISTRIBUTION

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8016	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8408	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9435	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.7913	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

(Continued)

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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