Risk Management and Financial Derivatives

A Guide to the Mathematics

Edited by

SATYAJIT DAS

McGraw-Hill
New York San Francisco Washington, D.C. Auckland Bogotá
Caracas Lisbon London Madrid Mexico City Milan
Montreal New Delhi San Juan Singapore
Sydney Tokyo Toronto

First published by The Law Book
Company Limited t/as LBC Information
Services, Sydney, Australia, 1997.

Library of Congress Cataloging-in-Publication Data

Risk management and financial derivatives : a guide to the mathematics
/ edited by Satyajit Das.
p. cm.
Includes index.
ISBN 0-07-015378-7
1. Derivative securities—Mathematical models. 2. Risk management—Mathematical models. 3. Investment analysis—Mathematical models. I. Das, Satyajit.
HG6024.A3R577 1998
332.64'5—dc21 97-46822
CIP

McGraw-Hill

A Division of The McGraw-Hill Companies

2 3 4 5 6 7 8 9 DOC/DOC 9 0 2 1 0 9 8

ISBN 0-07-015378-7

The sponsoring editor for this book was Stephen Isaacs, the editing supervisor was John M. Morriss, and the production supervisor was Suzanne W. B. Rapcavage.

Printed and bound by R. R. Donnelley & Sons Company.

This book is printed on recycled, acid-free paper containing a minimum of 50% recycled de-inked fiber.

Preface

1. BACKGROUND AND OBJECTIVES OF THE BOOK

Modern financial management entails an appreciation of a number of key mathematical concepts. This is particularly relevant to risk management and risk management products, such as financial derivatives.

The knowledge of the mathematical concepts and the inherent assumptions underlying these mathematical concepts has tended to be concentrated among a select group of quantitative analysts employed by individual organisations (the so-called *rocket scientists* or *quants*). However, increasingly, the central role played by these products in capital markets is forcing a broader range of personnel to be aware of and utilise these concepts. This may be in a purely supervisory capacity or in using the concepts and products in day-to-day activities. This trend will continue and increase over time.

This increasing emphasis on mathematical finance creates problems for individuals who may not be comfortable with the basic concepts underlying the actual techniques utilised. This is usually caused by either a lack of exposure to the techniques or the absence of use of the concepts since graduation from university or other institutions. Hence, there is a very strong and increasing demand for material which explains the mathematical basis of risk management and financial derivatives in a *non-technical* manner to allow non-specialists to gain an appreciation of the concepts that are utilised.

Risk Management and Financial Derivatives: A Guide To The Mathematics is directed to fill this gap in the literature.

The book consists of a collection of papers from leading market practitioners covering:

(1) the basic mathematics underlying risk management and financial derivatives products; and

(2) application of the basic techniques in a number of common settings to promote understanding of the actual use of the concepts. Applications covered include the most common applications of mathematics to finance, such as:

- yield curve modelling and bond/fixed income pricing;
- pricing derivatives, both forwards and options;
- investment management applications; and
- risk management, in particular, value at risk and portfolio stress simulations based on monte carlo techniques.

The style of the book is as practical as possible. The text avoids mathematical notation to the maximum degree feasible so that an intuitive grasp of the concepts can be gained. The book also includes numerous detailed worked examples, wherever possible, to help the reader understand the concepts and see how the practical calculations are undertaken.

The target audience for this book includes:

- financial institutions, particularly commercial and investment banks, as well as brokers, active in trading activities;
- liability and investment managers who utilise or are looking at utilising trading risk management techniques in the management of trading risk;
- service industries, consultants, IT firms, accountants et cetera, active in advising traders or users of these techniques; and
- regulatory agencies.

The book can also be used as the basis for practical in-house training programs, as well as in post-graduate programs such as MBA or Applied Finance courses in financial markets, either as the primary text or supplementary reading.

2. CONTENT AND STRUCTURE

The book is structured around several themes, which correspond to the parts of the book.

Part 1—Introduction

This consists of a single chapter (Chapter 1) designed to outline the fundamental basis of the application of mathematics to financial markets, in particular the essential risk reward basis of value and the use of risk neutrality and arbitrage as the basis for all valuation or pricing.

Part 2—Interest Rates and Yield Curves

This section and the three which follow it are focused on specific application areas of derivatives in risk management and derivative products. In this part, the focus is on fixed interest markets made: Chapter 2 covers interest rates, the pricing of bonds and measures on interest rate risk, utilising duration and convexity; and Chapter 3 covers yield curve modelling, with coverage of the derivation of zero-coupon rates and the generation of yield curves.

Part 3—Derivative Pricing

Part 3 covers the application of mathematical techniques in the pricing of financial derivatives. Chapter 4 covers the pricing of forward and futures contracts. Chapter 5 examines the pricing of option contracts. Chapters 6 and 7 focus on more advanced issues in option pricing—specifically, the pricing of interest rate options utilising term structure models and the valuation of non-standard or exotic options. Chapter 8 focuses on the estimation of volatility, which is a central issue in the valuation of options. Chapter 9 looks at more complex approaches to estimation of volatility using ARCH/GARCH approaches to modelling volatility changes. Chapters 10 and 11 examine the risk of options: Chapter 10 examines the measurement of option price sensitivity using the Greek alphabet of risk (delta, gamma, theta, vega and rho); and Chapter 11 focuses on the use of option price sensitivities to replicate option profiles (the practice of delta hedging).

Part 4—Investment Management

Part 4 focuses on investment management applications of mathematics. Chapter 12 examines the concept of efficient portfolios and diversification within investment portfolios and the optimisation of portfolio structures. Chapter 13 examines the immunisation of bond portfolio using duration-based hedging techniques. Chapter 14 covers the concept and practice of portfolio insurance, utilising basic option theory to guarantee minimum values of portfolios. Chapter 15 looks at the creation, structuring and management of indexed portfolios.

Part 5—Risk Management

Part 5 covers the mathematics of risk measurement and management. Chapter 16 focuses on the use of value of risk techniques to model trading risk. Chapter 17 examines the use of stress testing, primarily using monte carlo simulation methods to measure risk not captured by traditional models of risk. Chapter 18 extends the framework of risk management to cover credit risk.

Part 6—Mathematical Techniques

Part 6 sets out the mathematical techniques underlying the applications described in previous sections. In Chapter 19 the basic mathematics underlying the financial applications is described, together with material which allows the reader to further her or his knowledge of the techniques and so come to a more complete understanding of the subject matter. The chapter also includes cross-references to built-in mathematical functions found in spreadsheet packages, to allow the reader to program some of these applications.

The book is designed either to be read through from start to finish or as a reference source where individual sections are read as required.

4. ACKNOWLEDGMENTS

I would like to thank all the contributors to the book. They, all busy practitioners, add a practical dimension and real world application focus to the work.

I would like to thank the Publishers—LBC Information Services (Fiona Dixon, Carolyn Uyeda and Julie Burke); Irwin Publishing (Kevin Commins and Stephen Isaacs); and MacMillan Press (Jane Powell and Stephen Rutt). I would also like to thank Kim Paino, who edited the book.

I would like to thank my parents—Sukumar and Aparna Das—for their continued support and encouragement. In particular, I would like to thank my friend Jade Novakovic without whose help, support, patience, tolerance and encouragement this would never have been completed. This book is dedicated to these three people.

SATYAJIT DAS

Sydney
November 1997

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About the Authors

Dr Carol Alexander

After working in the mathematics department of the University of Amsterdam, and later as a bond analyst for Phillips and Drew (UBS), Carol Alexander joined the mathematics faculty at the University of Sussex in 1985. Since 1990 she has developed an international regulation for the time-series analysis of financial markets, specialising in risk measurement and investment analysis. In 1996 she moved to a part-time post at the university, when she became the Academic Director of Algorithmics Inc (www.algorithmics.com) and Editor in Chief of NetExposure, the electronic journal of financial risk (www.netexposure.co.uk).

During the past few years she has been consulting in risk management and time-series analysis for banks, corporates and other financial institutions. As a result, most of her academic research work now concentrates on applied financial econometrics, specialising in volatility and correlation analysis. Carol has developed a large number of general and in-house training courses covering the general areas of risk management and investment analysis for financial institutions.

With Algorithmics Inc, she is developing VAR models for large-scale risk management systems and new methods for historical simulation using pattern recognition. Her pattern recognition algorithm, developed with Dr Ian Giblin (Department of Mathematics, University of Pisa) won the first international non-linear financial forecasting competition in 1996. She also works with Dr Peter Williams (Department of Cognitive Science, University of Sussex) on using neural networks to estimate mixtures of normal distributions to model, "fat-tailed" distributions and term structures of kurtosis. Her research on emerging markets includes the analysis of equity and currency derivatives in the Asia-Pacific region, and the hedging of equities with new cointegration software.

She speaks at many international conferences and on mathematical techniques for risk and investment management and has written numerous articles in both academic and professional journals. Carol's books include the edited *Handbook for Risk Management and Analysis* (1st ed, April 1996; 2nd ed in two volumes, forthcoming February 1998, Wiley). More details of professional and academic publications are given on www.maths.sussex.ac.uk/Staff?COA.html.

Geoffrey Brianton

Geoffrey Brianton has an Honours degree in economics and statistics. He is an independent consultant in the area of investment risk management. For the last ten years he has worked as a Fund Manager in London, Sydney and Melbourne. During that time he has developed and managed a wide variety of quantitative funds, including indexed, capital protected and arbitrage funds. With a background in econometrics and operations research, he has a particular interest in the use of optimisation techniques, both in regard to the problems of portfolio construction and some of the broader problems found in finance such as optional hedging strategies.

Alan Bustany

Alan Bustany is a Principal Consultant in the Financial Services Industry Practice of Price Waterhouse Urwick. He is a Wrangler from Trinity College, Cambridge, and was part of the British team at the 1977 International Mathematical Olympiad in Belgrade. He is an Associate Fellow of the Institute of Mathematics and its Applications, and of the Securities Institute of Australia. Alan has used his mathematical background in a range of technology-related fields, including relational database theory; artificial intelligence; formal methods of software engineering; knowledge-based systems; and financial derivatives.

Mr Bustany has more than 15 years' experience in the computing industry in consulting, commercial, and product development roles. His current focus is strategic consulting in financial markets, risk management, and credit measurement. His clients include Bankers Trust, National Australia Bank, Macquarie Bank and Westpac.

Roger Cohen

Roger Cohen is currently employed at Deutsche Morgan Grenfell, in the Equity Derivatives and Trading Group. His main focus is on the development of structured equity products, pricing and risk management. Roger commenced his career in the financial markets in the Quantitative Applications Division at Macquarie Bank in 1992. Since then he has worked at Natwest Markets in the Global Markets area. He has presented papers at finance conferences in Australia and abroad. Prior to entering the financial markets, Roger was a lecturer in the Faculty of Engineering at the University of Sydney. His main research focus was in the computational modelling of fluid motion.

Frances Cowell

Prior to graduating from university, Frances worked in the biomedical library at the New South Wales Institute of Technology, and then as a Rehabilitation Counsellor for disabled people. After obtaining a Bachelor of Arts degree in psychology and statistics from the University of New South Wales, she worked in the wholesale liquor industry as a Marketing Manager, before completing a Master of Business Administration at the Australian Graduate School of Management.

Frances entered the investment management industry in January 1983 as the Research Analyst for Aetna Life & Casualty, then a major life office. Faced with analysing a portfolio of 300 stocks, she employed a novel approach to stock analysis, using a desktop computer and spreadsheet to simultaneously analyse expected risk-return profiles for a large number of stocks, thereby identifying apparently mispriced issues for closer analysis.

This was combined with industry-wide analyses to estimate industry growth rates and of competitive advantage to identify best performing stocks within industry groups.

In 1984, with the growth of the share price index futures market in Australia, Frances was attracted by the opportunities for SPI arbitrage and joined Australian Bank to exploit these. There she also established and managed an early portfolio protection operation which enabled the bank's borrowers to cap their interest rate risk. This portfolio protection program drew on a variant of Black-Scholes option pricing technology (delta hedging) to hedge the bank's sold option positions.

Later, Frances combined stock index arbitrage principles with delta hedging to exploit underpriced SPI futures contracts with minimal risk. This operation, which was unique at the time, was conducted within a major Australian institutional broking house.

In 1991 Frances began working on indexed portfolios for a major Australian superannuation fund. This work covered domestic and global equities and fixed interest. She managed the derivatives enhanced domestic equities index portfolio, which grew to over A$1 billion. From there she joined County NatWest to take over the management of index portfolios. At County, Frances further developed customising capabilities for indexation clients, with particular focus on adding value by managing after-tax returns to index portfolios.

Satyajit Das

Satyajit Das is an international specialist in the area of financial derivatives, risk management, capital markets, and treasury management.

He presents seminars on financial derivatives/risk management and treasury management/corporate finance in Europe, North America, Asia and Australia. He acts as a consultant to financial institutions and corporations on derivative instruments, risk management, and treasury/financial management issues.

Between 1988 and 1994, Mr Das was the Treasurer of the TNT Group, an Australian-based international transport and logistics company, with responsibility for the Global Treasury function, including liquidity management, corporate finance, capital markets, and financial risk management. He was also involved in the financial restructuring of the TNT Group in the early 1990s. During 1994, Mr Das acted as a consultant to the TNT Group in the areas of financial strategy and policy, capital allocation/management and strategic risk management.

Between 1977 and 1987, he worked in banking with the Commonwealth Bank of Australia, Citicorp Investment Bank and Merrill Lynch Capital Markets, specialising in fundraising for Australian and New Zealand borrowers in domestic and international capital markets and risk management, involving the use of derivative products, including swaps, futures and options.

In 1987, Mr Das was a Visiting Fellow at the Centre for Studies in Money, Banking and Finance, Macquarie University.

Mr Das is the author of *Swap Financing* (IFR Publishing Ltd/The Law Book Company Ltd, 1989); *Swaps and Financial Derivatives: The Global Reference to Products, Pricing, Applications and Markets* (IFR Publishing Ltd/The Law Book Company Ltd/Irwin Professional Publishing, 1994); *Exotic Options* (IFR Publishing Ltd/LBC Information Services, 1996); and *Structured Notes and Derivative Embedded Securities* (Euromoney Publications, 1996). He is also the editor of *The Global Swaps Market* (IFR Publishing Ltd, 1991). He has published on financial derivatives, corporate finance, treasury and risk management issues in professional and applied finance journals.

Mr Das holds Bachelors' degrees in Commerce (accounting, finance and systems) and Law from the University of New South Wales, and a Masters degree in Business Administration from the Australian Graduate School of Management.

Dr Garry de Jager

Garry de Jager holds a PhD in option pricing, a Masters of Business Administration and a Bachelor of Science in mathematics. He is currently the Senior Manager, International Capital Markets Research, for Chase Manhattan Bank, Sydney. His primary interests are modelling derivatives for commodities, foreign exchange and interest rate products. In his role supporting the trading desks, his special emphases are exotic options, hedging strategies, simulation analysis, volatility and correlation analysis, and provision of models for system development. He is a frequent traveller to Chase sites around the world, as well as a regular speaker at seminars on these topics.

Garry has wide commercial experience, having previously been the Director of Research and Development for the specialist financial option company Optech International; Marketing Representative for IBM; and Production Manager in the printing industry. Prior to joining Chase Manhattan he taught finance and Information Systems at the Queensland University of Technology.

Thomas R Gillespie

Thomas Gillespie graduated from the University of Sydney in 1986 with a Bachelor of Science, and in 1988 with a Bachelor of Arts majoring in mathematics and statistics. After graduation, Thomas joined Bankers Trust Australia and specialised in foreign exchange and fixed interest options. In 1989 he moved to James Capel Australia and there specialised in equity derivatives and arbitrage. This led to work with the HSBC group in Sydney, Tokyo and London on a number of projects, including Japanese derivatives arbitrage and automated trading systems. In 1994 Thomas decided to take up part-time studies again, earning a Graduate Diploma in Science in 1994, and is now pursuing a PhD at the University of Sydney. Thomas' current research interests are fitting alternative stochastic processes to financial time series, and the development of new estimation methods. Thomas is currently working with County NatWest Australia, specialising in equity derivatives and structured products.

John Martin

John Martin has worked, taught and published extensively in the areas of treasury, derivatives and financial risk management. He was closely involved in the development of the derivatives industry in both Australia and New Zealand in roles varying from market trader, risk manager, regulator, and educator. John's area of interest is in financial risk management, and he has written numerous articles on this topic, including his recently published book, *Derivatives Maths* (IFR Publishing Ltd, 1996).

Currently, John is a Divisional Director of Australian-based treasury advisory firm Oakvale Capital Ltd. He is responsible for the provision of specialist financial risk management advice to clients across a wide range of industries, including electricity, retailing, agriculture, and financial services.

Prior to joining Oakvale, John was the Risk Manager of the Sydney Futures Exchange from 1991 to 1994, and was then a Director of Financial Risk Management Consulting Pty Ltd. He has also held positions with the Reserve Bank of Australia, and with TNT as Manager, Treasury Planning.

John is regarded throughout the Asia/Pacific region as a leading author and speaker in the areas of treasury and financial risk management, and has lectured extensively to professional groups in the Asia/Pacific region.

John holds a Bachelor's degree in Economics with Honours from the University of Sydney.

Steuart Roe

Steuart Roe is a specialist in equities and finance for County NatWest Securities Australia Ltd, a member of the NatWest Markets Group. He specialises in derivative product development, implementation, and distribution.

Prior to joining County NatWest Securities Australia Ltd, he was an Associate Director, Structured Investments for County NatWest Investment Management Ltd. In this role he was responsible for the design and implementation of tailored investment solutions for individual clients. In particular, he was responsible for the management of in excess of A$2 billion in assets, the bulk of which was in portfolio insurance.

Steuart holds a Bachelor of Science in mathematics and statistics from the University of Melbourne, and a Master of Applied Finance from Macquarie University.

Dr Tim Rowlands

After completing a PhD in theoretical chemistry at Cambridge University in the United Kingdom in 1989, Tim returned to Australia and commenced work for Macquarie Bank in a quantitative analysis role. His work involved supporting the Fixed Interest Division with emphasis on developing and implementing new products, including being part of a team implementing a term structure option pricing model. Tim then moved to the State Bank of New South Wales in 1992, and, from 1993, led the Quantitative Group with work focusing on interest rate options, retail product enhancements (including options incorporating prepayment risk) and risk management methodologies.

In 1995 Tim became Head of Treasury Risk Management, with responsibilities across the full spectrum of market and credit risk, policy and compliance. In mid-1996 Tim moved to Westpac to take responsibility for methodology in the trading risk area. This role has included reviewing and refining the bank's VAR calculations and the implementation of historical simulation capability for risk measurement. In addition, Tim has been involved in research into applying VAR techniques to credit and operational risk to improve capital allocation and risk adjusted performance measurement within the business.

Lance Smith

Lance Smith has a PhD in mathematics from Duke University and was an Assistant Professor of Mathematics at Columbia University prior to joining Salomon Brothers in 1986. At Salomon he worked on the equity proprietary trading desk and was responsible for the development of all the desk's financial pricing and hedging models used for assessing and limiting the risk in a derivatives book. He also developed and implemented several proprietary trading strategies, as well as hedged the more exotic derivative OTC transactions undertaken by the desk. In 1993 he and some of his long-term colleagues founded Imagine Software Inc. Along with developing state of the art risk management technology, he continues to explore new methodologies in pricing derivative securities and quantifying risk in the financial markets.

Selected Bibliography

Chapter 3

Adams, Kenneth J, "Fitting Yield Curves To Market Data" (1994) 1(2) *The Pacific ALM Journal* 27.

Adams, Kenneth J and Van Deventer, Donald R, "Fitting Yield Curves And Forward Rate Curves With Maximum Smoothness" (1994) (June) *The Journal of Fixed Income* 52.

Cox, David, "Yield Curves And How To Build Them" (1995) 4 *Capital Market Strategies* 29.

Flavell, Richard, "Discount Function Construction" *IFR Swaps Weekly*, 27 July 1994, p 4.

—— "Fitting Interest Rate Curves" (1995) *Matlab Financial Computing Brief.*

McEnally, Richard W and Jordan, James V, "The Term Structure of Interest Rates" in Fabozzi, Frank J and Fabozzi, T Dessa (eds), *The Handbook of Fixed Income Securities* (4th ed, Irwin Professional Publishing, Chicago, 1995), pp 779-830.

Smith, David, "By The Bootstraps" (1990) 3(6) *Risk* 40.

—— "Zero Coupon Yields" (1995) 5 *Capital Market Strategies* 51.

Chapter 4

Das, S, *Swaps and Financial Derivatives* (2nd ed, The Law Book Company Ltd, 1994).

Francis, Toy and Whittaker (eds), *The Handbook of Equity Derivatives* (1995).

Hull, J C, *Options, Futures and Other Derivative Securities* (Prentice-Hall, 1993).

Martin, J S, *Derivative Maths* (IFR Publishing Ltd, 1996).

Chapter 5

Black, Fischer, "The Pricing of Commodity Contracts" (1976) 3 (March) *Journal of Financial Economics* 167.

Black, Fischer, "The Holes in Black-Scholes" (1988) 1(3) *Risk* 30.

Black, Fischer, "How We Came Up With the Option Formula" (1989) (Winter) *Journal of Portfolio Management* 4.

Black, Fischer, "How to Use the Holes in Black-Scholes" (1989) 1(4) *Continental Bank Journal of Applied Corporate Finance* 67.

Black, Fischer, "Living Up To The Model" (1990) 3(3) *Risk* 11.

Black, Fischer, Derman, E, and Toy, W, ''A One Factor Model of Interest Rates and Its Application to Treasury Bond Options'' (1990) (Jan-Feb) *Financial Analysts Journal* 33.

Black, Fischer, and Scholes, Myron, ''The Pricing of Options and Corporate Liabilities'' (1973) 81 *Journal of Political Economy* 637.

Cookson, Richard, ''Moving in The Right Direction'' (1993) 6(10) *Risk* 22.

Cox, J and Rubinstein, M, *Options Markets* (Prentice-Hall, Inc, Englewood Cliffs, NJ, 1985).

Derman, Emanuel, ''Model Risk'' (1996) 9(5) *Risk* 34.

Garman, Mark B and Kohlhagen, Steven W, ''Foreign Currency Option Values'' (1983) 2 *Journal of International Money and Finance* 231.

Grabbe, J Orlin, ''The Pricing of Call and Put Options on Foreign Exchange'' (1983) 2 *Journal of International Money and Finance* 239.

Hull, John C, *Options Futures and Other Derivatives Securities* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1993).

Hull, John C, *Introduction To Futures and Options Markets* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1995).

Jarrow, Robert A and Rudd, Andrew, *Option Pricing* (Richard D Irwin, Homewood, Illinois, 1983).

Leong, Kenneth, ''In the Eye Of The Beholder'' (1990) 3(7) *Risk* 38.

Leong, Kenneth, ''The Emperor's New Clothes'' (1990) 3(8) *Risk* 11.

Leong, Kenneth, ''Exorcising The Demon'' (1990) 3(9) *Risk* 29.

Leong, Kenneth, ''Price Versus Value'' (1991) 4(10) *Risk* 22.

Leong, Kenneth, ''Model Choice'' (1992) 5(11) *Risk* 60.

Mason, R C, ''Building On Black Scholes'' (1988) 1(11) *Risk* 13.

Meisner, James F and Labuszewski, John W, ''Modifying the Black-Scholes Option Pricing Model For Alternative Underlying Instruments'' (1984) (Nov-Dec) *Financial Analysts Journal* 23.

Ritchen, Peter, *Options—Theory, Strategy and Applications* (Scott, Foresman and Company, Glenview, Illinois, 1987).

Rowley, Ian, ''Pricing Options Using The Black-Scholes Model'' (1987) (May) *Euromoney Corporate Finance* 108.

Rowley, Ian, ''Option Pricing Models: How Good Is Black-Scholes?'' (1987) (June) *Euromoney Corporate Finance* 30.

Smithson, Charles W, ''Wonderful Life'' (1991) 4(9) *Risk* 37.

Smithson, Charles W with Shang Song, ''Extended Family (1) and (2)'' (1995) 8(10) *Risk* 19; (1995) 8(11) *Risk* 52.

Swiss Bank Corporation, *Understanding Derivatives* (Prospects Special, Basel, Switzerland, 1994).

Tompkins, Robert, *Options Explained* (MacMillan Press, England, 1994).

Chapter 6

Black, Scholes, (1973) 81 *Journal of Political Economy* 637.

Black, (1976) 3 *Journal of Financial Economics* 167.

Black, Derman, Toy, (1990) *Financial Analysts Journal* 33.

Brace, Gatarek, Musiela, (1995) (Aug) *Risk* 34.

Brennan, Schwartz, (1983) 4 *Finance* 119.

Cox, Ingersoll, Ross, (1985) 53 *Econometrica* 363.

Cox, Ross, Rubinstein, (1979) 7 *Journal of Financial Economics* 229.

Heath, Jarrow, Morton, (1992) 60 *Econometrica* 77.

Ho, Lee, (1986) 41 *Journal of Finance* 1,011.

Ho, (1995) (Summer) *Journal of Derivatives*.

Hull, *Options, Futures and Other Derivatives* (3rd ed, Prentice-Hall, 1997).

Hull, White, (1990) 3 *Review of Financial Studies* 573.

Jamshidian, (1989) 44 *Journal of Finance* 205.

Jarrow, *Modelling Fixed Income Securities and Interest Rate Options* (1996).

Longstaff, Schwartz, (1992) 47 *Journal of Finance* 1,259.

Rebonato, *Interest-Rate Option Models* (Wiley & Sons, 1996).

Rendleman, Bartter, (1979) 34 *Journal of Finance* 1,092.

Smithson, (1995) (Oct-Nov) *Risk*.

Vasicek, (1977) 5 *Journal of Financial Economics* 177.

Chapter 7

Black, F, "The Pricing of Commodity Contracts" (1976) 3(1 and 2) *Journal of Financial Economics* 167.

Black, F, Derman, E and Toy, W, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options" (1990) (Jan-Feb) *Financial Analysts Journal* 33.

Black, F and Scholes, M, "The Pricing of Options and Corporate Liabilities" (1973) 81 *Journal of Political Economy* 637.

Conze, A and Viswanathan, "Path Dependent Options: The Case of Lookback Options" (1991) 56(3) *Journal of Finance* 1,893.

Cox, J and Ross, S, "The Valuation of Options for Alternative Stochastic Processes" (1976) (Jan-Mar) *Journal of Financial Economics* 145.

Cox, J C, Ross, S and Rubinstein, M, "Option Pricing: A Simplified Approach" (1979) 7 *Journal of Financial Economics* 229.

Geske, R, "A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends" (1979) 7 *Journal of Financial Economics* 375.

Goldman, M B, Sosin, H B and Gatto, M A, "Path Dependent Options: 'Buy at the Low, Sell at the High' " (1979) 34(5) *Journal of Finance* 1,111.

Grabbe, J O, "The Pricing of Call and Put Options on Foreign Exchange" (1983) 2 *Journal of International Money and Finance* 239.

Heath, D C, Jarrow, R A and Morton, A, ''Contingent Claim Valuation with a Random Evolution of Interest Rates'' (1990) 9(1) *The Review of Futures Markets* 54.

Hull, J C and White, A, ''Efficient Procedures For Valuing European And American Path-dependent Options'' (1993) 1(1) *Journal of Derivatives* 21.

Merton, R C, ''Option Pricing When Underlying Stock Returns Are Discontinuous'' (1976) 3 *Journal of Financial Economics* 125.

Roll, R, ''An Analytical Valuation Formula for Unprotected American Call Options on Share with Known Dividends'' (1977) 5(2) *Journal of Financial Economics* 251.

Rubinstein, M and Reiner, E, ''Barrier Options'' in *Exotic Options*, unpublished manuscript by M Rubinstein (1991).

Whaley, R, ''On the Valuation of American Call Options on Stocks with Known Dividends'' (1981) 9 *Journal of Financial Economics* 207.

Chapter 8

Carrado, Charles and Miller, Thomas, ''Volatility Without Tears'' (1996) 9(7) *Risk* 49.

Chance, Don, ''Leap Into the Unknown'' (1993) 6(5) *Risk* 60.

Clarke, Roger C, ''Estimating And Using Volatility: Part 1'' (1994) (Fall) *Derivatives Quarterly* 40.

Cox, J and Rubinstein, M, *Options Markets* (Prentice-Hall, Inc, Englewood Cliffs, NJ, 1985).

Engle, Robert F, ''Statistical Models For Financial Volatility'' (1993) (Jan-Feb) *Financial Analysts Journal* 72.

Hargreaves, Guy D, ''Volatility in Interest Rate Options'', paper presented to a conference, ''Options on Interest Rates'', IIR Conferences, 9-10 March 1992, Sydney, Australia.

Hull, John C, *Options Futures and Other Derivatives Securities* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1993).

Hull, John C, *Introduction to Futures and Options Markets* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1995).

Leong, Kenneth, ''Estimates, Guesstimates and Rules Of Thumb'' (1991) 4(2) *Risk* 15.

Leong, Kenneth, ''Mean Streets'' (1991) 4(5) *Risk* 45.

Murphy, Gareth, ''When Options Price Theory Meets The Volatility Smile'' (1994) (Mar) *Euromoney* 66.

Tompkins, Robert, *Options Explained* (MacMillan Press, England, 1994).

Tompkins, Robert, ''Answer in the Cards'' (1995) 8(6) *Risk* 55.

Chapter 9

Alexander, C O, "Volatility and correlation forecasting" in C O Alexander (ed), *Handbook of Risk Management and Analysis* (Wiley, 1996).

Alexander, C O, "Evaluating the Use of RiskMetrics™ as a Risk Measurement Tool for Your Operation: What are its Advantages and Limitations" (1996) 2 *Derivatives Use, Trading and Regulation* 277.

Alexander, C O, " 'Splicing' methods for Value-at-Risk" *Derivatives Week*, Learning Curve, June 1997.

Alexander, C O and Leigh, C, "On the Covariance Matrices used in VAR Models" (1997) 4 (Spring) *Journal of Derivatives* 50-62.

Alexander, C O and Chibumba, A M, "Orthogonal factor GARCH", forthcoming, *Journal of Empirical Finance*, 1997.

Alexander, C O and Williams, P M, "Term Structure Forecasts of Foreign Exchange Volatility and Kurtosis: A Comparison of Neural Network and GARCH Methods", forthcoming, *Risk*, 1997.

Becker, S, "A Survey of Risk Measurement, Theory and Practice" in C O Alexander (ed), *Handbook of Risk Management and Analysis* (Wiley, 1996).

Bollerslev, T, "Generalized Autoregressive Conditional Heteroskedasticity" (1986) 31 *Journal of Econometrics* 307.

Broadie, M and Glasserman, P, "Estimating Security Price Derivatives Using Simulation" (1996) 42(2) *Management Science*.

Duan, J, "Cracking the Smile" (1996) 9(12) *Risk* 55-59.

Engle, R F, "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation" (1982) 50(4) *Econometrica* 987.

Engle, R F and Mezrich, J, "Grappling with GARCH" (1995) 8(9) *Risk* 112.

Engle, R F and Rosenberg, J, "GARCH Gamma" (1995) 2 *Journal of Derivatives* 47.

Figlewski, S, "Forecasting Volatility Using Historical Data", New York University Salomon Center (Leonard N Stern School of Business), Working Paper, Series No S-94-13, 1994.

Fitzgerald, M D, "Trading volatility" in C O Alexander (ed), *Handbook of Risk Management and Analysis* (Wiley, 1996).

Hull, J C, *Options, Futures, and Other Derivative Securities* (Prentice-Hall, 1993).

Taylor, S J, "Modeling Stochastic Volatility: A Review and Comparative Study" (1994) 4(2) *Mathematical Finance* 183.

Chapter 10

Brown, Stephen J, "Estimating Volatility" in Figlewiski, Stephen, Silber, William L and Subrahmanyam, Marti G, *Financial Options* (Business One Irwin, Homewood, Illinios, 1990), pp 516-537.

Chance, Don M, "Translating The Greek; The Real Meaning Of Call Option Derivatives" (1994) (July-Aug) *Financial Analyst's Journal* 43.

Cox, J and Rubinstein, M, *Options Markets* (Prentice-Hall, Inc, Englewood Cliffs, NJ, 1985).

Hull, John C, *Options Futures and Other Derivatives Securities* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1993).

Hull, John C, *Introduction To Futures and Options Markets* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1995).

Tompkins, Robert, *Options Explained* (MacMillan Press, England, 1994).

Chapter 11

Asay, Charles and Edelsburg, Charles, ''Can a Dynamic Strategy Replicate the Returns of an Option'' (1986) 6(1) *Journal of Futures Markets* 63.

Silber, William L, ''Marketmaking In Options: Principles and Implications'' in Figlewiski, Stephen, Silber, William L, and Subrahmanyam, Marti G, *Financial Options* (Business One Irwin, Homewood, Illinios, 1990), pp 485-516.

''The Dangers Of Neutrality'' (1991) 4(10) *Risk* 48.

Frye, Jon, ''Greek Alphabet Soup: A Recipe For Success'' (1988) 1(4) *Risk* 26.

Hull, John C, *Options Futures and Other Derivatives Securities* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1993).

Hull, John C, *Introduction to Futures and Options Markets* (2nd ed, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1995).

Leong, Kenneth, ''Solving Mystery'' (1990-1991) 4(1) (Dec-Jan) *Risk* 68.

Rubinstein, Mark and Leland, Hayne E, ''Replicating Options With Positions In Stock And Cash'' (1991) (July-Aug) *Financial Analyst's Journal* 63.

Rutherford, Janette, Sher, Answer, and Fitzgerald, Desmond, ''Building Blocks'' (1989) 2(7) *Risk* 42.

Smith, A L H, *Trading Financial Options* (Butterworths, London, 1986).

Westminister Equity, ''Building Confidence'' (1990) 3(4) *Risk* 33.

Chapter 12

Arnott, Robert D and Wagner, Wayne H, ''The Measurement and Control of Trading Costs'' (1990) (Nov-Dec) *Financial Analysts Journal*.

Black, Fischer, ''Universal Hedging: Optimising Currency Risk and Reward In International Equity Portfolios'' (1989) (July-Aug) *Financial Analysts Journal*.

Black, Fischer and Litternam, Robert, ''Global Portfolio Optimisation'' (1992) (Sept-Oct) *Financial Analysts Journal*.

Jeffrey, Robert H and Arnott, Robert D, ''Is your Alpha Big Enough to Cover Its Taxes?'' (1992) (Sept) *Journal of Portfolio Managament*.

Jobson, D and Korkie, B, ''Putting Markowitz Theory to Work'' (1981) (Summer) *Journal of Portfolio Management*.

Jobson, J D and Korkie, B, ''Improved Estimation for Markowitz Portfolios Using James-Stein Type Estimators'', proceedings of the American Statistical Association, Business and Economic Statistic Section, 1979.

Markowitz, Harry, *Mean-Variance Analysis in Portfolio Choice and Capital Markets* (Basil Blackwell Inc, Oxford, 1990).

Markowitz, Harry, ''Portfolio Selection: Efficient Diversification of Investments'', *Cowles Foundation Monograph No 16* (John Wiley and Sons, New York, 1959).

Perold, A F, ''Large Scale Portfolio Optimisation'' (1984) 30(10) *Management Science.*

Wolfe, P, ''The Simplex Method for Quadratic Programming'' (1959) 27(3) *Econometrica.*

Chapter 14

Black, F and Scholes, M, ''The pricing of options on corporate liabilities'' (1973) 81 (May-June) *Journal of Political Economy* 637.

Hull, J, *Options, Futures, and Other Derivative Securities* (2nd ed, Prentice-Hall International Inc).

Perold, A F and Sharpe, W F, ''Dynamic Strategies for Asset Allocation'' (1988) (Jan-Feb) *Financial Analyst Journal.*

Chapter 15

Chiang, A C, *Fundamental Methods of Mathematical Economics* (McGraw Hill, New York, 1974).

Kritzman, Mark, *The Portable Financial Analyst* (Probus, Chicago, Ill, 1995).

Lakonishok, Josef, Schleifer, Andre and Vishny, Robert W, *Study of the US Equity Money Manager Performance*, Brookings Institute Study, 1992.

Markowitz, H M, *Portfolio Selection: Efficient Diversification of Investment*, Cowles Foundation Monograph 16 (Yale University Press, New Haven, Ct, 1959).

Rosenberg, B, *Extra Market Components of Covariance in Security Returns* (1974) (Mar) *Journal of Financial and Quantitative Analysis* 263.

Rudd, Andrew, *Optimal Selection of Passive Portfolios* (1980) (Spring) *Financial Management* 57.

Rudd, Andrew and Clasing, Henry K Jr, *Modern Portfolio Theory* (2nd ed, Andrew Rudd, Orinda, CA, 1988).

Sharpe, William F, ''Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk'' (1970) 19(3) *Journal of Finance* 425.

Sharpe, William F, *Portfolio Theory and Capital Markets* (McGraw Hill, New York, 1970).

Chapter 16

Batlin, Carl, "Regulatory Approaches to Risk Management" (1996) 6 (June) *Financial Derivatives & Risk Management* 69.

Beckstrom, Rod A, Lewis, Don and Roberts, Chris, "VAR: Pushing Risk Management to the Statistical Limit" (1994) 3 *Capital Market Strategies* 9.

Beckstrom, Rod A and Campbell, Alyce R, "Value At Risk: Theoretical Foundations" (1996) 7 *Capital Market Strategies* 44.

Beckstrom, Rod A and Campbell, Alyce R (eds), *An Introduction to Value at Risk* (C.ATS Software Inc, Palo Alto, 1995).

Beder, Tanya Styblo, "VAR: Seductive But Dangerous" (1995) (Sept-Oct) *Financial Analysts Journal* 12.

Bock, Jerome T, "VAR Approaches to Market Risk Management" (1996) 6 (June) *Financial Derivatives & Risk Management* 15.

Cocks, Graham, "Measuring Market Risk Within a Bank" (1996) 1(2) *Journal of Applied Finance and Investment* 63.

Morgan, J P, *RiskMetric™—Technical Document* (3rd ed, J P Morgan, New York, 1995).

Morgan, J P, *RiskMetric™—Technical Document* (4th ed, J P Morgan, New York, 1996).

Lawrence, Dr Colin, Robinson, Gary and Stiles, Matthew, "Incorporating Liquidity Into The Risk Measurement Framework" (1996) 6 (June) *Financial Derivatives & Risk Management* 24.

Lawrence, Dr Colin and Robinson, Gary, "Value at Risk: Addressing Liquidity and Volatility Risks" (1996) 7 *Capital Market Strategies* 24.

Leong, Kenneth, "The Right Approach" (1996) (June) *Risk Value-At-Risk Supplement* 9.

Longerstaey, Jacques, "VAR, RiskMetrics™ And Market Risk Methodology" (1995) 6 *Capital Market Strategies* 9.

Longerstaey, Jacques and Zangari, Peter, "Commoditising the VAR Framework: Incoporating Gamma Risk" (1996) 6 (June) *Financial Derivatives & Risk Management* 49.

Matten, Chris, "The Capital Allocation Challenge for the Banks" (1995) 4-5 *Swiss Bank Corporation Prospects* 2.

Matten, Chris, "Risk Adjusted Performance Measurement" (1996) 6 (June) *Financial Derivatives & Risk Management* 37.

Ramaswami, Murali, "Why Capital at Risk is the Recommended Measure" (1996) 6 (June) *Financial Derivatives & Risk Management* 53.

Salzberg, Mike, "Firmwide Risk Management Systems" (1996) 6 (June) *Financial Derivatives & Risk Management* 62.

Sharma, Jitendra, "Practical Issues of Value-At-Risk" in (1996) *Corporate Finance Risk Management & Derivatives Yearbook* 8.

Shimko, David, "What is VAR?" (1995) 8(12) *Risk* 25.

Shimko, David, "VAR for Corporates" (1996) 9(6) *Risk* 28.

Shimko, David, "Investor's Return On VAR" (1996) 9(7) *Risk* 27.

Smith, Lance C, "Portfolio Simulation: Stress-Testing Techniques" (1996) 6 (June) *Financial Derivatives & Risk Management* 31.

Smithson, Charles and Minton, Lyle, "Value at Risk" (1996) 9(1) *Risk* 25.

Smithson, Charles and Minton, Lyle, "Value at Risk (2)" (1996) 9(2) *Risk* 38.

Spinner, Karen, "Adapting Value at Risk" (1996) (April) *Derivatives Strategy* 14.

Stambaugh, Fred and Cohen, Robert, "Value at Risk: Its Measurement and Uses" (1995) 4 (Dec) *Financial Derivatives & Risk Management* 45.

Taylor, Charles R and MacDonald, William A, "The Future of Market Risk Management" (1996) 6 (June) *Financial Derivatives & Risk Management* 10.

Turner, Chris, "VAR as an Industrial Tool" (1996) 9(3) *Risk* 38.

Wilson, Duncan, "Marriage of Ideals" (1995) 8(7) *Risk* 7.

Wilson, Duncan, "VAR in Operation" (1995) 8(12) *Risk* 24.

Chapter 19

Abramowitz, M and Stegun, I, *Handbook of Mathematical Functions* (9th ed, Dover, 1972).

Bhattacharya, R and Waymire, E, *Stochastic Processes with Applications* (Wiley, 1990).

Black, F and Scholes, M, "The Pricing of Options and Corporate Liabilities" (1973) 81 (May-June) *Journal of Political Economy* 637.

Feller, W, *An Introduction to Probability Theory and Its Applications* (Volume I, 3rd ed, Wiley, 1968).

Feller, W, *An Introduction to Probability Theory and Its Applications* (Volume II, 2nd ed, Wiley, 1971).

Garman, M and Klass, M, "On the Estimation of Security Price Volatilities from Historical Data" (1980) 53 (1) *Journal of Business*.

Hogg, R and Craig, A, *Introduction to Mathematical Statistics* (5th ed, Prentice-Hall, 1995).

Huber, P, *Robust Statistics* (Wiley, 1981).

Hull, J, *Options Futures and Other Derivative Securities* (2nd ed, Prentice-Hall International, 1993).

Hull, J and White, A, "The Pricing of Options on Assets with Stochastic Volatilities" (1987) 42 (June) *Journal of Finance* 281.

Jarrow, R and Rudd, A, *Option Pricing* (Irwin, 1983).

Jarrow, R and Rudd, A, "Approximate Option Valuation for Arbitrary Stochastic Processes" (1982) 10 *Journal of Financial Economics* 347.

Jarrow, R and Rudd, A, "Tests of an Approximate Option Valuation Formula" in Brenner, M (ed), *Option Pricing* (Lexington Mass, DC Heath, 1983, pp 81-100).

Knuth, D, *The Art of Computer Programming: Seminumerical Algorithms* (2nd ed, Vol 2, Addison-Wesley, 1981).

Kreyszig, E, *Advanced Engineering Mathematics* (John Wiley and Sons, 1993).

Launer, R and Wilkinson, G, *Robustness in Statistics* (Academic Press, 1979).

Lawrence, C, Zhou, J and Tits, A, *Users Guide for CFSQP Version 2.4: A C Code for Solving (Large Scale) Constrained Nonlinear (Minimax) Optimisation Problems, Generating Iterates Satisfying All Inequality Constraints* (Electrical Engineering Department and Institute for Systems Research, University of Maryland, College Park, MD 20742 USA, 1996).

Malliaris, A and Brock, W, *Stochastic Methods in Economics and Finance* (North-Holland, 1982).

McKean, H, *Stochastic Integrals* (New York, Academic Press, 1969).

Mendenhall, W, Scheaffer, R and Wackerly, D, *Mathematical Statistics with Applications* (4th ed, Duxbury Press, 1990).

Microsoft Corporation, *Microsoft Excel User's Guide* (1993).

Moro, B, ''The Full Monte'' (1995) 8(2) *Risk.*

Neter, J and Wasserman, W, *Applied Linear Statistical Models* (Irwin, 1974).

Neter, J, Kutner, M, Nachtsheim, C and Wasserman, W, *Applied Linear Statistical Models* (4th ed, Irwin, 1995).

Papageorgiou, A and Traub, J, ''Beating Monte Carlo'' (1996) 9(6) *Risk.*

Parkinson, M, ''The Extreme Value Method for Estimating the Variance of the Rate of Return'' (1980) 53(1) *Journal of Business.*

Press, W, Flannery, B, Teukolsky, S and Vetterling, W, *Numerical Recipes in c* (2nd ed, Cambridge University Press, 1992).

Ross, S, *An Introduction to Probability Models* (4th ed, Academic Press, 1989).

Schuss, Z, *Theory and Application of Stochastic Differential Equations* (Wiley, 1980).

Taha, H, *Operations Research: An Introduction* (5th ed, Macmillan Publishing, 1992).

Tezuka, S, *Uniform Random Numbers: Theory and Practice* (Kluner Academic Publishers, 1995).

Risk Management and Financial Derivatives:

A Guide to the Mathematics

Part 1

Introduction

Chapter 1

Risk-reward Relationships—Foundations of Derivatives

by Lance Smith

1. INTRODUCTION

Classical portfolio theory examines risk/reward tradeoffs from a ''mean-variance' framework. In this model, the ''risk'' of an individual security is encapsulated by the variance (or, equivalently, *standard deviation*) of its returns. The higher the variance, the more uncertain the return, and therefore the greater the risk. Diversification by assembling a portfolio of securities enables investors to decrease their variance while maintaining their expected profitability target.

The analysis of the risk in derivatives, as well as the pricing methodology, differs from the classical framework in many respects. In fact, in the case of equities, the only point they have in common is the assumption that the risks of an underlying security are characterised by the mean and variance of its returns. At this point, the analysis diverges.

First of all, a derivative security (such as an option) has an asymmetrical return pattern, so that although the risks of the *underlying* security may be summarised by a mean variance model, this description is clearly insufficient to adequately capture the risks in the *option*. We will see that the value of the derivative security is determined chiefly by the price of the underlying and its variance. However, in order to understand this connection, we must first introduce two concepts: *hedging* and *arbitrage*.

2. HEDGING AND ARBITRAGE

The process of reducing or eliminating a particular risk in a portfolio through a trade, or a series of trades, (or contractual agreements) is called *hedging*. The corresponding trades or contractual agreements are referred to as *hedges*. In a typical investment portfolio, there is little or no hedging; rather, the investor simply tries to achieve the greatest upside potential, given an acceptable level of risk. This is the framework of modern portfolio theory.

The pricing of derivatives goes to the other extreme: here, the requirement is to construct a portfolio (the hedging portfolio) that eliminates all of the risks introduced by the derivative security being analysed. In particular, the hedging portfolio is required to replicate a return pattern identical to that of the derivative security, so that, from the point of view of an investor, the two alternatives—replicating portfolio and derivative security—are indistinguishable.

This latter point introduces the notion of *arbitrage*. If the replicating portfolio and the derivative security produce the same return pattern, then they should have the same value. If they currently have different values in the marketplace, there is then an opportunity for *arbitrage*, that is, one can sell the higher-valued representation and purchase the lower-valued one, securing a risk free profit.

In the real world, things seldom work out as neatly, and there may be other reasons for an apparent arbitrage opportunity. Most pricing models ignore these secondary issues and focus on the primary risks. It is important, therefore, to understand the underlying assumptions and the implications for the pricing model. In the next section, we will illustrate these points by carefully inspecting two examples.

3. REPLICATING PORTFOLIOS

This section will attempt to provide an intuitive understanding of the "building blocks" of derivatives from a trading and risk management perspective, as opposed to an abstract mathematical one. The basic point of view is that in order to determine the price of a derivative security, one needs to understand how to *hedge* the security, and that the theoretical value is then determined by calculating the cost of the hedge. In this context, *hedging* will refer to a trade, or a series of trades in an appropriate *underlying security* in such a way as to offset the corresponding risk in the derivative security. In practice, there may be alternative choices of hedging security, but these can be compared to this basic case.

By examining the hedging process in some detail, we can arrive at a better appreciation of the risks that are not being adequately hedged; that is, we can better understand "where the model breaks down". The trader's job is then to determine a price for these unhedged risks given the trader's current portfolio. The risk manager's job is to understand and quantify the total unhedged risk in the firm's position and ensure that it is maintained within acceptable limits, given the firm's risk/reward profile.

We will illustrate these points by carefully examining two basic examples.

The first, that of a forward contract on a stock, requires only a *static* hedge. The second, an option on a stock, requires a *dynamic* hedge. In both cases we will see that the price of the security is determined by two considerations:

1. **cost of the hedge (or value of the replicating portfolio)**; and
2. **compensation for unhedged risk.**

3.1 **Example 1:** A one year forward contract on a stock

A customer wishes to purchase 100,000 shares of a particular stock from you, one year from today. He wants to determine the price today at which he will purchase the stock in one year.

We will determine the price, the *forward price* of the stock, by examining the cost of the hedge. We seek a trade or series of trades that will exactly offset the risk inherent in the forward contract. We know that in one year we need to have in hand 100,000 shares of stock. One way to achieve this is to purchase the shares of stock today and set them aside for delivery in one year. What will this cost? Let us assume that the current (spot) price of the stock is $100, so that we can purchase the stock at $100 per share today. What must we charge in one year in order to break even? The $100 per share that I have spent on my hedge could have been invested elsewhere, such as in money market securities which earn interest. This is a relatively "risk free" transaction. We wish to place a hedge so that the forward contract is equally risk free. Put another way, as a trader, I will be charged interest on the $100 per share that I have tied up in my hedge. If the rate at which I borrow money for one year is 10% (simple annual), then I will incur $10 per share of interest charges during the life of the forward contract, so I need to receive $110 per share at maturity in order to break even. So, at first pass, the forward price of the stock should be $110.

However, if the stock pays a dividend of, say, $3 over the next year, then I will receive $3 per share in my hedging portfolio. This reduces the cost of my hedge and the customer will expect to be rebated by adjusting the forward price to $107. That is:

Total cost of hedge = $107 per share.

At this point we should stop and investigate if there are any risks that we have ignored. One risk is that the $3 of dividends is not guaranteed. We can only *forecast* $3, recognising that the company of the stock could adjust this amount either up or down. For this reason we might modify our price to provide a cushion against this event. An alternative is to simply agree to *pass through* the dividends at the maturity of the contract. In this way neither ourselves nor the customer bear the dividend risk, which is basically unhedgeable. In this case the forward price would remain at $110 per share, and the customer receives any interim dividends.

We have also ignored the effect of changes in interest rates. If we finance our stock hedge overnight, then we are at risk to changes in interest rate levels because our forward price was determined assuming a fixed rate of 10%. This may be overcome by taking term financing for one year at 10%, instead of overnight.

Another risk which has been ignored by our neat analysis is *counterparty risk*; that is, the risk that the counterparty may default on moneys owed to us. Suppose that over the next three months the stock plummets to $25 per share. Let us assume we have a pass-through forward so that the price of the original forward contract has been set at $110. At this point our counterparty will owe us about $85 more per share above the current market price, in nine months (the remaining term of the forward contract). On 100,000 shares this comes out to be $8,500,000. This is essentially the (unrealised) amount of money that we have lost on our hedge, so that if the counterparty defaults we are really out this amount. For this reason there may be collateralisation requirements built in to the contract. We note that futures contracts are exchange-traded forward contracts with a daily collateralisation requirement (that is, variation margin).

Finally, another risk that has been ignored is that of *liquidity*, or the practical limitation of implementing a hedge in the marketplace. In order to hedge our position in this example, we must purchase 100,000 shares of stock. Unless the forward price is calculated off of our average cost of acquiring the stock hedge, we will also be at risk in that the purchase of this much stock may impact on the price.

The main point we are making here is that the theoretical determination of the forward price is as we initially calculated. However, by thinking through what is required to actually hedge the position, we arrive at a better understanding of the risks involved.

Exercise:
Suppose the customer wishes to *sell* a forward contract on the stock. Suppose that in addition the stock can be borrowed or loaned for an annual fee of 2.0%. What should the forward price be?

We next turn to an example of a European style (no early exercise) equity option. Our goal is to understand the hedging process for such a security and its impact upon the theoretical valuation of the option. We will proceed as intuitively as possible.

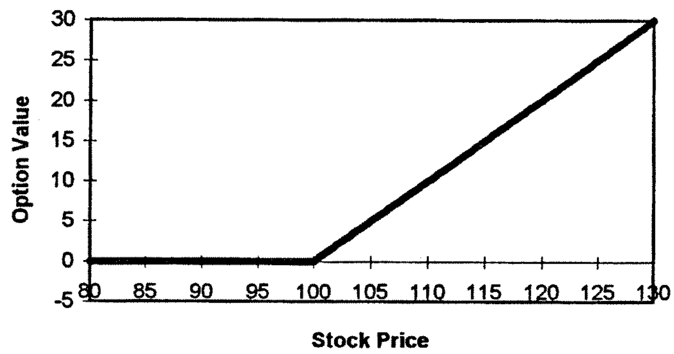
3.2 **Example 2:** European style call option on a stock

A customer wishes to purchase a three month call option on 100 shares of stock. The current stock price is $100, as is the exercise (strike) price of the option.

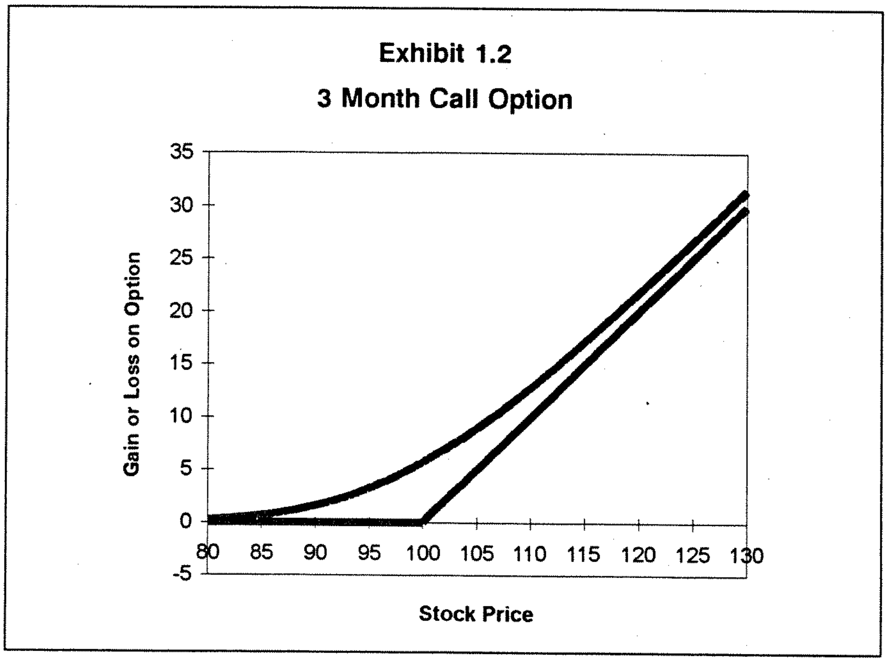
We will begin the analysis with the *expiration diagram* of this security (Exhibit 1.1). That is, a graph of the value of this security at expiration. Clearly, to the extent that the stock price exceeds the exercise price, the excess will be the value of the option, and if the stock price falls below the exercise price, the option will expire worthless. Therefore, the expiration diagram is simply a "hockey stick" as presented below.

Exhibit 1.1

Expiration Value



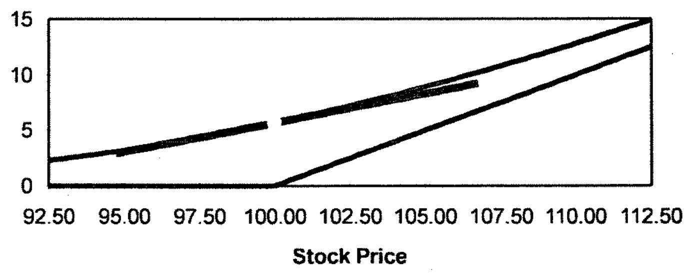
We currently have three months to go on this security. What should the graph of the call option value look like when the three months are taken into account? A number of things are easy to see. First of all, for very low stock prices, say around 80, the option has very little value as the possibility of the stock rising 20 points is rather remote (again, we are speaking *intuitively* here). For stock prices around 100, the option has value in that if the stock price drops, there is nothing to lose, but if it rises, there is much to gain, in fact, point for point with the stock price. Finally, for high stock prices, say 120, the option is of course, worth at least 20 points but not a whole lot more because the option has as much to lose as it has to gain depending upon whether the stock falls or rises; put another way, the one-sidedness of the expiration diagram is not as evident with the stock at 120. These intuitive samples indicate that the graph should look something like the following:



Note that the graph begins to parallel the hockey stick at both extremes. We see that, on the downside, the graph should approach the expiration diagram. However, on the upside, it will approach a value that is a fixed amount above the expiration diagram. We now will set about to hedge our position. Obviously, as the stock price increases, so does the value of the call option. As we are short the option we will suffer a loss unless we have positioned a hedge that compensates for this loss. One way is to purchase some shares of stock. Then, as the stock price increases, our hedge will make money to offset the losses on the option position. The flip side is of course that if the stock price *drops* our hedge will *lose* money, but our short option position will *gain*. The first question is: How much stock should we purchase? If we make a brief return to first semester calculus we will recall the notion of *tangent line*. The amount of stock we should purchase will correspond to the *slope* of the tangent line. For high stock prices this slope approaches 100%, and for low stock prices it approaches 0%.

Exhibit 1.3

Delta Hedging



Let us suppose that the current slope, corresponding to the current stock price of 100, is 50%. We should then purchase 50 shares of stock, since the option is actually on 100 shares, not just 1. Suppose next that the stock rallies to 102, where the slope is now 60%. We are underhedged! In order to get hedged again we should purchase another 10 shares, paying 102 per share. Now suppose the stock falls back to 100. Now we are *overhedged* and should sell off 10 shares at 100. Of course we are now back to where we began, except that we have lost 2 points on 10 shares for a $20 loss. If the stock continues to see-saw in this fashion we will continue to "buy high and sell low" in the process of actively hedging our position. This is an odd way to make money, and in fact we appear to be losing. However, the other side of the coin is that, as time passes, the value of the option should "decay". That is, the graph eventually should approach the hockey stick. To the extent that the option decays, we will make back some of the money we are losing by hedging. It is a race between the two effects: trading losses versus time decay. In other words, if the stock is *volatile*, we will be whipsawed more and lose the race, and therefore lose money. If the stock is very quiet, we will win the race and make money. The famous Black-Scholes option pricing formula basically calculates what these hedging costs will be over the life of the option and then asserts that the (present value) of these costs should equal its price. That is, just as in the calculation for the forward contract, *the value should equal the cost of the hedge*. This time it is more complicated, of course, because the hedging process (termed "delta-hedging") is *dynamic*. We summarise this discussion below.

The theoretical price of an option is equal to the cost of hedging the option.

With this point in mind, we can now examine what *information* is required to price an option. Clearly, the terms of the option (exercise price, expiration date, et cetera) are required. We also need to know various market parameters such as the stock price, dividend information and interest rates, as these are all relevant to calculating the cost of the hedge, just as for the stock forward. However, because of the dynamic nature of the hedging process there is one more parameter which we need: the *volatility* of the stock (or its square, the *variance*). This is a statistical measurement, calculated as an (annualised) standard deviation of the stock price returns that quantifies the extent of the "whipsawing" we can expect when dynamically hedging the option position. The more volatile the stock, the more expensive our hedge, so that the price of the option should increase with the volatility of the stock. Typical values range from 10% to 50%. The lower end corresponds to a stock index, while a blue chip stock is typically in the 20%-35% range.

We summarise the required information in a table:

Inputs for Option Valuation
Contractual Terms
Exercise Price
Expiration Date
Style (American or European)
Market Information
Stock Price
Interest Rates
Dividend Information
Statistical Information
Volatility

At this point it is important to note that all of the inputs are known *except for the volatility* (and to some extent the dividends). That is, the cost of hedging will be influenced by the volatility of the stock *during the life of the option*. This is not really known, but must be predicted. We can certainly measure the past volatility of the stock, but there is no guarantee of the future volatility. This has been compared to "skiing down a slope backwards", watching the trees go by. Because of the inherent intractability of the volatility parameter, there is always a degree of uncertainty in what the hedging costs will be. In short, if we are hedging only by dynamically adjusting our stock hedge, this parameter risk is unhedged. In a typical situation a trader will have a portfolio of options, both long and short of varying maturities, so that the risk of the unknown volatility is netted across the entire portfolio. Then the risk manager needs to be concerned with issues such as three month volatility versus six month volatility; that is, the "term structure" of volatility. We will not explore this aspect here; it is a natural extension of the ideas presented so far.

Exercise:

For very high stock prices, we have stated that the option value approaches a fixed amount above the expiration diagram. Calculate this value by calculating the cost of the hedge (note that the slope is 100%).

4. GLOSSARY OF TERMS

Before we can continue with our discussion, it is important that we define some frequently used technical terms.

A Glossary of Greeks

Delta	Greek symbol δ. This is simply the slope of the tangent line. This risk is *hedgeable* with the underlying stock.
Gamma	Greek symbol γ. This measures how rapidly the slope changes. For a trader, it is used to anticipate how much rehedging will be required for a given move in the stock price. This risk is *unhedgeable*, except with other option-like securities.
Theta	Greek symbol θ. This measures the time decay. In a sense (referring to the "race" mentioned earlier), theta is the flipside of gamma. This risk is also *unhedgeable*, except with other option-like securities.
Sigma	Greek symbol σ. Measures the *volatility* of the stock (expressed as an annualised standard deviation of returns). Also *unhedgeable*, except with other option-like securities.
Kappa	Greek symbol κ (also called "vega"). This measures the sensitivity to the volatility assumption, σ. Again, *unhedgeable* except with other option-like securities.
Rho	Greek symbol ρ (also called "dv01"). Measures the sensitivity to interest rates for (usually) a one "basis point" change (that is, .01%). This is *hedgeable* by trading, for example, an appropriate bond.

5. RISK NEUTRALITY

Up to now, we have examined derivative pricing from the perspective of hedging. As it turns out, there is an alternative method of calculation that is actually a byproduct of the hedging approach, termed the "Risk Neutrality Hypothesis". It states that the price of the derivative security can be calculated by making simplifying assumptions about the underlying process, and then computing its "expected value" under the simplified process, discounting as if it were a known cashflow. We will first illustrate this principle in the case of an ordinary stock option, and then indicate why it is true with a brief foray into the binomial world.

This time we begin again with the hockey stick expiration diagram. We next calculate the "expected value" of this by simply taking each point on the diagram and multiplying by a probability, and then adding them all up. This sounds tedious; fortunately, computers are very good at this. The resulting number is then present valued to today. The probability distribution is obtained by taking the original process for the stock price (that is, *lognormal*—we have carefully avoided actually writing it down) and replacing the *expected return* of the stock—a very subjective number—with the "risk free" rate (actually, our financing rate). We have retained the stock volatility, σ, which is in general a less subjective number than the expected return.

In order to understand why the risk neutral calculation is equivalent to calculating the hedging costs, we will consider a simple one step binomial "tree". That is, suppose that over the next time period the stock, currently at a price of S, can either go up to a price S^+, or go down to a price S^-. We will assume that $S^+ = uS$ and $S^- = dS$ with $0<d<1<u$ (that is, u stands for "up" and d stands for "down"). *We do not assume that we know the probability of either event.* This is tantamount to not knowing the expected return on the stock. Next, we attempt to hedge an option on this stock over the next time period, assuming that at the price S^+ it will equal C^+ and at S^- it will equal C^- (for example, if the option expires in the next time period). We wish to construct a hedge of stock and cash (accruing at a rate of r%) which will "hedge" this option. To that end, we assume that we hold a position of m shares of stock and n units of cash (m will be a fractional share in this calculation). We certainly are not concerned about roundlots right now. Thus, the value of our hedging portfolio is currently

$$mS + nB$$

We will assume that over the next time period that B will grow to FB; F is a "future value" factor; and F—1 is the interest earned per unit of cash during this time period. Our task is to determine m and n so that we are hedged in the two events $S = S^+$ and $S = S^-$. That is, we must determine m and n so that:

$$mS^+ + nFB = C^+$$

and

$$mS^- + nFB = C^-$$

These are two equations in two unknowns which are easily solved. For example, we find that m is just

$$m = (C^+ - C^-)/(S^+ - S^-)$$

which corresponds exactly to the tangent line slope originally discussed. In a similar fashion we can solve for n, and substitute back in to find that, after rearranging terms

$$mS + nB = (pC^+ + qC^-)/F$$

where

$$p = (F - d)/(u - d)$$

and

$$q = (u - F)/(u - d).$$

We note that

$$p + q = 1$$

and

$$0 < p < 1$$
$$0 < q < 1$$

as long as $F < u$. This last assumption simply asserts that the stock must have a chance of outperforming the cash instrument in any time period. (Otherwise, why would anyone ever buy the stock!) These equations imply that p and q can be interpreted as *probabilities* (termed *arbitrage probabilities*) and the value of the hedging portfolio today, which equals the option price today, is simply equal to the expected value of its price over the next time period, using the arbitrage probabilities), discounted by the factor F. This is nothing more than a binomial description of the risk neutrality principle.

The power of the risk neutrality methodology is that once it has been demonstrated that a particular derivative security can be hedged—usually by considering a similar one step binomial tree (or in the case of more complicated multifactor securities, a multinomial tree)—an elaborate mathematical machinery can then be called into play to actually perform the calculation. The main *drawback* of this methodology is that one can be easily seduced into the risk neutral world where none of us actually live, and forget some of the model assumptions that brought us there. This is why, from the point of view of risk management, it is important to understand how the models can break down, and how to best *stress test* them.

6. APPLICATIONS OF THE RISK NEUTRALITY PRINCIPLE

The risk neutrality calculation methodology can be applied to a wide variety of derivative securities. The key point is to verify the hedgeability of the relevant risks; the problem is then reduced to an "expected value" calculation, which is usually quite tractable or at least amenable to a wide variety of mathematical techniques.

6.1 **Example**: A three month "look back" option

This is a security that will payout in three months the difference between the *maximum* stock price reached over the next three months, S^*, and today's stock price, K. That is, the payment will be:

$$S^* - K.$$

K is currently known, while S^* is not. Is this hedgeable? We resort to binomial logic. With one period to go to expiration, we will know the current maximum S^*, and therefore, the maximum at expiration in either case $S = S^+$

(where the maximum will be the greater of S^+ and S^*), or $S = S^-$ (where the maximum will remain S^*). This means that we can then construct our hedge just as before.

This reasoning can be extended to earlier periods as well, establishing that the security is hedgeable. However, in order to *calculate* the theoretical price, we can now invoke the Risk Neutrality Hypothesis and employ alternative means. Now any mathematician can apply advanced techniques (partial differential equations, Green's functions, stopping times, Monte Carlo as a last resort, et cetera) to solve the problem. Some of these techniques have been around for over 100 years and have become practical with the advent of computer technology. For American style (early exercise) securities, this calculation is performed over a small time interval, after which the security can be tested for early exercise, just as in a binomial option pricing model.

Once the model is in hand, it should be *stress tested* in order to reveal hidden risks that are *unhedgeable* (except with other options or option-like securities).

7. ARBITRAGE—A CLOSER LOOK

As we have noted, the basic premise of all pricing models is that the value of the derivative security should equal that of a replicating portfolio; conversely, if the values differ, than arbitrageurs can play one against the other and obtain a riskless profit. In an efficient market this arbitrage activity should force convergence of market prices to theoretical prices.

All of this is approximately true, and more true for some derivative securities than for others, but there are often legitimate reasons for a "mispricing", usually due to features of the marketplace that have not been adequately incorporated into the pricing model. Obvious features are transaction costs such as brokerage fees and stamp duties. Some others are listed below:

7.1 Counterparty risk (for OTC transactions)

This has been illustrated in the example of the forward contract.

7.2 Cash flow risk

The pricing models usually assume that many can be borrowed whenever it is needed. This may not be really true. If we return to the example of the forward contract, recall the scenario where the stock has fallen to \$25 a share. If we have purchased the stock on margin, this would trigger a massive margin call; if we have insufficient funds, our hedge will be liquidated.

7.3 Parameter risk

As discussed in the example of the European style call option, the cost of the hedge will depend largely upon the experienced volatility during the hedging. This parameter can only be estimated.

7.4 Horizon risk

This is a more subtle risk, and has to do with the *rate of convergence* between the market price and the theoretical price; that is, the rate at which the trade increases in profitability. In the case of a three month call option, it will take at most three months. But for, say, a convertible bond, it could take as long as 15 years. This lengthy horizon can easily disincentivize arbritrageurs from stepping in (they tend to have notoriously short time horizons).

8. CONCLUSION

This chapter has discussed the pricing of equity derivatives, but the same techniques apply to other financial derivatives as well. The general procedure is to:

(i) identify the primary risks and a mathematical model for their evolution through time;

(ii) identify underlying "hedging instruments" for the risks;

(iii) verify the hedgeability (this may be done with an appropriate binomial tree, or *multinomial* tree in the case of multiple risks);

(iv) calculate the value by invoking the Risk Neutrality Hypothesis; and

(v) consult a mathematician to actually perform the calculation. Also ask for a list of the model parameters. These will give rise to corresponding sensitivities which should be stress tested in order to develop a better understanding of the unhedged risks.

Part 2

Interest Rates and Yield Curves

Chapter 2

Interest Rates, Bond Pricing, Duration and Convexity

by Roger Cohen

1. WHAT IS AN INTEREST RATE?

The term interest rate is commonly used to describe the growth or earning potential associated with an amount of money. Common occurrences of interest rates include the advertised return on money deposited in a bank account, home loan mortgage rates, financing costs and so forth. The types of rates, and the context in which they are used is often confusing. In this chapter, various representations of interest rates will be introduced and explained. The context in which interest rates are used in the financial markets will be explained.

An interest rate refers to the rate of growth or decay of an asset over time. It is a measure of the value of the asset at the present relative to its value in the future. Although the asset is usually cash, it need not be. Interest rates allow us to quantify questions such as "*is it worth more now or in the future?*" or "*what will this be worth in ten years time?*". They are a measure of the earning (or expense) associated with deferred consumption. For example, an individual may have a sum of money that is not needed at the present time. Another individual may need money (perhaps to buy food, or to build a house). The two individuals can enter into an agreement where the latter gets the use of the money at the present time. The full amount plus an additional sum will be repaid at a future date. This additional sum is the interest paid by the borrower to the lender. By deferring consumption of the money, the lender is rewarded with extra cash or interest. This is the governing principle of most financial transactions. The financial markets just formalise the mechanics of such transactions. They provide an efficient framework for transferring capital. The underlying principal is that a lender receives interest for the use of capital by a borrower. This applies where the lender is an individual, an organisation or even an entire country. The converse applies to the borrower.

Although varying in complexity, interest rate transactions involve a borrower and a lender. A bank deposit, for example, is effectively a loan by an individual to a bank. The underlying rationale is that the bank is able to use this money to earn income greater than the expense associated with the interest paid to the depositor. An example of this is that home loan mortgage rates (where the bank is the lender and an individual is the borrower) are invariably higher than deposit rates (where the bank is now the borrower). The appropriate rates of interest are dependent on the parties involved in a transaction, the risk of the transaction (that is, whether the borrower will be able to repay the debt), and the amount, structure and timing of the transaction. This will not be discussed further here.

Interest rates are also used to quantify the growth or decay of commodities other than money. This includes gold and other precious metals, and certain financial instruments such as stocks. Interest rates need not be positive. If there are storage costs, or time wastage associated with holding a commodity, then its interest rate may be negative. This means that a lender of such a quantity will pay a fee to the borrower.

1.1 Representations of interest rates

There are many representations of interest rates. Usually (but not always) rates are a positive amount expressed as a percentage. They are measures of the growth or decay of an asset. For example, an advertised rate of 10% paid annually, means that $100 will be worth $110 after one year. Generally a rate is expressed as a percentage, and a basis. The percentage gives the amount of growth that is expected, the basis tells the period over which this growth is compounded. If the 10% of the above example were paid semi-annually (that is, twice a year), then after six months $100 would be worth $105.[1] If the interest is compound, then after a further six months, the $105 is increased by 5% giving $110.25 after one year. From this we can see that 10% semi-annually is then equivalent to 10.25% as an annual rate. Similarly, 10% quarterly would mean a growth of 2.5% is applied each quarter. Simple rates are applied over a period without compounding. An annual simple rate of 10% would increase $100 to $110, $120 and $130 over 1, 2 and 3 years respectively. If the 10% were compounded annually, then $100 would increase to $110, $121 and $133.10 respectively.

1.2 Interest rate arithmetic

There is nothing special about how an interest rate is expressed. Rates can be changed from one basis to any other. The premise for doing this is to realise that, at the end of a period, the final amount must be the same when any rate basis is used. To convert from one basis to another,

$$(1+\frac{r_1}{b_1})^{b1} = (1+\frac{r_2}{b_2})^{b2}$$

where r_1 and r_2 are interest rates with bases b_1 and b_2.

Example:

What is the quarterly equivalent of 7.5% semi-annual?

$$(1+\frac{r}{4})^4 = (1+\frac{0.075}{2})^2$$

This solves to give $r = 0.07431$ or 7.431% as a quarterly rate.

1. By convention, non-annual rates are divided by the number of periods in a year. Thus 10% semi-annually means interest is 5% every half year; 10% quarterly would be 2.5% per quarter and so forth.

An interest rate can be converted from any basis to any other.[2] It is convention to express the periodicity of rates in years. There are various conventions for determining the number of days in a year. These will be discussed below.

Commonly used interest rate bases include

1. *Simple*: This is where the rate is expressed exactly over the period required. For example, 8% over 45 days would mean that \$1 grows to \$1(1 + 0.08*45/365) = \$1.0099 after 45 days.
2. *Daily*: These rates are compounded daily. Cash or overnight rates set by most central banks are daily effective. A daily rate of 10% means that after n days, \$100 increases to $\$100(1+0.1/365)^n$.
3. *Monthly, quarterly and semi-annual*: Rates are compounded 12, four and two times per year.

1.3 Year basis

In financial calculations, the number of days per year used when calculating interest periods is not necessarily the actual number of days in the period. Convention uses either a 365 day or a 360 day year. Months may use the actual number of days or be considered as having a fixed number of days—usually 30. Arithmetic for these conventions can be found in DAS, *Swaps and Financial Derivatives* (2nd ed, 1994), pp 174-178.

1.4 Continuous rates

Another interest rate basis is that of continuous or continuously compounding rates. These never actually appear explicitly in financial instruments or transactions. They are a representation most often used internally in financial calculations. They provide a simple means for performing interest rate arithmetic. Within many financial models, rates are converted to continuous rates. They are converted back to their original basis after manipulation.

In the section above, we showed various compounding periods ranging from one year down to one day. If the compounding period is decreased further, then in the limit (where the period becomes infinitesimal), we have continuously compounding rates. If we had a rate r compounded continuously, then after one year, we would have the following growth:

$$\lim_{n \to \infty}(1+\frac{r}{n})^n = e^r$$

An amount $\$A$ would grow to $\$Ae^r$ after one year. For the general case, using rate r for time t (in years), the growth would be e^{rt}. The advantage of continuously compounding rates will become apparent when we introduce

2 It is interesting to note that when rates are advertised, they are commonly expressed in a basis that makes them look attractive to an investor. Deposit rates are often expressed as annual effective even when they are compounded (10.25% annual effective looks more attractive to the uninformed than 10.00% paid semi-annually). For borrowing rates, the converse is often the case. Mortgage rates are often compounded monthly or even daily (a rate of 10% daily is the same as 10.516% annual effective).

forward rates later in this chapter. It is also essential in many models used for option pricing.

Exhibit 2.1

Value of $1 in 1 Year, at 10% Interest Using Various Bases

Basis	Periods Per Year	Rate	Formula	Value
Annual Effective	1	10%	$(1+0.1)$	$1.10000
Semi Annual	2	10%	$(1+0.1/2)^2$	$1.10250
Quarterly	4	10%	$(1+0.1/4)^4$	$1.10381
Monthly	12	10%	$(1+0.1/12)^{12}$	$1.10471
Daily	365	10%	$(1+0.1/365)^{365}$	$1.10516
Daily (360)	360	10%	$(1+0.1/360)^{360}$	$1.10516
Continuous	∞	10%	$e^{0.1}$	$1.10517

10% continuously compounded is the same as 10.517% annual effective.

It is a common perception that continuously compounded rates are complex. This is not true. The reality is that working with continuously compounding rates is simpler than with discrete rates. This is due to the exponential growth and decay used when valuing with continuous rates. The perception of difficulty arises as rates need to be converted to continuous before they can be used, and then often back to simple rates after manipulation. To convert a rate to continuous we use the relationship

$$e^{rt} = (1 + R)^T$$

where r is the continuous rate applied over time t years, and R is the periodic rate applied over T periods. The relationship between t and T is $t = T.f$ where f is the number of periods per year over which R is effective.

Example: continuous and discrete rates

What is the continuous and annual equivalent of a semi-annual rate of 7.5%?

When converting a rate from one basis to any other, we will still get the same return. After one year, the semi annual rate R will gross one dollar up to $\$1(1 + R/2)^2$ dollars. A continuously compounded rate r will gross a dollar up to $\$1.e^r$ dollars. For equivalence, these must be equal. Similarly, the annual rate r' will also gross up to the same dollar value.

$$(1+\frac{R}{2})^2 = e^r = (1+r')$$

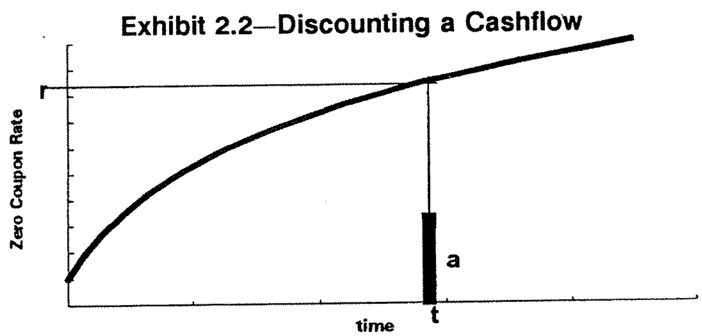
Solving when $R = 7.5\%$ gives an annual effective rate of 7.6406% and a continuous rate of 7.3628%.

$$(1+\frac{0.075}{2})^2 = e^{0.073628} = (1+0.076406)$$

The continuously compounded rate can be used in calculations. It is easier to work with exponentials and logarithms rather than the discrete representations.

1.5 Valuing cashflows

From the discussion of interest rates above, we are now in a position to value any cashflow to or from the present day. If we are valuing future cashflows to the present, this is often referred to as *discounting* or obtaining the *net present value* or *NPV*. The reverse—finding the value in the future of an amount of cash at present—is referred to as *grossing up*.



Any cashflow can be valued by discounting at an appropriate rate

$$Val = \frac{a}{(1+r)^t} = e^{-Rt} = a.DF(t)$$

The concepts of discounting and grossing up form the basis of most financial transactions. Investors are trying to maximise their return or the NPV of their assets, while borrowers are seeking the minimum cost for their borrowings. This is all done within a framework where the risks involved and the structure of the transactions are considered.

1.6 Forward rates

As well as being able to express interest rates in many different bases, it is also important to specify the exact period over which the interest rates apply. Rates in the above sections are considered to apply from the present to a date in the future, or vice versa. These are referred to as *spot rates*. The spot date is usually the present date, or the date out of which a transaction begins.

This need not always be the case. Where rates apply over a period that does not involve the present (or spot date), we refer to these rates as *forward rates*. An example—the three month rate in three months time is referred to as the three month forward. There is a deterministic relationship between spot and forward rates. This relationship stems from the principle that a cashflow should have the same NPV no matter how it is discounted. If this does not hold, then there is a basis for increasing the NPV by revaluing the cashflow differently. Such an occurrence is called an *arbitrage* (see the section below). In the financial markets, professional arbitrageurs constantly exploit such occurrences. Because of this, they do not often exist, and if they do then it is only for very short periods.

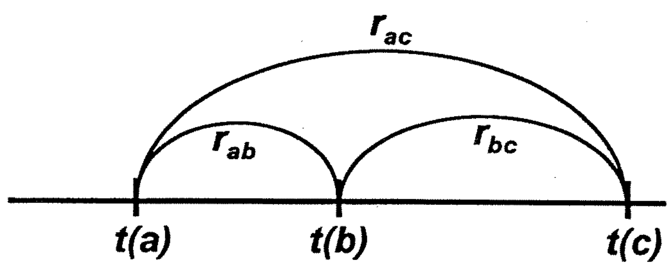
If it is assumed (usually this is the case) that there is no arbitrage, then over any period, the value of a cashflow will be invariant whether it is discounted by a single rate or a series of forward rates. The general principle

$$\frac{1}{(1+r_{ac})^{t(ac)}} = \frac{1}{(1+r_{ab})^{t(ab)}} \times \frac{1}{(1+r_{bc})^{t(bc)}}$$

where r^{ij} is the rate applicable over period $t(ij)$.

Exhibit 2.3

Calculating Forward Rates



In the market, there may be slight differences in value depending on the path for discounting or grossing up. Usually this will be less than the margin lost if the difference were to be exploited.

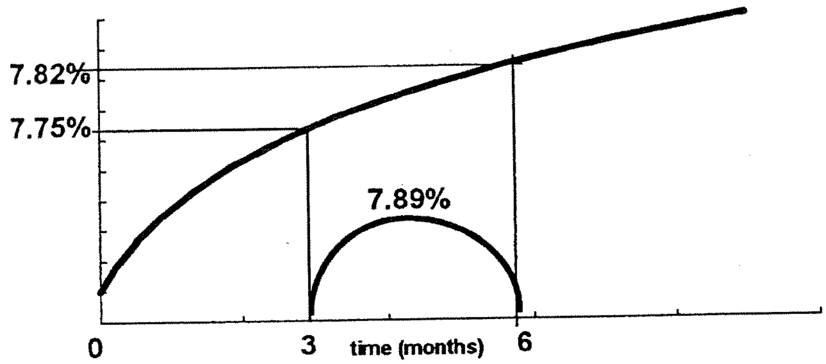
Example: forward rates

What is the forward rate from three months to six months given the following spot rates?

	Spot Rate
1 month	7.49
3 Month	7.75
6 Month	7.82

Exhibit 2.4

Exhibit Spot and Forward Rates



To solve for the forward rate from three to six months, we note that under the no arbitrage principal, one dollar in six months must have the same NPV if it is discounted by the six month spot rate, or by the forward rate from three to six months then the three month spot rate. If these are annual effective rates, then

$$\frac{1}{(1+r_{06})^{\frac{6}{12}}} = \frac{1}{(1+r_{03})^{\frac{3}{12}}} \times \frac{1}{(1+\mathbf{r}_{36})^{\frac{3}{12}}}$$

The subscripts on the rates above refer to the start and end of the period over which the rates apply. Spot rates all have a subscript starting with zero (the spot date). Solving this gives $\mathbf{r}_{36} = 7.8900\%$

Continuously compounded make calculations of forward rates extremely simple. In the example above, we can do the calculation using continuous rates.

	Spot Rate	Continuous
1 month	7.49	7.2228
3 Month	7.75	7.4644
6 Month	7.82	7.5293

The forward rate from three to six months in the continuous representation is given by

$$e^{-r_{06}\frac{6}{12}} = e^{-r_{03}\frac{3}{12}} \times e^{-\mathbf{r}_{36}\frac{3}{12}}$$

This simplifies to

$$e^{-\mathbf{r}_{36}\frac{3}{12}} = e^{-(r_{06}\frac{6}{12} - r_{03}\frac{3}{12})}$$

or

$$\mathbf{r}_{36} = \frac{12}{3}(r_{06}\frac{6}{12} - r_{03}\frac{3}{12})$$

The forward rate is a simple arithmetic expression, which gives the continuously compounded forward rate from three to six months as 7.5942%. Converted back to an annual effective rate we get 7.8900%, which is exactly the same as when calculated using the annual effective rates directly. Once the conversion to the continuously compounding domain is made, calculations are generally simpler.

Futures contracts are common manifestations of forward rates. In most markets participants have access to a strip of bill futures. These are usually three month instruments which start at various dates in the future. Forward rate agreements or FRAs are instruments which provide a guaranteed forward rate.

Digression: an arbitrage

The calculations from above are based on the principle of *no arbitrage*. To illustrate what happens when there is an arbitrage opportunity, we use the spot rates of the example above, but instead of solving for the forward rate of 7.89%, assume that there exists some instrument which will pay a rate of, say, 8.5% for the forward period from three months to six months.

	Rate
1 month	7.49
3 month	7.75
6 month	7.82
3-6m fwd	8.5

Using these rates, consider the value of a cashflow (say $1,000,000) in six months time. If it were discounted by the six month rate of 7.82%, its NPV would be $963,053. If the NPV were calculated by discounting from six to three months at the forward rate of 8.5%, then from three months to spot at the three month spot rate 7.75%, the NPV is now $961,697 (or $1,356 less than using the six month spot rate). If these rates were all available to a market participant, and the risk associating with borrowing or lending at them were equivalent, then an investor can borrow money for six months—the first three at the three month spot rate, the latter at the forward. This money would be lent at the six month spot rate. At the end of six months the investor would be $1,356 better off, with no net outlay. This represents an arbitrage opportunity. In reality if such an opportunity were to occur, it would quickly be exploited. This would cause the rates to be adjusted until the arbitrage disappears.

Theoretically, there should be no difference to the NPV of a future cashflow, no matter what path the discounting follows. Over one year 365 one day rolls should be the same as one year roll and so forth. This forms the basis of arbitrage free pricing theory. It is a foundation for the pricing of many complex instruments, including options.

1.7 Discount factors

Whenever we use an interest rate, we need to specify the period over which the rate applies, and the basis in which the rate is expressed. This applies to both spot and forward rates. Often this leads to undue complexity. Another way to express rates is as *discount factors*. A discount factor or DF is just the value of one dollar when discounted over the required period. As an example, if the NPV of a dollar at some future date is 94 cents, then the discount factor at this date is 0.94. Discount factors have the advantage that they do not depend on any specific basis. Using discount factors means the complexity of keeping track of whether a rate is annual, semi-annual or continuous is no longer necessary. The only aspect that needs to be tracked is the period over which the discount factor applies. Discount factors can be related to both spot and forward rates.

For compounding rates the discount factor is

$$df = \frac{1}{(1+r)^t}$$

where r is the applicable rate over time t (r must be expressed in the same basis as t—for example, if r is semi annual, then t must be the number of semi annual periods over which r applies). Where the rate r is simple (annual), the discount factor is

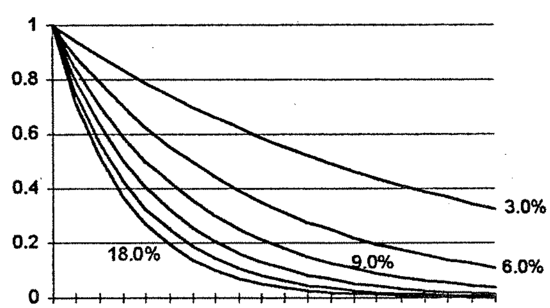
$$df = \frac{1}{(1+r\frac{d}{365})}$$

where d is the number of days over which discounting takes place. For continuous rates, the discount factor is $df = e^{-rt}$.

A curve can be drawn which gives the discount factor corresponding to any rate. These discount curves show the NPV of $1 at any time when discounted by the applicable rate.

Exhibit 2.5

Constant Rate Discount Factors for Rates of 3% to 18%



Discount factors are also applicable to forward rates. If r_{ab} is a forward rate from time t_a to t_b, then a forward discount factor df_{ab} can be generated. It provides the discounted value at t_a of one dollar at t_b. The relationship between spot and forward discount factors is straightforward.

$$df_{0b} = df_{0a} \times df_{ab}$$

where the subscripts are as described above.

1.8 Putting it all together—the term structure of interest rates

The interest rates used in the above sections are all considered in isolation. In the marketplace, we are constantly bombarded with different rates or types of rates. These are represented in a multitude of different bases. It is only meaningful to compare these rates if they are converted into a uniform basis, and have a consistent underlying type. When this is done, we obtain an *interest rate term structure* or *yield curve*. The interest rate term structure is just a time consistent view of interest rates. Usually it is a curve showing the rates which are effective over different time periods. For example, if we have a set of annual effective rates specified for periods of one, two, three years and so forth, then a curve through these is a representation of an interest rate term structure. Please refer to Chapter 3 for examples.

There is no single universal term structure. Rates differ depending on the currency, types of instrument and the way the rates are represented. Within a single currency, there are different term structures for classes of interest rates. The interest rate term structure may also be derived from market rates rather than using them directly. These aspects will be discussed further in the chapter on the yield curve.

1.9 Summary—interest rates

In this chapter so far we have covered various forms of interest rates. We have shown the different bases that rates can be expressed in, and how to convert from one base to another. The process of discounting and grossing up have been illustrated. These allow cashflows at any time to be valued at any other time. The concept of net present value shows the worth of future cashflows at the spot date. Continuously compounding rates have been introduced as a tool to simplify calculation. Discount factors are also used to represent the value of cashflows at various times. The difference between spot and forward rates is shown. Using the principle of no arbitrage, we can derive forward rates from spot rates and vice versa.

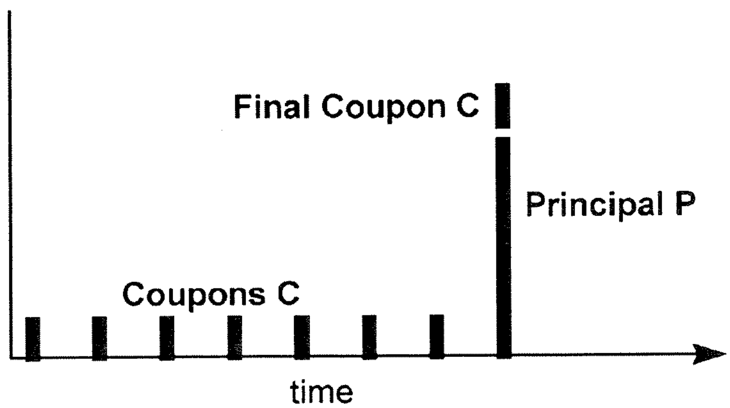
2. BOND PRICING, DURATION AND CONVEXITY

2.1 Bonds

A bond is a medium to long-term financial instrument. It is usually issued by a party in order to raise funds. Over the life of the bond, the issuer makes periodic interest payments. These are referred to as coupons. At maturity, the issuer is under an obligation to repay the original principal of the bond. The life of a bond is usually not less than one year (bills or short-term notes are used for shorter periods). It is not uncommon for bonds to have maturities in excess of 20-30 years. Coupon payments are generally made quarterly, semi-annually or annually.

Exhibit 2.6

The Cashflow Structure of a Bond



The value of a bond is determined by the size and timing of the cashflows. It is also highly dependent on the quality of the bond. For example, government or sovereign bonds are usually regarded as high quality instruments. They will be more expensive than lower quality bonds such as those issued by a corporation or an individual.

Some of the factors that affect the value of a bond include:

1. *Coupon size and timing*: the larger and more frequent the coupons, the greater the value of the bond.
2. *Maturity*: this is the period over which coupon payments are made.
3. *Issuer quality*: the risk associated with default is relatively lower if the issuer is of high quality. Such bonds will attract a premium.
4. *Current interest rates and outlook*: these affect the value of the bond on the secondary market.
5. *Liquidity*: there can be a premium for bonds that are easily tradeable.

Bonds are referred to as *fixed interest or fixed income instruments*. This is because, once the bond has been issued, all future payments are known. The only disruption to these will be if there is some sort of crisis event where the issuer cannot make an interest or principal payment, or a payment is delayed. The probability of such a default event is priced into most bonds. As this probability of default changes, the premium or discount associated with it will vary.

Many bonds, once issued, trade in the secondary market. Their market value will depend on prevailing economic conditions as well as the current state of the issuer. To enable bonds to trade, the market has developed conventions for their valuation. These all derive from the principle of discounting cashflows (which is discussed earlier in this chapter). This makes sense, as a bond is just a collection of cashflows. The only additional parameter is the risk to the bond holder associated with actually receiving these cashflows.

2.2 Pricing bonds

In the market place, bonds are usually quoted on the basis of either a yield, or a capital price. Either method can be derived from the other. The choice of method of quotation is just a convention of each particular market.

2.3 Bond yields

The yield of a bond—commonly referred to as its yield to maturity—is basically a measure of the return the bond holder can expect for the outlay involved in purchasing the bond. Yield is quoted as a rate in the same basis as the coupon payments that make up the bond. Most government and investment bonds pay semi-annual coupons—thus the quoted yield is semi-annual. There are a number of representations of the formula for pricing bonds. These generally differ in terminology only. They represent the net present value of the cashflows that make up the bond when they are all discounted by the bond yield.

Bonds are commonly priced as a value per $100 of principal. If the current price is above $100, the bond is said to be valued at a premium. If it is less, then the bond is at a discount. Whether a bond is at a premium or a discount depends on the relativity of the current yield to the coupon size of the bond. Where the yield is greater than the coupon, the bond will be at a discount. Where it is less, the bond is said to trade at a premium.

The price per $100 principal is

$$P = v^{\frac{f}{d}}(c(x + a_n) + 100v^n)$$

where

c = the periodic coupon payment in dollars per $100 principal *c = coupon/(coupons per annum)*

n = the number of complete periods from the next coupon to maturity

f = the number of days to the next coupon date

d = the number of days from the last coupon date to the next

v = *1/(1+i)* where i is the periodic effective interest rate (*i = yield/frequency* ie: *i = yield/2* for semi annual coupon bonds)

x = 0 if ex-interest, $x = 1$ if cum-interest[3]

a_n = $(1-v^n)/i$

To value a bond, the above formula is used. Given a yield and the structural details, the value of the bond can be determined.

Example: pricing a bond

What is the price per $100 of a bond with the following characteristics:

Maturity: 15 January, 2001

Settlement: 14 March, 1997

Coupon: 8.5% paid semi-annually

Yield to maturity: 7.14%?

The following quantities can be derived from the above data:

Next Coupon	15/7/97
Previous Coupon	15/1/97
i	3.570%
v	0.965531
n	7
a_n	6.098603
f	123
d	181
c	4.25

Substituting these into the bond formula, and using $x = 1$ (the bond is *cum*-interest), the price of the bond is $105.844 per $100 principal. This means that an investor would pay $105.844 to receive semi annual coupons of $4.25 every six months from 15 July, 1997. A final payment of $104.25 (the $100 principal plus a final coupon) is received by the investor on 15 January, 2001. At this stage, the bond ceases to exist.

The effect on the price of the bond at different yield levels can be seen in the table below.

Yield	Price
7.00%	$106.321
7.14%	**$105.844**
7.50%	$104.628
8.00%	$102.969
8.50%	$101.343
9.00%	$ 99.748
9.50%	$ 98.185

As the yield increases, the cashflows are discounted at a higher rate. This means that the bond will have a lower net present value.

3. Ex-interest means that the next interest payment is not included in the price of the bond. By convention, there is an ex-interest period before any coupon payment. The duration of this is set by market convention. If the bond trades during this period, the original holder, rather than the purchaser, still gets the coupon.

2.4 Characteristics of bonds

To understand the characteristics of a bond, it is useful to understand how the pricing formula works, and the sensitivity of the bond to various parameters. This allows the risks of the bond to be quantified and managed. It also allows the effect of market conditions to be observed and reacted to.

2.5 Principal and interest

As time progresses, a bond will accrue interest. At coupon dates, this interest is paid out, and the process repeats. The value of a bond can be split into principal and interest components. Interest accrues by the same amount each day. The daily accrual is equal to *c/d* using the terminology from the example above. Thus, on any day (before the bond goes ex-interest), the accrued interest is:

$$Accrued\,Interest = c[\frac{d-f}{d}]$$

The capital price is just the gross price less the accrued interest

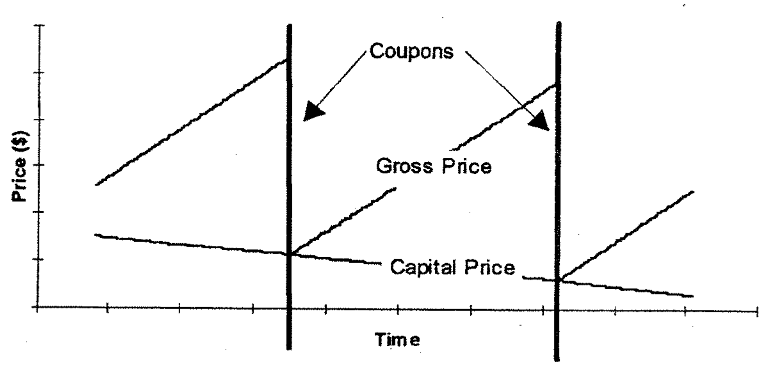
$$Gross\ Price = Capital\ Price + Accrued\ Interest$$

For the bond in the example above, with $f = 123$ and $d = 181$, the accrued interest is $1.362. The capital price is $104.482.

Because of the process of accruing interest over a period, then paying it at coupon dates, the gross price of a bond will sawtooth with time if the yield is constant. The capital price will move much more smoothly.

Exhibit 2.7

Gross and Capital Price of a Bond



In many markets, bonds are quoted using capital price rather than yield. From the example above, this bond would be quoted as having a (capital) price of $104.482. The capital price is usually rounded to an even multiple (often eighths, sixteenths or thirty-seconds) of a dollar. The convenience of doing this is that the settlement proceeds can easily and unambiguously be agreed upon, without the use of a pricing formula.[4] In cases where bonds are quoted this way, participants in the market still need to obtain yields. This allows them to compare different bonds, and to gauge their returns. In order to calculate the yield given the price of a bond, the formula for the bond price needs to be solved iteratively. There is no closed form solution for the yield of a bond given its price.

2.6 The bond pricing formula

It is useful to examine the bond pricing formula in more detail. This gives an understanding of the fundamental principle of how bonds are valued. It provides a basis for valuing bond-like instruments, where there may be special conditions on the cashflows (for example, uneven coupon periods or partial principal repayment).

The bond formula discounts all cashflows at the yield to maturity. This is done in two stages. First, cashflows are discounted to the next coupon payment date. These are then discounted from the payment date to the settlement date. The term

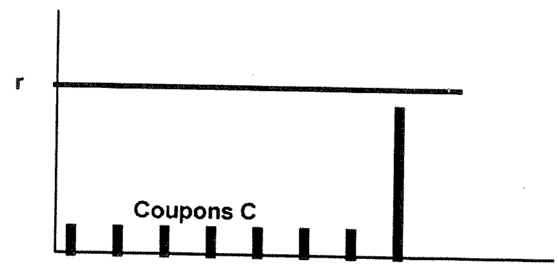
$$v^{\frac{f}{d}} = \left(\frac{1}{1+i}\right)^{\frac{f}{d}}$$

is a discount factor from the next coupon date to the settlement date. The rest of the equation discounts all cashflows to this date.

4. The reason for using capital price to quote bond prices is mostly historical. Before computers or financial calculators were available, market participants rarely agreed on settlement proceeds when only a yield was agreed. To remove this problem, price was quoted directly. Accrued interest is easily and unambiguously determined. It is still the case that for complex instruments such as prepayable securities—where the notion of yield is ambiguous or dependent on other factors or assumptions—that capital price is a more convenient way to quote instruments.

Exhibit 2.8

Using the Bond Formula to Discount at the Bond Yield

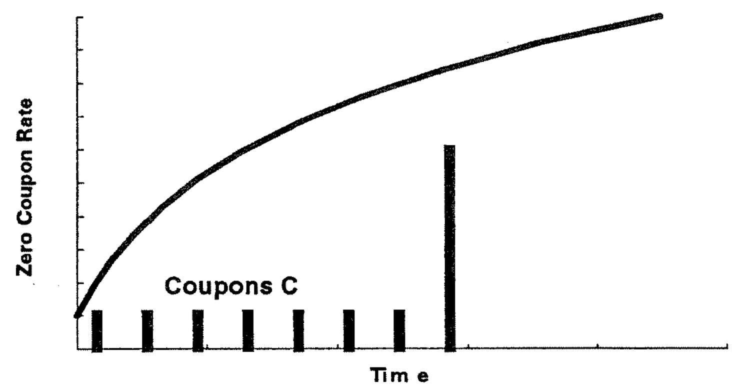


$$\text{Price} = \left[C + \frac{C}{(1+r/2)^{1}} + \frac{C}{(1+r/2)^{2}} + \ldots + \frac{(100+C)}{(1+r/2)^{n}} \right] V^{f/d}$$

Discounting each bond cashflow by the yield to maturity of the bond gives its market value in dollars. It assumes that each cashflow is reinvested at the yield to maturity. In reality, the cashflows will all be reinvested at rates available in the market. If a bond were issued with a ten year maturity, then the first cashflow would be paid nine and a half years to maturity, the second at nine years and so on. The first cashflow could thus be reinvested at a 9.5 year rate, the second at a 9 year rate. The penultimate coupon would be reinvested for just six months. The rates for all these periods are definitely not the same. They may not even equal the bond yield. To reflect a more accurate reinvestment assumption, the zero coupon curve is used. A zero coupon curve constructed out of bonds of similar characteristics would be appropriate for this purpose. The short end of the curve would reflect cash and bill rates, further out, the bonds would be used.

Exhibit 2.9

Pricing a Bond on a Zero Coupon Curve



$$\text{Price} = \frac{C}{(1+r1)^{t1}} + \frac{C}{(1+r2)^{t2}} + \ldots + \frac{(100+C)}{(1+rm)^{tm}}$$

If the zero coupon curve is used, then each cashflow is discounted at the appropriate zero coupon rate. If the curve is constructed for bonds of the type being valued, then this price should be exactly the same as the market price of the bond calculated using the yield to maturity and the bond formula. If there is a difference between the price using the formula, and that using the zero coupon curve, it either represents a difference in the type of instrument being valued compared to those on which the curve is based,[5] or a mispricing in the market.[6]

The zero coupon framework for valuing bonds is not used in the marketplace directly by traders. This is because it requires significant information in its derivation. It is well suited to risk management, relative value, arbitrage analysis and other purposes. It is also very useful for the valuation of non-standard bonds. At the expense of losing the convenience of using a single yield to maturity (or capital price), a more realistic reinvestment assumption can be applied. This is much closer to what would be attained in reality if all cashflows were to be reinvested.

When the zero curve is upward sloping, the yield to maturity will be lower than the zero coupon yield at maturity. This is because, when discounting on the zero curve, earlier cashflows are discounted at low yields. The latter ones will require a higher yield if the market price of the bond is to be retrieved. The converse is true for an inverse yield curve.

5. For example, a corporate bond valued on a government bond curve will have its price overstated. The government curve does not reflect the higher credit risk—and hence discount—associated with corporate bonds compared to government issues.
6. This may be where a bond is relatively cheap compared to similar bonds. If this occurs, an *arbitrage* opportunity may be exploited.

2.7 Risk parameters for bonds

The holder of a bond or bond portfolio is exposed to changes in the valu of the instruments that make up the portfolio. These changes may be due t structural issues such as liquidity or the creditworthiness of the issuer. Othe risks are market-related, such as the effect of changing yields, or variations i the interest rate term structure. These risks are quantified and managed b calculating various sensitivity parameters for bonds.

2.8 Risk management

An investor holding a bond or a portfolio of bonds is exposed to change in value of the components of the portfolio. Unless the investor plans to hol all instruments to maturity, and is not reinvesting the interest cashflows fron coupon payments, then exposure to prevailing rates and other marke conditions must be considered. There are also structural exposures such a the creditworthiness of the issuer which will not be considered further here The focus is on interest rate risk-related issues.

The fundamental measures which are used by most portfolio managers t quantify the interest rate risk or exposure in their portfolio is the sensitivit to rate or yield changes, and the timing or length of the cashflows in thei portfolio. The former is usually quantified by calculating the sensitivity to a yield shift of one basis point. This is referred to in the market as the PVBF or *present value per basis point*. The PVBP is calculated for bonds b shifting the yield to maturity and measuring the dollar change in the bonc price. It is usually quoted as an amount per million dollars face value of the bond.[7] A measure commonly used to express the length of a bond o portfolio is the *modified duration*.

2.9 Duration and modified duration

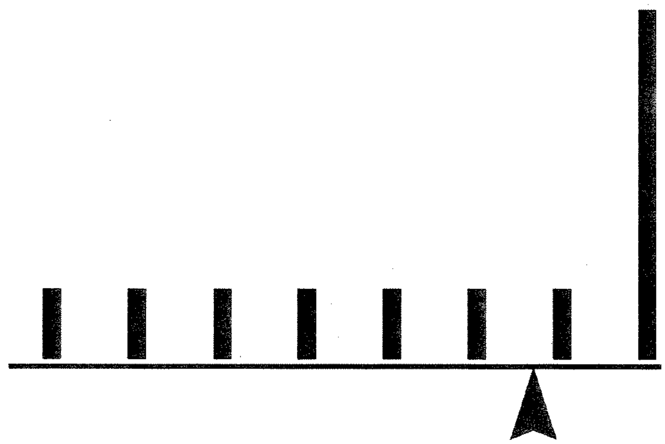
Measures of duration are useful in that they give a simple means fo judging the length of time exposure of a bond or portfolio. These measures aggregate all the bond or portfolio cashflows. Because of this aggregation, they give no information about the structure and timing of cashflows. They are a first order measure, whose use is widespread in the financial industry.

Duration is a concept introduced by Frederick R Macaulay and is one that bears his name.[8] *Macaulay duration* is essentially the time weighted average of the cashflows of a bond. Graphically, it is illustrative to consider the duration as the fulcrum point on a timeline where the cashflows balance. At the duration point, exactly half the dollar value of a bond will have been paid when referenced to the present.

7. In some markets, the PVBP is referred to as the DV01 or *dollar value of one basis point*.
8. For examples, see Das, *Swaps and Financial Derivatives* (2nd ed, 1994), pp 1055-1060, and references contained therein.

Exhibit 2.10

Duration Point

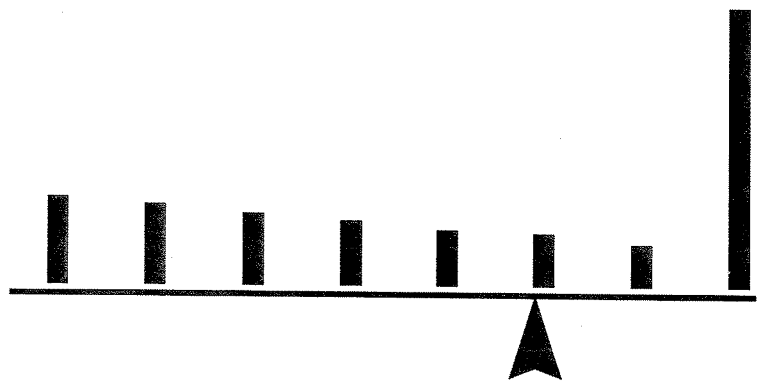


Duration is useful in that it illustrates where the cashflows occur. A zero coupon bond would have a duration exactly equal to its time to maturity. Using the fulcrum analogy, the pivot point would be where the cashflow occurs. If coupons are included, then, as the coupon size increases, the fulcrum would need to be brought closer to the settlement date. An annuity (a stream or equal cashflows) would have the fulcrum halfway to maturity if the first cashflow were on the settlement date. There is no effect on duration calculations due to the yield to maturity or reinvestment of coupons.

A more useful duration-type measure is the *modified duration*. Extending the balance analogy above, this is the pivot point where the net present value of the cashflows balance. The modified duration is the point where an investor would receive half the present market value of a bond.

Exhibit 2.11

Modified Duration



Under the discounting process, cashflows at later dates will be worth less proportionally than near flows. This serves to bring the balance point closer to the settlement date. The modified duration is less than the Macaulay duration for any coupon bond (they are equal for a zero coupon bond).

Numerically, both modified and Macaulay duration are simple calculations.

$$DMac = \frac{\sum_i c_i t_i}{\sum_i c_i}$$

where

DMac = Macaulay duration

c_i = cashflow that occurs at time t_i.

$$DMod = \frac{\sum_i t_i \frac{c_i}{(1+r)^{ti}}}{\sum_i \frac{c_i}{(1+r)^{ti}}} = \frac{\sum_i t_i \frac{c_i}{(1+r)^{ti}}}{price}$$

DMod is the modified duration. All symbols are as above, and r is an appropriate rate by which the cashflows are discounted. The *price* can be either the capital price or the gross price of the bond. The modified duration will vary depending on which is chosen.[9] If the modified duration is calculated using zero coupon rates then r is replaced by the zero coupon rate

9. *Capital price modified duration* is often chosen as its behaviour with time is more regular than when gross price is used. This is because of the discontinuity in the gross price of a bond when a coupon is paid. When this occurs, the gross price modified duration will increase discontinuously.

appropriate to each cashflow. Modified duration based on yield to maturity will differ slightly from that based on zero coupon rates.

Example: duration and modified duration

What is the duration and modified duration of the following bond

Maturity: 25 January, 2001

Settlement: 25 June, 1996

Coupon: 8.25% paid annually

Yield to maturity: 7.90%?

Date	Time (y)	Cashflow (c_i)	$c_i t_i$	Discounted Value $(DV_i)_i$	$DV_i t_i$
13/06/96	0	-	-	-	-
25/01/97	0.619	8.250	5.108	7.871	4.873
25/01/98	1.619	8.250	13.358	7.294	11.811
25/01/99	2.619	8.250	21.608	6.760	17.706
25/01/00	3.619	108.250	391.776	82.209	297.528
	Total	**133.000**	**431.851**	**104.134**	**331.918**

This gives a value for the Macaulay duration of 431.851/133.0 = 3.247 years, and a modified duration of 331.918/104.134 = 3.187 years. Note that the bond price from above is $104.134.[10] The maturity of the bond is 3.619 years. As the coupon size decreases, the duration of the bond will approach maturity.

Exhibit 2.12

Effect of Coupon Size on Duration and Modified Duration

Coupon	DMac	Dmod	Maturity
0%	3.619	3.619	3.619
2%	3.508	3.488	3.619
4%	3.412	3.376	3.619
6%	3.329	3.280	3.619
8%	3.256	3.197	3.619
8.25%	**3.247**	**3.187**	**3.619**
10%	3.191	3.124	3.619

10. Using the bond formula, the price is $104,147. This difference is exactly one day's worth of accrued interest. This is because of the leap year 1996, which means an extra day's accrual. This is picked up in the formula, but not by direct discounting of cashflows.

2.10 Duration of portfolios

As for a single bond, duration and modified duration can be calculated for portfolios. The method of calculation is exactly as for the single bond case. A portfolio duration is just the weighted average of the component durations.

$$Duration\ (total) = \frac{\sum_n Duration_n price_n}{\sum price_n}$$

This applies for both Macaulay and modified duration (the two must not be combined).

It is useful to know the (modified) duration for a portfolio, as this gives an approximation for its aggregate life. Duration is important for portfolio managers who are benchmarked against bond indices.

2.11 Interest rate sensitivity—PVBP

When yields (or zero coupon rates) change, bonds change in value. The PVBP quantifies sensitivity to this change. Usually this is done by shifting the yield and valuing the bond. The difference between the shifted and original price suitably normalised gives the PVBP. When hedging bonds with futures or switching from one bond to another, traders often make sure that they stay PVBP matched. This means that their sensitivity to yield changes (assuming that the yields all change by the same amount) is not altered, provided the changes are small.

The theoretical value for PVBP is

$$PVBP = \frac{\partial P}{\partial i}$$

This can be related to the modified duration:

$$DMod = \frac{PVBP}{Capital\ Price} = \frac{1}{Capital\ Price}\frac{\partial P}{\partial i}$$

Modified duration is a useful risk management measure. It gives an aggregate length for a bond or portfolio. The interest rate sensitivity will be similar to that for a zero coupon bond maturing at the modified duration. The sensitivity will not be exactly the same as the cashflow timing can be very different. This is especially prevalent if a portfolio contains bonds across the complete maturity spectrum. The similarity is only first order. Portfolio managers are often set modified duration limits within which their portfolios must be maintained. This sets the basic risk profile of the portfolio, whilst still leaving the manager the flexibility to choose specific instruments or maturity profile.

2.12 PVBP for zero coupon bonds

The value of a zero coupon bond is

$$\frac{\$100}{(1+r)^t}$$

So, per $100:

$$PVBP \approx 100\,[\frac{1}{(1+r)^t} - \frac{1}{(1+r+\Delta r)^t}]$$

where

Δr is a perturbation to the zero coupon rate.

Using a Taylor series expansion of this, it can be shown that $PVBP \propto t$. As the length of the zero coupon bond increases, its PVBP will increase proportionally. This is a useful rule of thumb. Most coupon bonds pay coupons substantially less than the final payment at maturity, so the PVBP for such bonds also increases with time to maturity.

2.13 Convexity

Modified duration and PVBP are useful measures of the sensitivity of bonds to changes in the underlying yield. They are calculated for small yield changes. What happens when the change in yield is large? It is normally the case that most of the time markets only move a few basis points from day to day. Occasionally, there are shocks where the market can move tens or even hundreds of basis points. What happens to the value of bonds under such moves?

The relationship between bond price and yield is not linear. A yield move up will not produce the same magnitude of price change as the same move downwards. The price change for a ten basis point move will not be exactly ten times more than for a one basis point move. The PVBP is not constant over all yields. Convexity is a measure of how the PVBP changes with yield. It is a measure of the non-linearity of the price behaviour of bonds:

$$Convexity = \frac{\partial^2 P}{\partial i^2} = \frac{\partial PVBP}{\partial i}$$

Convexity becomes useful when bonds with similar PVBP are being compared. A larger convexity implies that the PVBP will change more with yield than does a lower convexity. This is a useful property, and can provide benefit to a portfolio manager. This will be illustrated in the section on portfolio management.

Exhibit 2.13

Convexity: Price is not Linear with Yield

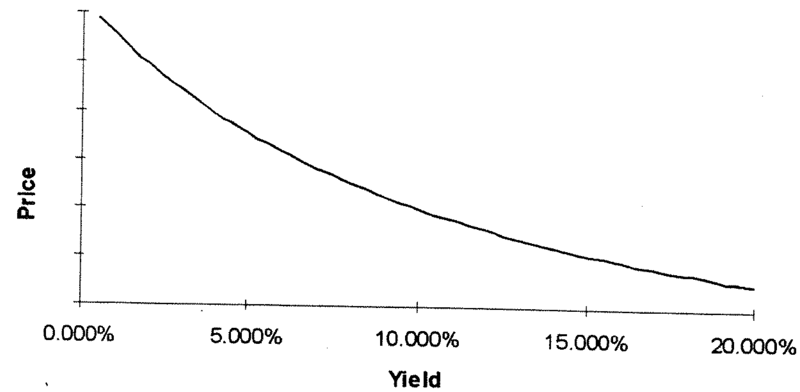


Exhibit 2.14

Convexity for a 20% Coupon Bond

Yield	Yield Change (bp)	Gross Price ($)	Price Change ($)	PVBP per $m
10.00%	—	181.004	—	—
10.01%	1	180.888	0.116108	116.108
10.10%	10	179.848	1.155831	115.583
11.00%	100	169.950	11.053433	110.534

The more convex a bond or portfolio is, the more the PVBP will change as yield changes become large. This has major implications when hedging bond portfolios.

2.14 Evaluating Duration, PVBP and Convexity

Although there exist derivative formulae for modified duration, PVBP and convexity, in practice it is more convenient to evaluate them numerically. This also has the advantage of quantifying them over a yield change that is realistic in the market.[11] The following are suggested for these calculations

$$PVBP = \frac{1}{2\Delta i}[P(i - \Delta i) - P(i + \Delta i)]$$

11. The market is more likely to move, say, 1, 2 or 5 basis points than the infinitesimal move that the differential formulae apply to.

$$Convexity = \frac{1}{\Delta i^2}[P(i - \Delta i) - 2P(i) + P(i + \Delta i)]$$

$$DMod = \frac{PVBP}{Capital\,price}$$

The bond price needs to be calculated at only three yield levels to obtain all these quantities. This is simple to implement, and gives values that are relevant to market moves.

2.15 Time effects

As a bond moves towards maturity, it accrues interest. On coupon payment dates, this interest is shed. The capital price will also change. When the bond matures, its capital price will be exactly \$100. If a bond is trading at a premium, its capital price will decrease; while the converse is true for discount bonds. This drift towards a capital price of \$100 is called the *pull to par*.

Exhibit 2.15

Pull to Par for Various Coupon Bonds (Yield 8% Semi-Annual)

Maturity (y)	Coupon		
	6%	8%	10%
10	86.410	100.000	113.590
8	88.348	100.000	111.652
6	90.615	100.000	109.385
4	93.267	100.000	106.738
2	96.370	100.000	103.630
0	100.000	100.000	100.000

Over time, components that change the price of a bond are the pull to par, interest accrual and yield changes. It is useful to split any change into these components. Pull to par depends on the difference between the coupon and the yield to maturity. Where this difference is large, the pull will be greater. The pull to par also increases as the bond gets closer to maturity.

2.16 Non-standard bonds

2.16.1 Constant accrual bonds

It is often the case that a bond is not exactly the same as the standard bonds described above. The standard bond pays a fixed coupon every period. Interest accrues each day, and is paid at the coupon date. The coupon size is always the same. If there are differences in the number of days per period (there invariably are as a standard year has an odd number of days), the interest accrued per day will differ slightly.

With constant accrual bonds, the daily accrual is specified. This means that the actual interest payment will differ depending on the number of days in the period.

Example: constant accrual bond

A bond pays a coupon—nominally 10% per annum. The coupon is paid semi-annually, with the same accrual per day.

With this bond, the accrual is based on a standard year of 365 days. The daily accrual (per \$100 face value) is \$10/365 = \$0.027397 per day. If the coupon periods are 182 days and 183 days long in a normal year, then the interest payments are \$4.986 and \$5.014 respectively. In a leap year, both periods have 183 days. The payments are 5.014% for both periods. In leap years, the bond pays \$10.028 in interest.[12]

Pricing of these bonds is commonly done with the bond formula modified slightly for the first coupon.

$$P = v^{\frac{f}{d}}((c_n dx + ca_n) + 100v^n)$$

Here c_n is the daily interest accrual. The next coupon will be $c_n d$. Pricing assumes that subsequent coupons are all c. Other terms are as per the standard formula. Technically, the value of each coupon should be accrued separately. This is not done, as it will make only a small difference to the bond price, with a large amount of extra complexity.

2.16.2 Bonds with long or short first coupon

The coupon of a standard bond is paid periodically. It is often the case that, when a bond is first issued, the time to the first coupon payment is not an exact period. In this case, the first coupon will accrue interest over either more or less days than if the bond were continuing from a normal payment. In such cases, the initial coupon is usually based on a daily accrual (as described in the section above). The term d (the number of days in the period) is adjusted for the non-standard period. Once this coupon is paid, the bond is priced as a standard bond.

$$P = v^{\frac{f}{d1}}((c_n d1.x + ca_n) + 100v^n)$$

for the first period of $d1$ days, after which the standard formula is used.

Example: Short first period bond

A bond paying an annual coupon of 8.25% each year on 15 July is initially issued on 1 March. How much will the first coupon payment be?

The standard accrual for this bond is \$8.25/365 = \$0.022603 per day. The first period is 136 days long, so the first coupon will be \$3.074 instead of \$8.250. After this is paid, all subsequent coupons will be \$8.25.

12. For a conventional bond, the interest payment would be \$5.00 irrespective of the period length. In 183 day periods, the daily accrual is \$5.00/183 = \$0.02732, and for 182 day periods it is \$0.02747.

2.16.3 *Short/long last period*

As with a non-standard first period, the last coupon may be delayed or paid early. The final principal repayment may also be shifted. In this case, the formula is adjusted for this. If the last coupon is shifted by l days, and the final payment is shifted by m days, the standard bond formula becomes:

$$P = v^{\frac{f}{d}}\left(c(x + ca_{n-1}) + cv^{(n+l/365)} + 100v^{(n+m/365)}\right)$$

2.16.4 *Bonds with amortising principal*

The bonds considered so far all pay periodic interest. The principal is paid at maturity. It is often the case that some or all of the principal will be repaid during the life of the bond. If the repayment schedule is known, the bond can be priced as a series of bonds of different periods. This is best illustrated by example.

Example: An amortising bond

Consider a bond with the following characteristics:

Maturity	5 years
Coupon	10%
Frequency	Semi-Annual
Amortisation	20% every year

The cashflows per $100 principal for this bond are

Period	Interest	Principal Paid	Remaining Principal	Total Cashflow
1	$ 5.00		$100.00	**$ 5.00**
2	$ 5.00	$ 20.00	$100.00	**$ 25.00**
3	$ 4.00		$80.00	**$ 4.00**
4	$ 4.00	$ 20.00	$80.00	**$ 24.00**
5	$ 3.00		$60.00	**$ 3.00**
6	$ 3.00	$ 20.00	$60.00	**$ 23.00**
7	$ 2.00		$40.00	**$ 2.00**
8	$ 2.00	$ 20.00	$40.00	**$ 22.00**
9	$ 1.00		$20.00	**$ 1.00**
10	$ 1.00	$ 20.00	$20.00	**$ 21.00**

To price this bond, we can split it into a series of five bonds each of face value $20. These bonds mature ar the end of years 1, 2, 3, 4 and 5 respectively. The cashflows for these bonds are as follows:

Period	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Total Cashflow
1	$ 1.00	$ 1.00	$ 1.00	$ 1.00	$ 1.00	**$ 5.00**
2	$ 21.00	$ 1.00	$ 1.00	$ 1.00	$ 1.00	**$ 25.00**
3		$ 1.00	$ 1.00	$ 1.00	$ 1.00	**$ 4.00**
4		$ 21.00	$ 1.00	$ 1.00	$ 1.00	**$ 24.00**
5			$ 1.00	$ 1.00	$ 1.00	**$ 3.00**
6			$ 21.00	$ 1.00	$ 1.00	**$ 23.00**
7				$ 1.00	$ 1.00	**$ 2.00**
8				$ 21.00	$ 1.00	**$ 22.00**
9					$ 1.00	**$ 1.00**
10					$ 21.00	**$ 21.00**

Each of these is a standard bond of $20.00 face value. The price of the complete amortising bond is the sum of the prices of each of the component bonds.

Any fixed amortisation of principal can be priced by decomposing the bond into a series of standard bonds of different maturities.

2.16.5 Unknown amortisation of principal

There is a major class of amortising securities where the exact schedule for repayment of the principal is not specified when the bond is issued. There will probably be minimum and maximum boundaries for repayment, but within these, repayment is not known. Such instruments are usually asset-backed securities. These are based on interest payments provided by asset holders. The most common asset-backed securities are mortgage-backed bonds. A mortgage-backed bond is funded by mortgage repayments from property owners. Because a property holder can repay principal at a rate faster than the minimum required under the loan agreement, the holder of a bond backed by these repayments has an uncertain schedule for principal repayment. There are many different structures for mortgage-backed securities. These will not be covered here.

In order to price securities where the principal repayment schedule is unknown, the repayment rate needs to be modelled. This can be a very complex process. In the case of mortgage-backed securities, some models will examine the underlying pool of assets, the demographics of the mortgagees and the prevailing economic environment.

Asset-backed securities that are prepayable contain an additional risk to the investor over the normal risks associated with bonds. This is prepayment risk. This is because the actual repayment rate will differ from that used in the model by which prepayable bonds are priced. If these securities are trading at a premium, then where prepayments are faster than anticipated, the bond holder will be penalised. Where the bond is at a discount, faster repayment will be of benefit to the holder.

Chapter 3

Interest Rate and Yield Curve Modelling

*by Satyajit Das (with a contribution from Roger Cohen)**

1. INTRODUCTION

Interest rates and the process of discounting future cash flows to price and value financial transactions is fundamental to capital markets. Accurate, consistent and reliable interest rates are therefore essential to all financial transactions. This is true irrespective of the type and complexity of the instrument.

In essence, interest rates are the pure price of time designed, through the discounting process, to equate cash flows occurring at different future dates to facilitate valuation and comparison of different transactions. In practice, interest rates are used for the following range of transactions:

1. Pricing and valuation of financial instruments or transactions—entailing the valuation of instruments, such as bonds or derivatives on fixed income instruments, by allowing analysis of the returns from different sets of cash flows through comparison of their discounted present value. In addition, the use of interest rates to value and price derivatives in other asset classes, such as currency, equity, and commodities.
2. Relative value and arbitrage—covering the use of interest rates and discounting to assess the relative value of traded or untraded instruments and to identify arbitrage opportunities.
3. Assessing or forecasting economic expectations—involving the analysis of various types of information available from the yield curve, such as forward rates, to assess market expectations of the path of future interest rates and the term structure of future interest rates which allow the formation and testing of expectations about future economic activity and inflation rates.

However, the process of deriving interest rates or discount factors is far from simple and unambiguous. A whole body of work has developed to assist in the determination of the interest rates to be utilised for the various identified purposes. In this chapter the basic issues relating to the derivation of interest rates and yield curves for use in the pricing and valuation of transactions is examined.

The structure of this chapter is as follows: the interest rates to be utilised are first considered, including identification of the concept of discount factors and the various types of interest rates (par, forward and zero) and their interrelationship, as well as the calculation of forward and zero coupon rates from the available yield curve. The second part of the chapter deals with the

*This chapter is written by Satyajit Das. Roger Cohen contributed Exhibits 3.14 and 3.21.

problems of deriving a suitable yield to enable the calculation of the various interest rates, including approaches to interpolation (linear and splines) and the issues in practical curve construction. The chapter concludes with a review of current best practice in interest rate and yield curve modelling.

2. INTEREST RATES

2.1 The concept of interest rates and discount factors

The concept of interest rates and discount factors or present value are interrelated. Central to the concept of interest rates and discount factors is the fact that value in financial transaction is given by cash flow, which is defined in terms of two vectors: amount and the time at which the cash flow occurs.[1]

The concept of discounting these cash flows which may occur at different points of time is designed to enable the value of these individual items to be calculated at a determined point of time (for example, today) to allow comparability. In essence, this requires the cash flow to be moved in time to determine an *equivalent* cash flow as at the relevant date. Using the fundamental homogeneity and uniformity of cash, the current value of cash can be given by its present value, which is intuitively an amount which, if invested at the relevant interest rate, will give a value equivalent to the stated cash flow as at the date on which the cash flow occurs in the futures.

This can be stated more precisely as:

$$C_{t0} = C_{t1} * DF_{t1}$$

Where

C_{t0} = Cash flow at time t0

C_{t1} = Cash flow at time t1

DF_{t1} = Discount factor (or present value) for cash flows as at time t1.

The discount factor (DF) generated or utilised to calculate a discount factor is essentially the present value of $1 at a specific future time. In theoretical terms, this is merely the price of the relevant zero coupon bond, discounted at the zero coupon rate (or, if appropriate, the par or coupon yield to maturity).

Exhibit 3.1 sets out mathematically the relationship of discount factors to the relevant interest rate for each future period. Discount factors, under conditions of positive interest rates, will be less than 1 and greater than 0 and are inversely related to yield with reference to maturity. *Exhibit 3.2* sets out the shape of the interest rate curve and the discount rate curve.

1. To be strictly accurate an additional vector which is required to define value is any contingency or conditionality relating to the cash flow, eg, in the case of an option.

Exhibit 3.1

Discount Factors and Interest Rates

1. Simple Interest

$DF = 1/(1 + R_{t1} * t/N)$

Where

DF = Discount factor for time t1 at rate R_{t1}

R_{t1} = Interest rate as at time t1

t = Number of days (or t1 − t0)

N = Number of days in a year (360 or 365)

2. Compound Interest

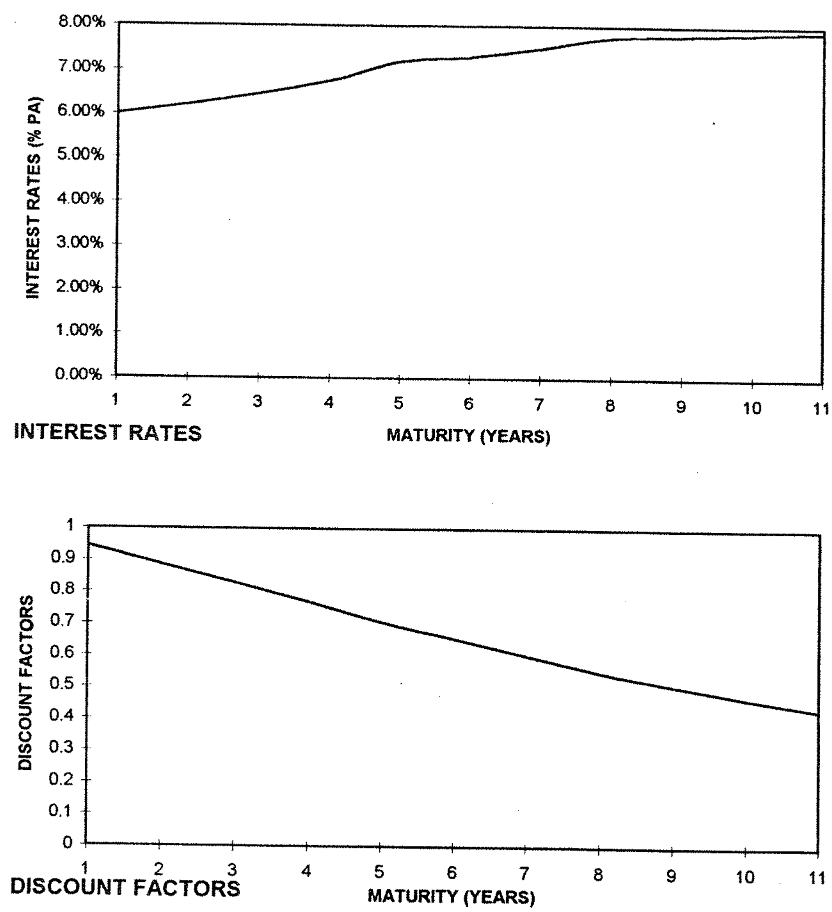
$DF = 1/(1 + R_{t1})^{(t/N)}$

3. Continuously Compounded Interest

$DF = e^{-Rt1 * t/N}$

Exhibit 3.2

Relationship Between Interest Rates and Discount Rates



The advantage of discount factors is that each discount factor is unique. It is not affected by market quotation conventions prevalent in relation to interest rates such as the periodicity of compounding (annual, semi-annual, quarterly or continuous) or the day count basis utilised (actual/365, actual/360 or bond basis).

In effect, the problem of yield curves can be expressed as the problem of either deriving term structure of interest rates or discount factors.

2.2 Types of interest rates

2.2.1 Overview

There are in practice three separate types of interest rates:

1. *Par rate*—which is defined as the interest rate on a coupon paying instrument out of today which is the standard interval rate of return formulae which discounts all payments on a coupon bond or instrument at the same interest rates.
2. *Forward rate*—which is defined as the interest rate on a coupon paying instrument out of a nominated date in the future.
3. *Zero rate* (also known as zero coupon rates, spot rates or pure interest rates)—which is defined as the interest rate on an instrument which pays no coupon and entails the exchange of a cash flow today for another (larger) cash flow at a nominated future date.

The par rate is the only observable interest rate in markets. Forward rates and zero rates are more difficult to observe directly. However, in jurisdictions where there are traded markets in interest rate futures and/or zero coupon securities,[2] it may be possible to directly observe certain forward and zero rates.

The three types of rates are clearly and unambiguously interrelated as they are, in effect, different perspectives on the same set of interest rates. In practice, the par interest rates which are observable are utilised to calculate the implied forward and zero rates.

The formal interrelationship between par, forward and zero rates can be stated as follows:

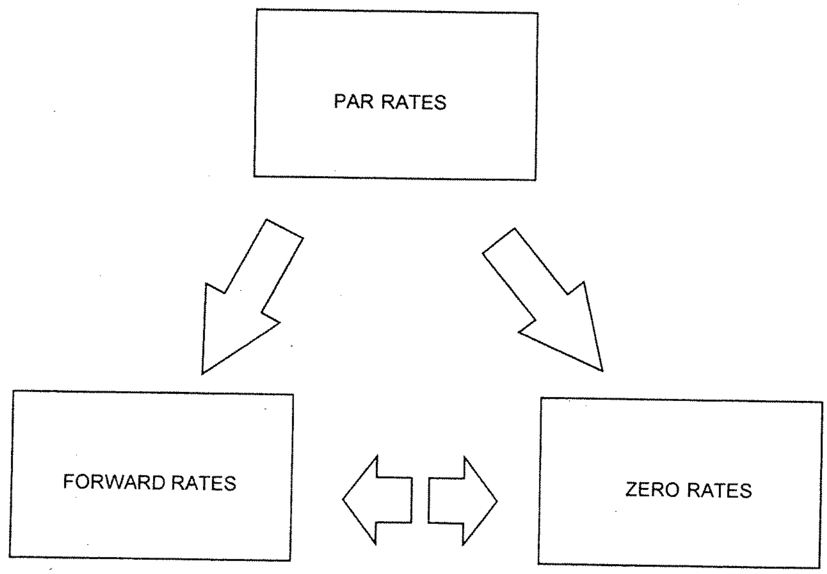
- forward interest rates are the interest rates at which par yields are reinvested;
- zero rates are the par interest rates with the reinvestment risk removed; and
- forward rates as between two points in time are implied by the zero rates at those two points.

Exhibit 3.3 sets out diagrammatically this relationship.

2. For example, the US Treasury STRIPS market which allows the unbundling of a bond into individual coupon and principal components and allows separate trading in these components.

Exhibit 3.3

Relationship Between Par, Forward and Zero Rates



The difference between the rates (in particular, par and zero rates) can be seen from a consideration of their role in valuation. Valuation of all financial transactions assumes the use of identified and specific interest rates or discount/present value factors to discount or present value cash flows identified with individual transactions. The two alternative types of interest rates available are:

1. par rates to maturity; and
2. zero rate to maturity.

As noted above, the par rate to maturity is usually directly observable, being the market quoted rate for the relevant securities of the required maturity. In the case of derivative or swap transactions, the relevant par rate to maturity is, typically, the quoted swap rate for the relevant maturity or, if unavailable, the interpolated yield based on available swap yield curve information. In contrast, the zero rate is not directly observable and is usually estimated from the existing par rate curve for the relevant instrument.

Traditionally, financial instruments have been valued utilising par yield to maturity. However, use of par rates creates a number of problems:

- coupon effect;
- assumptions on reinvestment rates; and
- absence of an unambiguous and unique interest rate for each maturity.

The coupon effect refers to the phenomenon observed in markets that the par interest rates of bonds or other financial instruments with the same maturity but different coupons may vary significantly. These differences may be caused by factors such as the interest rate risk or differential interest rate volatility of the securities, tax or clientele effects.

Utilisation of the par rate implies that the actual realised return only equals the normal redemption par yield to maturity if reinvestment rates on all intermediate cash flows, typically the coupons, are actually equal to the redemption yield. The realised yield, therefore, would only be equal to the par yield where the security is a zero coupon security, that is, a security which has no intermediate cash flows, as there is no potential reinvestment risk in the transaction. In practice, reinvestment rates on coupon cash flows will not equal the redemption yield. Theoretical forward rates are the only true measure of available reinvestment rates, and even then, the forward rates implicit in the yield curves at any point in time do not guarantee that these reinvestment rates are actually achieved.

In addition, the use of coupon of par rates creates an ambiguous relationship between yields and maturities. The use of par rate technology does not facilitate the identification of an *unique* interest rate and, by implication, discount factor for a particular maturity.

For example, assume the following yield curve exists:

Maturity (years)	**Par yield % pa**
0.25	7.25
0.50	7.55
1.00	7.92
1.50	8.23
2.00	9.05

Under these circumstances, a two year security, which pays intermediate coupons, say, every six months, will be valued by discounting all payments at 9.05% pa. However, for an identical security, with a maturity of 1.5 years, all cash flows, including intermediate coupons, would be discounted at a different rate, namely, 8.23%. Consequently, the rates applicable for years 0.5, 1.00 and 1.5 can be, either, 9.05 or 8.23% pa depending, solely, on the final maturity of the security. Because of these problems, par rate to maturity valuation does not imply an unambiguous relationship between the interest rate and the relevant maturity.

The identified problems of par rate to maturity technology are substantially overcome by utilising zero rates. As noted above, the zero rate can be defined as the interest or discount rate which applies between a cash flow now and a cash flow at a single date in the future, which is equivalent to the yield on a pure discount bond or zero coupon security (hence, the reference to zero rate). Utilising zero rates allows, for example, a two year yield to be directly related to a pure two year security, being a pure zero coupon security with a single cash flow in two years time.

The zero rate eliminates the coupon effect. The use of zero rates to discount or present value cash flows does not involve any assumptions as to the reinvestment rate applicable to any intermediate cash flows. In addition, the zero coupon rate has the advantage that each maturity is identified with a single unambiguous interest rate, being the rate of a pure single payment instrument. These factors allow zero rates to be utilised to value and ultimately manage entire portfolios of financial instruments (bonds and derivatives) as a series of cash flows, each of which is valued at a unique rate.

2.3 Derivation of forward rates

Forward interest rates can be calculated from the current yield curve. If suitably spaced yields and either synthetic or actual securities are available, then forward rates can be estimated.

The forward rates can be calculated based on the theoretical construct that securities of different maturities can be expected to be substitutes for one another. Investors at any time have three choices. They may invest in an obligation having a maturity corresponding exactly to their anticipated holding period. They may invest in short-term securities, reinvesting in further short-term securities at each maturity over the holding period. They may invest in a security having a maturity longer than the anticipated holding period. In the last case, they would sell the security at the end of the given period, realising either a capital gain or a loss.

According to a version of the pure expectations theory of interest rate term structure (see discussion below), investors' expected return for any holding period would be the same, regardless of the alternative or combination of alternatives they chose. This return would be a weighted average of the current short-term interest rate plus future short rates expected to prevail over the holding period; this average is the same for each alternative.

Forward rates may be calculated from the currently prevailing cash market yield curve, as any deviation from the implied forward rates would create arbitrage opportunities which market participants would exploit. This arbitrage is undertaken by buying and selling securities at different maturities to synthetically create the intended forward transaction. By simultaneously borrowing and lending the same amount in the cash market but for different maturities it is possible to lock in an interest rate for a period in the future. If the maturity of the cash lending exceeds the maturity of the cash borrowing the implied rate over the future period, the forward-forward rate, is a bid rate for a forward investment. Similarly, if the maturity of the cash borrowing exceeds the maturity of the cash lending, then the resulting forward-forward rate is an offer rate for a forward borrowing. This process of generating forward rates is set out in *Exhibit 3.4*.

Exhibit 3.4

Relationship of Forward Rates to Par Rates

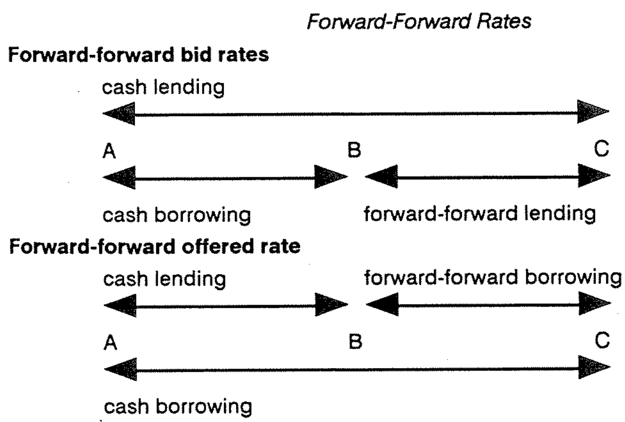


Exhibit 3.5 sets out the mathematical relationship between par interest rates and forward rates. *Exhibit 3.6* sets out examples of calculating forward rates.

Exhibit 3.5

Derivation of Forward Rates

$(1 + R_{t1})^{t1} * (1 + R_{t1 \times 2})^{t2 - t1} = (1 + R_{t2})^{t2}$

Where

R_{t1} = Par interest rate to time t1

R_{t2} = Par interest rate to time t2

R_{t1x2} = Forward interest rate between time t1 and t2

t1,t2 = Time to maturity in days from the present divided by 365

Rearranging to solve for the forward interest rate:

$R_{t1 \times 2} = [(1 + R_{t2})^{t2} / (1 + R_{t1})^{t1}]^{1/(t2 - t1)} - 1$

The above assumes that all rates are expressed in consistent time units, usually annual effective rates.

Exhibit 3.6

Derivation of Forward Rates—An Example

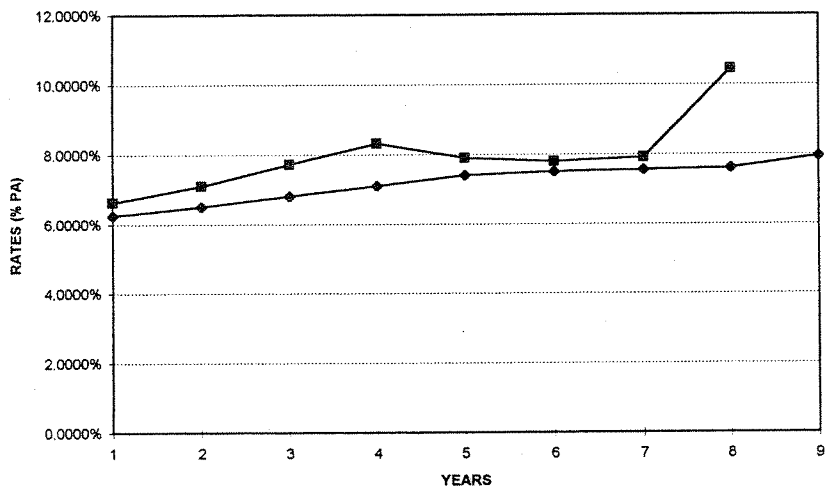
CALCULATION OF FORWARD/FORWARD INTEREST RATES 23-Jul-96

DAYS	(3) RATES (ANNUAL)	(4) (1+Rt2)^t2	(5) (1+Rt1)^t1	(6) (4)/(5)	(7) 1/(t2−t1)	FORWARD RATES FOR PERIOD t1 TO t2 (6)^(7)−1
0						
31	6.2500%	1.01600	1.00516	1.01078	5.98361	6.627%
92	6.5000%	1.03372	1.01600	1.01744	3.96739	7.101%
184	6.8000%	1.05284	1.03372	1.01850	4.05556	7.716%
274	7.1000%	1.07400	1.05284	1.02010	4.01099	8.308%
365	7.4000%	1.09478	1.07400	1.01934	3.96739	7.898%
457	7.5000%	1.11570	1.09478	1.01911	3.96739	7.799%
549	7.5500%	1.13682	1.11570	1.01894	4.05556	7.906%
639	7.6000%	1.16532	1.13682	1.02507	4.01099	10.440%
730	7.9500%					

DAYS	RATES (ANNUAL)	FORWARD RATE MATRIX							
0									
31	6.2500%	6.627%							
92	6.5000%	6.912%	7.101%						
184	6.8000%	7.209%	7.405%	7.716%					
274	7.1000%	7.507%	7.705%	8.013%	8.308%				
365	7.4000%	7.592%	7.754%	7.974%	8.102%	7.898%			
457	7.5000%	7.628%	7.763%	7.930%	8.000%	7.848%	7.799%		
549	7.5500%	7.669%	7.786%	7.925%	7.977%	7.867%	7.852%	7.906%	
639	7.6000%	8.026%	8.161%	8.340%	8.464%	8.503%	8.708%	9.172%	10.440%
730	7.9500%								

Exhibit 3.6—*continued*

INTEREST RATE FORWARDS



The following characteristics of forward rates should be noted:

1. Forward rates lie above (below) the par rates where the yield curve is positive (negative). This means that the forward rates cross from one side to the other of the par curve where the yield curve changes shape.
2. Forward rates have greater momentum than par rates; that is, the rate of change of the forward rates is more attenuated than that of the par curve.
3. Forward rates can be more volatile than par rates; that is, a small change in par rates can lead to a proportionately larger change in the forward rate.

It is important to note that forward rates, when regarded as forecasts of future short-term interest rates, require a number of theoretical and practical assumptions. From a theoretical perspective, this approach assumes the absence of transaction costs and assumes the validity of the pure expectations theory of the term structure of interest rates. In particular, the forward rate as calculated from the current cash market yield curve contains no compensation for risk and, in particular, includes no liquidity premium. In practice, the last condition is violated as forward rates are generated from the observed interest rate term structure, which, typically, incorporates a liquidity premium.

2.4 Derivation of zero rates

2.4.1 Basic methodology

The actual computation of the zero rate yield curve is complex. In theory, for each future payment of a coupon security, there exists a zero rate that discounts that payment to its present value. These rates constitute the zero rate curve, points along which represent the yield to maturity of a zero coupon bond for the appropriate maturity rate. This zero coupon yield curve is estimated from the existing par or coupon yield curve. This is completed by calculating equilibrium zero coupon rates which value each component of the cash flow of conventional coupon securities in an internally consistent fashion, such that all par bonds would have the same value as the sum of their cash flow components.

The zero coupon rates are calculated using an iterative methodology whereby the zero coupon rate is determined from a known yield curve for the successive points in time (often referred to as bootstrapping). An alternative technique for deriving the zero rates is using the implied forward rates.

2.4.2 Calculating zero coupon rates through bootstrapping

The bootstrapping approach involves a series of distinct steps:

- separate a coupon bond into a series of zero coupon bonds;
- utilise available zero rates to price components; and
- solve for the unknown zero rate within the constraint that the market value of the bond must be equal to the value of the components using zero rates.

Exhibit 3.7 shows the simple calculation of a zero coupon rate. Given that a one year bond has a coupon and yield to maturity of 8.00% pa semi-annual, a total price of $1m is derived. However, if the six month discount security has a yield of 7.00% (not 8.00%) and the first coupon is discounted accordingly at 7.00% (which is a known zero coupon rate), then the 12 month payments must be discounted at a rate higher than the rate (8.02% pa semi-annual in this case) to maintain the equilibrium price of $1m.

In a similar way, break even zero rates for each subsequent maturity can be derived through iteration. Known zero rates are used to derive the succeeding zero using the same logic to generate a complete yield curve of zero rates from the par rate curve.

The zero rate could, in theory, also have been derived from the discount factor. The relevant discount factor for the 1 year rate is in fact the present value of the final cash flow in 1 year divided by the final cash flow which can be used to solve for the zero rate. This is shown in *Exhibit 3.7*.

Exhibit 3.7

Derivation of Zero Rates—Bootstrapping Technique

CALCULATING BREAK-EVEN ZERO RATES—BOOTSTRAPPING

YEARS	PAR RATES (%PA SA)	PAR RATES (%PA A)	CASH FLOWS			ZERO RATE		PRESENT VALUE AT		DIFFERENCE	DISCOUNT FACTORS FROM	
			PRINCIPAL	COUPON @8.000%	TOTAL	(%PA ANNUAL)	(%PA S/A)	ZERO RATE	PAR RATE 0.0816		ZERO RATES	BOOTSTRAPPING
0.00			(1,000,000)		(1,000,000)							
0.50	7.00%	7.12%		40,000	40,000	7.12%	7.00%	38,647	38,462	186	0.966184	
1.00	8.00%	8.16%	1,000,000	40,000	1,040,000	8.18%	8.02%	961,353	961,538	(186)	0.924378	0.924378
								1,000,000	1,000,000			

2.4.3 Calculating zero coupon rates through implied forward rates

As an alternative method, it is possible to determine the zero coupon rate curve by using the forward rates implicit in the current yield curve and assuming compounding of intermediate cash flows at the implicit forward rates. The basic concept is to use forward rates to reinvest intermediate cash flows to synthesise a zero coupon bond and derive the zero rate which equates the two cash flows. *Exhibit 3.8* sets out an example using the same data as in *Exhibit 3.7*. In theory, both approaches should yield identical results provided a consistent yield curve is utilised.

The zero rate using the forward rate through the pyramid technique can also be derived using discount factors. This is done by taking the six month discount factor and multiplying it by the discount factor calculated from the forward rate for 6 × 12 forward rate (in the above example). The product is effectively the zero rate discount factor for 1 year. This is shown in *Exhibit 3.8*.

Exhibit 3.8

Derivation of Zero Rates—Implied Forward Technique

CALCULATING BREAK-EVEN ZERO RATES—FORWARD RATES

YEARS	PAR RATES (%PA SA)	PAR RATES (%PA A)	FORWARD RATES		CASH FLOWS			ZERO RATE		DISCOUNT FACTORS FROM	
			(%PA A)	DISCOUNT FACTOR	BOND	RE-INVEST-MENT	FINAL	(%PA ANNUAL)	(%PA S/A)	ZERO RATES	FORWARD RATES
0.00					(1,000,000)		($1,000,000)				
0.50	7.00%	7.12%	9.208%	0.956916	40,000			7.12%	7.00%	0.966184	
1.00	8.00%	8.16%			1,040,000	$1,801	$1,081,801	8.18%	8.02%	0.924384	0.924556

2.4.4 *Characteristics of zero coupon rates*

Exhibit 3.9 sets out examples of zero rates for hypothetical yield curves.

Exhibit 3.9
Zero Rate Curves—Examples

The accompanying graphs set on the derivation of zero coupon rates from a given yield curve utilising the iterative methodology specified. The tables are calculated using the following assumptions:

1. linear interpolation is used to determine the full yield curve; and
2. coupons (payable semi-annually) are assumed to equal the par yield applicable to a specified maturity.

YEAR	DAYS TO PAYMENT	PAR RATE	ZERO RATE
0.50	182	6.0000%	6.0000%
1.00	365	6.1500%	6.1523%
1.50	547	6.2000%	6.2031%
2.00	730	6.2500%	6.2548%
2.50	912	6.3750%	6.3871%
3.00	1095	6.5000%	6.5210%
3.50	1278	6.5750%	6.6011%
4.00	1461	6.6500%	6.6825%
4.50	1643	6.6750%	6.7076%
5.00	1826	6.7000%	6.7336%
5.50	2008	6.7750%	6.8202%
6.00	2191	6.8500%	6.9080%
6.50	2373	6.9250%	6.9972%
7.00	2556	7.0000%	7.0880%
7.50	2739	7.0667%	7.1695%
8.01	2922	7.1333%	7.2525%
8.50	3104	7.2000%	7.3373%
9.01	3287	7.2667%	7.4239%
9.50	3469	7.3333%	7.5124%
10.01	3652	7.4000%	7.6030%

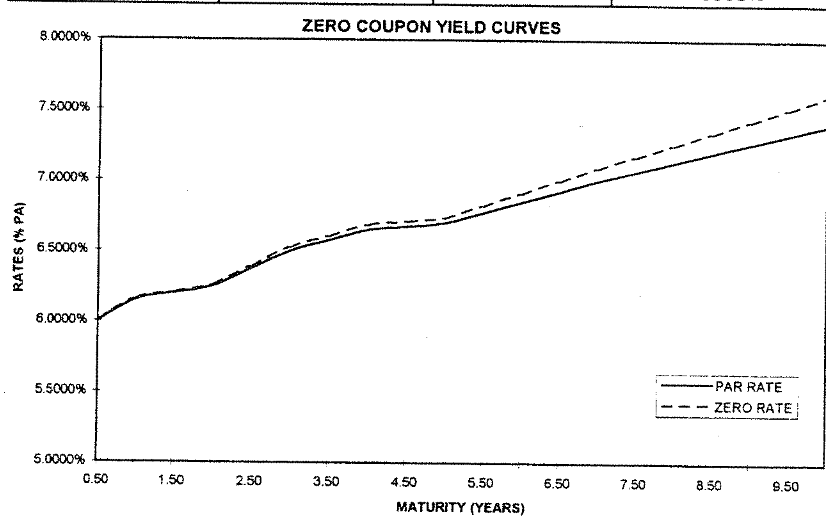
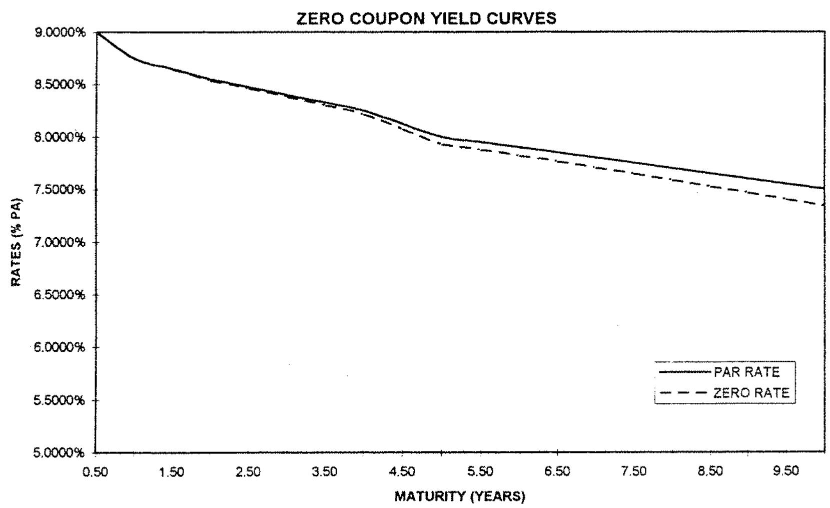


Exhibit 3.9—*continued*

YEAR	DAYS TO PAYMENT	PAR RATE	ZERO RATE
0.50	182	9.0000%	9.0000%
1.00	365	8.7500%	8.7445%
1.50	547	8.6500%	8.6420%
2.00	730	8.5500%	8.5375%
2.50	912	8.4750%	8.4585%
3.00	1095	8.4000%	8.3781%
3.50	1278	8.3250%	8.2964%
4.00	1461	8.2500%	8.2135%
4.50	1643	8.1250%	8.0710%
5.00	1826	8.0000%	7.9275%
5.50	2008	7.9500%	7.8735%
6.00	2191	7.9000%	7.8181%
6.50	2373	7.8500%	7.7616%
7.00	2556	7.8000%	7.7039%
7.50	2739	7.7500%	7.6452%
8.01	2922	7.7000%	7.5856%
8.50	3104	7.6500%	7.5250%
9.01	3287	7.6000%	7.4635%
9.50	3469	7.5500%	7.4012%
10.01	3652	7.5000%	7.3381%



The following characteristics of zero coupon rate and the corresponding zero coupon yield curve should be noted:

1. Theoretical zero coupon rates are always above (below) the relevant par or coupon yield curve for a normal or positively (inverse or negatively) sloped yield curve. This reflects the fact that a coupon bond is a collection of zero coupon bonds and the yield to maturity on a coupon bond is simply the average of the zero coupon rates on the constituent zero coupon securities. Consequently, if yield is increasing in a normally sloped yield curve, then each constituent zero element of the coupon bond will have a yield which is less than or equal to that on a zero with a

maturity that is the same as the coupon bond dictating that the yield on the coupon bond must be less than a zero of the maturity. A reverse logic is applicable in the case of negatively sloped or inverse yield curves.

2. The steeper the curve the more steep is the zero coupon rate curve.
3. Zero coupon rates can be more volatile than par rates as each zero rate is dependent on each forward rate leading up to the maturity of the zero coupon rate. A movement in any of the rates results in a movement in the zero coupon rate.

3. YIELD CURVE MODELLING

3.1 Overview

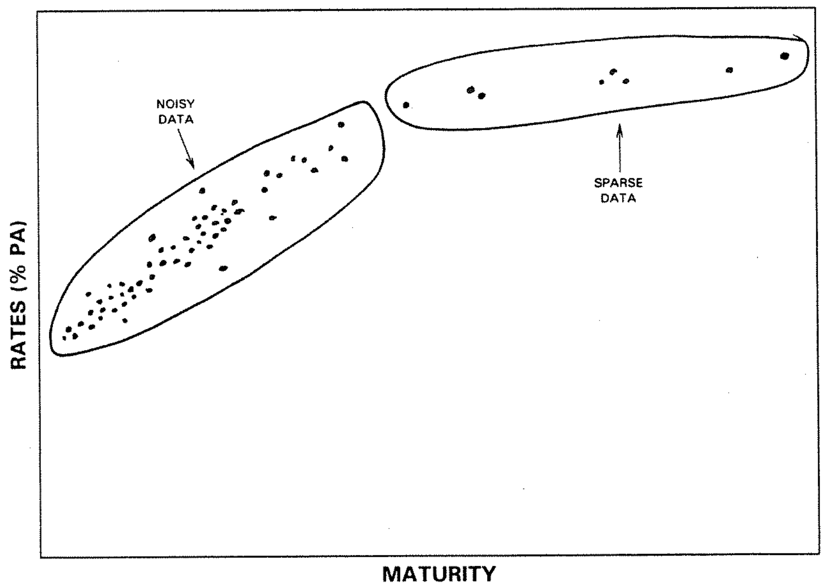
All par rates and zero rates assume and require the existence of a known yield curve. However, much of the problems in derivation of interest rates and discount factors revolves around the issues in generating a complete yield curve of rates for generation of the suitable zero rates used for valuation.

Exhibit 3.10 sets out a typical yield curve which illustrates the difficulties with defining the complete yield curve under most conditions. The curve highlights the following problems:

- data noise—the yield curve may be obscured by the presence of noisy data points whereby there may be a number of yields for similar or identical maturities, which has the effect of increasing the difficulty of determining the *true* interest rate for any maturity; and
- data sparseness—the yield curve may be sparse; that is, it may have significant gaps between observable interest rates. This makes it difficult to specify interest rates for maturities *between* the observed data points.

Exhibit 3.10

Typical Yield Curve



The yield curve described is fairly typical with the problem of noise at the shorter maturity and sparseness in longer maturities. The problems described may be caused by a number of factors including the institutional structure of the market, such as the regulatory framework and liquidity factors, as well as tax factors which may affect trading and valuation of financial instruments.

It should be noted that the above assumes homogeneity in terms of default risk or credit quality of the complete set of interest rates; that is, the rates are all risk free rates or of an identical credit quality. In practice, the interest rates may not be homogenous. This problem is considered below in the context of yield curve construction in practice.

The yield curve modelling problem is separable into two separate and distinct problems:

1. interpolation—that is, the generation of a complete yield curve from the data available; and
2. term structure of interest rates—that is, an understanding of the yield curve shapes and interest rate evolution over time.

The second naturally influences the first process.

3.2 Interpolation techniques

As noted above, the process of interpolation requires the use of available interest rates at various points in the yield curve to generate a complete term structure of yields from which the relevant zero rates can be stripped.

There are a number of interpolation techniques:

1. *Stepped model*—all points on the yield curve are given by the nearest actually observed interest rates.
2. *Linear interpolation*—all points on the yield curve are created by a straight line drawn between each actually observed interest rate.
3. *Non-linear interpolation*—all points on the yield curve are fitted to actually observed interest rates using either regression or spline techniques.

In practice, linear and non-linear techniques are the most important interpolation practices used. Each of these is discussed below.

The need and importance of creating an accurate and consistent yield curve using the choice of interpolation techniques available is best achieved when the yield curve generated satisfies the following criteria:

1. *Fit*—that is, the yield curve generated is consistent with and closely tracks *observed market interest rates*.
2. *Low in noise*—that is, the curve has the appropriate degree of fit in that it is not volatile in response to noisy data (usually where the curve is over fitted).
3. *Consistent*—that is, the par, forward and zero rate derived from the curves are consistent with the observed and theoretical behaviour of these rates. In essence, they are arbitrage free.
4. *Smoothness*—that is, the par, forward and zero rates derived are smooth in that they do not show sudden and unexpected changes and volatility.

In practice, all the criteria identified are unlikely to be satisfied *simultaneously*. In addition, the appropriate trade-off between the criteria is not readily definable. This necessarily introduces a substantial degree of subjectivity in the choice of method and the generation of the yield curve.

3.3 Linear interpolation

Linear interpolation requires the use of straight lines as between any two points of the observed yield curve to estimate the interest rate between these points. *Exhibit 3.11* sets out an example of linear interpolation.

Exhibit 3.11

Linear Interpolation

Given the following seven and ten year interest rates, the benchmark interpolated bond rate for an eight year rate is calculated as follows:

Maturity	Yield
7 Years	7.30% pa
10 Years	7.47% pa

Interpolated yield is calculated as follows:

Interest Rates	Maturity	Days (between)	Blending Factor
7 year rate	15/4/19X4		556/(389 + 556) = 0.588
		389	
8 year rate	9/5/19X5		
		556	
10 year rate	15/11/19X6		389/(389 + 556) = 0.412

Maturity	Interest Rate (%pa)	Blending Factor	Blended Rate (%pa)
7 year rate	7.30	0.588	4.292
10 year rate	7.47	0.412	3.078
		8 year Interpolated Yield	7.37

It is important to note that linear interpolation *on interest rates* is equivalent to *exponential interpolation* on discount factors. Consequently, it is usually done with interest rates rather than discount factors.

The major advantage of linear interpolation is its simplicity and ease of calculation. The disadvantages include:

- the tendency to produce inaccurate rates where the yield curve is changing slope reflecting an inherent tendency for discontinuity (kinks) at each maturity point where the yield curve is not linear in slope;
- the difficulty of generating rates where there is sparse or noisy data; and
- the prospect of generating yield curves which are inconsistent with term structure models of interest rates and also inconsistent with the concept of yield curves which change shape continuously.

3.4 Non-linear interpolation models

3.4.1 Introduction

The concept of non-linear interpolation is predicated on the use of mathematical techniques to generate a fitted yield curve through observed interest rate points. This is undertaken with the objective of fitting a yield curve which reflects the optimality criteria identified and is consistent with the term structure of interest rate assumptions usually made. As noted above, two types of models are generally utilised: regression-based models; and cubic spline-based models.

3.4.2 Regression-based models

A number of models have emerged which seek to use regression techniques, usually non-linear least square regression techniques, to create a fitted yield curve. The models are generally similar in approach, differing in:

- the form of the equation; and
- the number of terms.

Two popular models are the Bradley-Crane model (described in *Exhibit 3.12*) and the Elliot-Echols model (described in *Exhibit 3.13*).

Exhibit 3.12

Regression-based Yield Curve Models—Bradley-Crane Model

The Bradley-Crane model has the following form:

$$\ln(1 + R_M) = a + b_1(M) + b_2\ln(M) + e$$

Where

R_M = Observed interest rate for maturity M

M = Maturity of the interest rate

The model implies that the natural logarithm (ln) of one plus the observed yields for term to maturity of length M are regressed on two variables, the term to maturity and the natural log of the term of maturity. The last term (e) represents the unexplained yield variation. Once the estimated values of a, b_1 and b_2 are obtained, specific maturities of interest can be substituted to obtain estimated yields at these maturity points.

Source: Stephen P Bradley and Dwight B Crane, "Management of Commercial Bank Government Security Portfolios: An Optimisation Approach under Uncertainty" (1973) (Spring) *Journal of Bank Research* 18.

Exhibit 3.13

Regression-based Yield Curve Models—Elliot-Echols Model

The Elliot-Echols model has the following form:

$$\ln(1 + R_i) = a + b_1 (1/M_i) + b_2 (M_i) + b_3 (C_i) + e_i$$

Where

R_i = Yield to maturity

M_i = Term to maturity

C_i = Coupon rate of the ith bond

The model implies that the natural logarithm (ln) of one plus the observed yields are regressed on three variables, the inverse of maturity, the term to maturity and the coupon. The last term (e) represents the unexplained yield variation. Once the estimated values of a, b_1, b_2 and b_3 are obtained, specific maturities and coupons can be substituted to obtain estimated yields at these maturity points.

The Elliot-Echols Model is useful where it is sought to fit yield curves directly to yield data for individual bonds rather than to an homogenised yield series. This might be desirable as a means of avoiding possible distortions created in the process of arriving at the synthetic yield series.

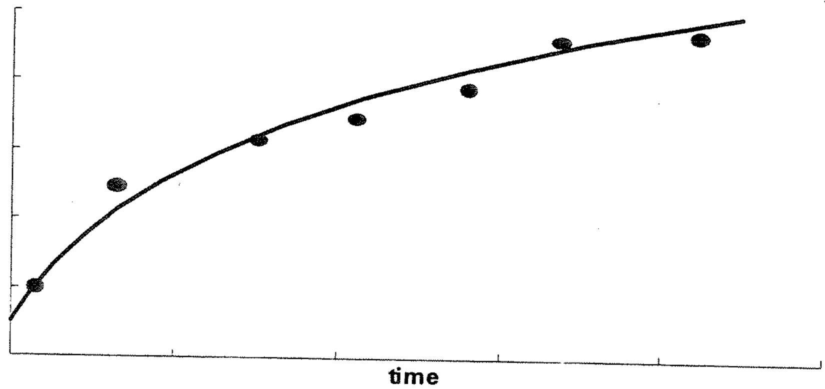
Source: Michael E Echols and Jan Walter Elliot, "A Quantitative Yield Curve Model for Estimating the Term Structure of Interest Rates" (1976) *Journal of Financial and Quantitative Analysis* 87.

An example using a regression-based model is set out in *Exhibit 3.14*.

Exhibit 3.14

Example of Regression-based Model

Regression-based models require the fitting of a functional form to the yield curve. The function is chosen so that it reflects the general shape of the term structure. Parameters that specify the exact form of the function are evaluated to minimise the difference between observed market data and the values given by the function.



Fitting a Curve

The yield curve is specified as a function of rates *r1, r2, r3 . . . rn* and time *t*.

$$df(t) = F(t, r1, r2, r3, \ldots, rn)$$

Where *df(t)* is the discount factor at time *t*. The form of the function is found by a least squares minimisation of the differences between the observed market rates and the values given by the function. This requires minimisation of:

$$\sum_{i=1}^{n} [F(ti, r1, r2, \ldots, rn) - Market\ Price\ (ti)]^2$$

This now gives a function for which discount factors can be obtained for any term.

Example: An Exponential Curve for Yield

Here we specify the curve of the form:

$$df(t) = a_1 e^{-r1t} + a_2 e^{-r2t} + \ldots + a_n e^{-rnt}$$

To obtain the term structure function, values for the coefficients $a_1 . a_2, \ldots a_n$ must be found. This is done using the least squares minimisation shown above. Once these are obtained, we have a function for the yield at any time.

To illustrate this consider a term structure function containing seven terms

$$df(t) = a_1 e^{-r1t} + a_2 e^{-r2t} + \ldots + a_7 e^{-r7t}$$

The values of r_1, r_2,, r_3., r_4,, r_5,. r_6 and r_7 are specified. Market rates for 1 through 10 years are used as observations. To define the yield curve, only the coefficients a_1, a_2,, a_3., a_4,, a_5,. a_6 and a_7 need to be found. This is done using the minimisation above.

Exhibit 3.14—*continued*

Time (y)	Yield
1	7.52
2	7.57
3	7.7
4	7.8
5	7.88
6	7.95
7	7.995
8	8.03
9	8.05
10	8.06

Market Observed Rates

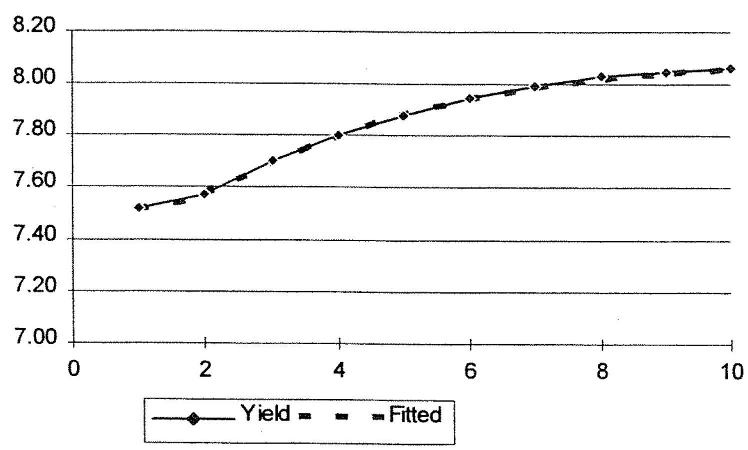


Exhibit 3.14—*continued*

Coefficient (I)	ai	ri
1	5.627834	0.00274
2	4.530894	0.019178
3	−1.28394	0.082192
4	−0.25905	0.246575
5	−0.69042	1
6	2.108328	2
7	4.998892	10

This gives a smooth curve through the known data points. At each known rate, the difference between the market and that given by the curve is small (less than 0.006).

The defined discount function is:

$$df(t) = 5.627834e^{-0.00274t} + 4.530894e^{-0.019178t} - 1.28394e^{-0.082192t} - 0.25905e^{-0.246575t} - 0.69042e^{-t} + 2.108328e^{-2t} + 4.998892e^{-10t}$$

Using this function, discount factors—and hence yields—can be obtained for any time *t*.

These regression-based models are generally useful in avoiding some of the problems of linear interpolation techniques. In practice, the models represent a compromise between too few terms (which tends to create a smooth curve) and too many terms (which tends to overfit the curve creating a noisy curve).

3.4.3 Cubic splines

A number of models have emerged which use polynomial functions to model and create a fitted yield curve. The basic technique used is that of a cubic spline.

The concept of spline techniques is based on creating a yield curve which does not oscillate to a significant degree (that is, it is not noisy) and is relatively smooth. In practice, this is created using splines which are pieces of elastic material which are constrained so as to pass through a given series of points but are allowed to assume other shapes in between the points specified. In theory, the spline will take the shape that minimises its strain energy which is consistent with the mathematical definition of smoothness specified in determining the fit of the yield curve.

Using splines there are two choices in fitting a yield curve:

1. Use a single high order polynomial—this is generally not favoured because there is an inherent tendency for the curve to take untractable shapes between data points.
2. Use a number of lower order polynomials which are then linked to create a complete yield curve—this is generally the favoured methodology because of its inherent flexibility and its inherent satisfaction of the condition that it go through all observed interest rate data points.

The latter technique is referred to as a piecewise cubic spline technique. Using this technique, the complete yield curve is generated as follows:

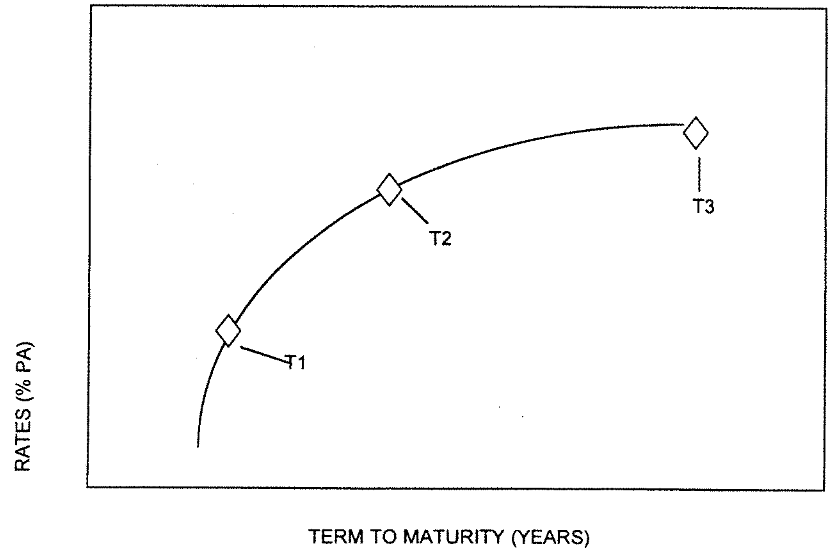
1. The observed yield curve in terms of observed data points is divided into a series (n-1 where n is the number of observed data points) of pairs; in effect, a series of paired adjacent yield curve points and rates.
2. The yield curve between any of these observed pairs is then specified as a polynomial which is unique.
3. Each polynomial which specifies the yield curve shape between two unique points is related to the adjacent or neighbour polynomial so that the slope and/or the rate of change of slope is equal at the common data point between the two polynomials. In effect, the first and (optionally) the second derivative of the two polynomials are equated.

Exhibit 3.15 sets out the mechanics of implementing a piecewise cubic spline methodology of interpolation.

Exhibit 3.15

Piecewise Cubic Spline Technique of Interpolation

The piecewise cubic spline is calculated for the following three interest rates (R) as at time t_1, t_2 and t_3. The technique is also applicable to discount factors. The yield curve segment is set out in the following diagram:



The yield curve is divided into two separate pieces, which are described by unique splines.

The section between t_1 and t_2 is described by:

$$R t_1 = a + b t_1 + c t_1^2 + d t_1^3$$

The section between t_2 and t_3 is described by:

$$R t_3 = m + n t_3 + p t_3^2 + q t_3^3$$

There are eight constants within the two polynomials which must be calculated. It is now necessary to ensure that the two polynomials go through the common data point and that their slope and the rate of change of slope is equal. This is done as follows:

The section at t_2 is described by both the above polynomials:

$$R t_2 = a + b t_2 + c t_2^2 + d t_2^3$$

$$R t_2 = m + n t_2 + p t_2^2 + q t_2^3$$

The first and second derivatives at t_2 must be equal, therefore:

$$b + 2 c t_2 + 3 d t_2^2 = n + 2 p t_2 + 3 q t_2^2$$

$$2 c + 6 d t_2 = 2 p + 6 q t_2$$

To ensure that the change in slope is equal to zero at the ends, the following equation must also be satisfied:

$$2 c + 6 d t_2 = 2 p + 6 q t_2 = 0$$

There are now six equations that can be solved using multiple regression techniques to generate the optimal piecewise cubic spline.

The above process generates a spline which passes through the data points which by definition is the smoothest interpolated function which fits the observed data.

The major advantages of piecewise cubic splines include:

- the fitted yield curve passes through observed data points and avoids the discontinuity or kinks where the yield curve changes slope;
- the curve is generally smooth; and
- the curve provides a robust estimation technique in a variety of market conditions.

The disadvantages of piecewise cubic splines include:

- the need for sufficient data points to allow a good fitted curve to be generated;
- the computation is somewhat complex; and
- the solution of cubic polynomials with multiple regression techniques exhibit the problems of multi-collinearity (that is, the function introduces uncertainty due to the linkages between each segment of the yield curve) and when solved using multiple regression techniques the definition of accuracy of the result is not unambiguous.

3.4.4 Other forms of yield curve interpolation

Two other forms of interpolation models are sometimes used: basis splines and Laguerre functions.

Basis splines are similar to the piecewise cubic spline technique described. The major benefit in using basis splines is that they avoid some of the problems with piecewise cubic splines described above. This is the result of the fact that basis splines go to zero at defined points reducing the linkage issues identified above. Typically, the third order basis splines are used, as they satisfy the required criteria of smoothness.

Basis splines are generated in a systematic manner with second order splines being generated from first order splines and third order splines being generated from second order splines. *Exhibit 3.16* sets out the mechanics of using basis splines to create a yield curve. The process follows the following logic:

- Each spline function is specified with a defined range. Outside this range it has a zero value. The points at which the spline is zero are referred to as knot points.
- One spline will end where another spline commences across the yield curve.
- Where knot points have been specified, each spline is weighted using multiple regression techniques. This is predicated on the fact that market bond prices can be expressed in terms of the sum of the discounted bond cash flows and the discount factor at each point in the yield curve coinciding with a cash flow is capable of definition in spline functions and weights. This will allow the bond price today to be expressed as a function of unknown function weights and the product of the spline function values and the bond cash flows. This allows the regression to be performed

nominating the bond price as the dependent variable and the function cash flows products as the independent variable. The regression process is then used to estimate the function weights.

Exhibit 3.16

Basis Spline Technique of Interpolation

The basis spline is calculated by specifying the following:

First, discount factors are specified as the sum of weighted spline functions:

$$DF(t) = \sum_{l=1}^{e} W_1 S_1(t)$$

Where

DF (t) = Discount factor for time t

W_1 = Function weight

S_1 = Spline function 1

Secondly, the bond price is specified as the sum of the discounted bond cash flows:

$$P_i = \sum_{j=1}^{1} C_i \sum_{l=1}^{e} W_1 S_1(t)$$

Where

P_i = Price of bond i

C_i = Bond I cash flow at time j

Finally, prices are expressed as the function weights and cashflow function product which allow determination of the unknown weights:

$$P_i = \sum_{e=1}^{e} W_i \sum_{j=1}^{n} C_i S_1(t) \quad l=1$$

Source: David Cox, "Yield Curves And How To Build Them" (1995) 4 *Capital Market Strategies* 29-33.

The basis spline, some commentators have argued, has better properties for fitting yield curves than the simpler piecewise cubic spline techniques. However, the basis spline techniques have a number of disadvantages:

- they are complex and computationally difficult;
- the shape of the fitted yield curve is sensitive to the location of the knot points. It appears necessary to ensure an even number of bonds are available between the knot points, which, in practice, is difficult to satisfy; and
- basis splines demonstrate instability and volatility and provides inaccurate estimates where there are gaps in the yield curve and sparse data.

An alternative interpolation technique is to utilise Laguerre functions. *Exhibit 3.17* sets out a description of interpolation techniques using Laguerre functions.

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Basis Spline Technique of Interpolation

The basis spline is calculated by specifying the following:

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Where

DF (t) = Discount factor for time t

W_1 = Function weight

S_1 = Spline function 1

Secondly, the bond price is specified as the sum of the discounted bond cash flows:

$$P_i = \sum_{j=1}^{1} C_i \sum_{l=1}^{e} W_1 \, S_1 (t)$$

Where

P_i = Price of bond i

C_i = Bond I cash flow at time j

Finally, prices are expressed as the function weights and cashflow function product which allow determination of the unknown weights:

$$P_i = \sum_{e=1}^{e} W_i \sum_{j=1}^{n} C_i \, S_{1\,l=1} (t)$$

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Where

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C_i = Bond I cash flow at time j

Finally, prices are expressed as the function weights and cashflow function product which allow determination of the unknown weights:

$$P_i = \sum_{e=1}^{e} W_i \sum_{j=1}^{n} C_i\ S_1\ (t) \quad l = 1$$

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An alternative interpolation technique is to utilise Laguerre functions. *Exhibit 3.17* sets out a description of interpolation techniques using Laguerre functions.

Exhibit 3.17

Polynomial-based Yield Curve Models—Laguerre Model

Laguerre functions consist of a polynomial multiplied by a polynomial decay function in the following form:

$I_t = (a_0 + a_1 * t_1 + a_2 * t_2 + ... + a_n * t_n) * e^{-b * t}$

Where

I_t = Interest rate for maturity t

t_n = Time to maturity

a_n, b = Constants

Where Laguerre functions are utilised for term structure modelling the decay function eventually dominates the polynomial component. This means that the long term rates stabilises, as predicted by a Laguerre function. This property provides Laguerre models with an advantage over other models where the estimates of long term rates continues to increase or decrease with time.

The advantages of Laguerre functions include:

- they provide a range of flexible shapes which are consistent with observable interest rate data; and
- they are consistent with theoretical work on yield curve shape and there is some evidence for their applicability to interest rate data.

Source: B F Hunt, "Modelling The Term Structure", paper presented at Conference on Options on Interest Rates (organised by IIR Pty Ltd), Sydney, March 1992.

3.4.5 Interest rate models

A newer approach to modelling the yield curve entails the use of term structure models. The key feature of these models is the use of assumed stochastic processes to drive the term structure of interest rates.

These models have the following characteristics:

- the models entail explicit recognition of the uncertain element in interest rate structure; that is, interest rates are probabilistic rather than deterministic;
- the models entail linking the term structure of interest rates to specified stochastic processes and nominated stochastic factors;
- the evolution of these factors over time in accordance with the assumed process determines interest rates; and
- the model generated interest rates satisfy certain no arbitrage conditions.

There are a large number of competing models.[3] *Exhibit 3.18* sets out an example of these types of interest rate models.

3. See John Hull, *Futures Options and Other Derivatives* (3rd ed, Prentice Hall, Englewood Cliffs, New Jersey, 1997), Ch 17.

Exhibit 3.18

Interest Rate Models

The Vasichek model specifies the following stochastic model for interest rates:

$$dr = \alpha (\gamma - r) dt + \sigma dz$$

Where

dr = Change in the short-term interest rate

α = Parameter (greater than 0) which describes the speed at which r revert to a long-run average value

γ = Long-run value of r

r = Short-term interest rate

dt = Short-term interval

σ = Volatility of r

dz = Random variable chosen from a normal distribution with mean 0 and variance dt

The process specified identifies that the change in the short term rate r over the intermal dt will have two components:

1. A deterministic component ($\alpha (\gamma - r) dt$) whereby r will revert to a long run value at a speed parameter (α).
2. A stochastic component (σdz) which will change randomly.

The structure of the first term implies that if r is close to (away from) its long run value, the deterministic term will be small (large). This term reflects the premise of mean reversion whereby interest rates tend towards some normal rate. The stochastic term will be larger as the time over which change occurs increases. The structure is designed to be consistent with the general pattern of evolution of interest rates in capital markets.

The specified process for interest rate changes allows the derivation of valuation formula for a discount bond, which in turn facilitates the solution for the value of interest rate derivative products.

Source: O A Vasichek, "An Equilibrium Characterisation of the Term Structure" (1977) 5 *Journal of Financial Economics* 177-188.

The model described is a relatively simple single factor model incorporating mean reversion. The major variations include two factor models (such as a short term and a long term interest rate), the inclusion or exclusion of mean reversion, and the imposition of arbitrage free conditions. Models commonly utilised include the Heath-Jarrow-Morton[4] model and the Hull-White model.[5]

The major application of these models is in pricing interest rate derivatives, in particular, options. The research into interest rate models is largely predicated on these demands. They are also related integrally to the non-linear interpolation techniques identified above. This relationship is predicated on the fact that an appropriate curve is fitted to observed market data. The fitted curve then allows the construction of an interest rate yield curve model which obtains estimates which are consistent with the market

4. See D Heath, R Jarrow and A Morton, "Contingent Claim Valuation With A Random Evolution Of Interest Rates" (1991) *Review of Futures Markets* 54-76; "Bond Pricing and the Term Structure of Interest Rates: A New Methodology" (1992) *Econometrica* 60 at 1, 77-105.
5. See Hull, op cit n 3.

data. In this sense, the interest rate models reflect an extension of the interpolation techniques to allow the provision of solutions, both analytical or numerical, for the value of interest rate derivative products.

4. TERM STRUCTURE OF INTEREST RATES

Interest rates deal with the process of valuation of cash flows at different future times. Term structure deals with the pure price of time in the application of different interest rates at different future times. The process of interpolation assumes implicitly or explicitly a term structure model of interest rates. Understanding of the term structure of interest rates is essential in yield curve modelling.

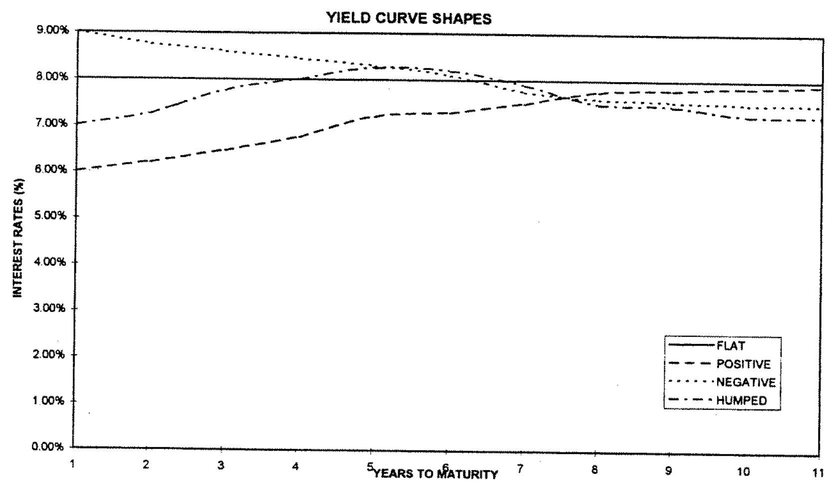
The term structure of interest rates can be defined as the structure of interest rate applicable for cash flows of a homogenous credit quality for different maturities. The types of term structure (or yield curve shapes) observed (see *Exhibit 3.19*) include:[6]

- positive—interest rates increase with maturity;
- negative—interest rates decrease with maturity;
- flat—interest rates are the same across all maturities; and
- humped—interest rates increase with maturity but peak and decrease from their maximum level with further increases in maturities.

6. For a summary of interest rate term structure theories see Richard W McEnally and James V Jordan, "The Term Structure of Interest Rates" in Frank J Fabozzi and T Dessa Fabozzi (eds), *The Handbook of Fixed Income Securities* (4th ed, Irwin Professional Publishing, Chicago, 1995), pp 779-830.

Exhibit 3.19

Yield Curve Shapes/Interest Rate Term Structures



The following theories of the determinants of the term structure of interest rates are usually considered:

1. *Expectations Hypothesis*—in its pure form, the expectations hypothesis states that expected interest return from securities of all maturities are equal. This implies that the return for a particular maturity represents the expected holding period return from investing in shorter term securities which are reinvested at maturity at the implied forward rates. This is based on the implicit assumption that the implied forward rate is the market consensus expected short term interest rate as of the future date. In effect, this theory is consistent with the assumption that longer term interest rates are an average of expected short term interest rates.
2. *Liquidity Preference Hypothesis*—this states that investors prefer shorter maturities in preference to longer maturities and require a premium in the form of a higher interest rate or lower price (the liquidity premium). This is consistent with the assumption of increased price risk with increases in maturity reflecting the potential instability in their capital values upon liquidation prior to maturity if required. The liquidity premium is usually assumed to increase with maturity but at a decreasing rate.
3. *Preferred Habitat/Market Segmentation Hypothesis*—this assumes that the market for shorter dated securities is segmented from that for longer dated securities reflecting the investment preferences of the underlying investors arising from their asset liability matching requirements and risk preferences. This implies that differences in the market structure as embodied in differential demand/supply equilibrium for securities of different maturities dictate the interest rate term structure.

The theories of interest rate term structure are not mutually exclusive. Newer attempts to develop models to encompass the key determinants of term structure usually combine elements of each of the theories. For example, the biased expectations hypothesis states that interest rates reflect the combined impact of future interest rate expectations and also liquidity premia for increased maturities.

The interest rates models described above are not true term structure theories but represent new ways of modelling the yield curve. In reality, the interest rate models are consistent with the key theoretical approaches identified because each represents a special case of stochastic processes used to generate the yield curve.

Exhibit 3.20 sets out a summary of the relationship between the observable interest rate term structures and the theoretical models identified.

Exhibit 3.20

Summary of Interest Rate Term Structure

Type Of Term Structure	Flat	Positive	Negative	Humped
Expectation Theory	Short-term interest rates are expected to remain the same	Short-term interest rates are expected to increase	Short-term interest rates are expected to decrease	Short-term interest rates are expected to increase and then decrease
Liquidity Premium	No liquidity premia	Positive liquidity premia	Negative liquidity premia	Positive liquidity premia followed by negative liquidity premia
Preferred Habitat/Market Segmentation	Equilibrium in demand supply across all maturities	Excess of supply over demand in longer maturities	Excess of supply over demand in shorter maturities	Excess of supply over demand in intermediate maturities
Biased Expectations	Short-term rates are expected to decrease but are offset by increasing liquidity premia	Short-term rates are expected to remain the same or increase moderately and are accentuated by increasing liquidity premia	Short-term rates are expected to decrease but the rate of decrease is offset by an increasing liquidity premia	Short-term rates are expected to stay the same or increase and then decrease (the decrease being sharper than the increase), with the increase being accentuated by and the fall retarded by an increasing liquidity premia

In practice, the yield curve interpolation model utilised must be consistent with the observed interest rate term structure prevalent in the relevant market to avoid inaccuracies and poor predictive performance.

5. YIELD CURVE CONSTRUCTION IN PRACTICE

The process of derivation of appropriate interest rates and discount factors can, as shown above, be reduced into two separate and distinct processes. The first process entails the use of yield curve modelling processes to derive a complete set of interest rates. The second process entails the generation of zero rates from the yield curve which can be utilised for the valuation of financial instruments.

In practice, a number of additional considerations are relevant. A major factor underlying these uncertainties is the incomplete and imperfect nature of financial markets generally and the difficulties in nominating objective criteria to select optimal models for constructing accurate yield curves. These problems include the following:

- The difficulty in identifying yield curves which are homogenous in terms of credit or default risk. In practice, a series of interest rates derived from similar but not perfectly credit homogenous yield curves are combined.
- The problems of defining the characteristics of fit for an estimated yield curve because the criteria of fit, consistency and smoothness can be applied at different levels. For example, there are in reality multiple curves which can be fitted to satisfy the smoothness criteria: the par curve; the forward curve; the zero curve; or the discount factors generated off any of these curves. Linear interpolation often creates irregular forward curves while splines can create regular smooth curve, for one or more of these sets of interest rates or discount factors, *but not for all curves*.

The problems identified are not necessarily capable of perfect solution. In practice, practitioners use compromises reflecting trade-offs between market structure, data integrity, estimation accuracy, computational efficiency and cost effectiveness. A major criteria is the issue of hedge-ability; that is, the capacity to hedge the components of the yield curve and the interest rate risks assumed in the course of pricing, valuing and trading financial instruments off the selected yield curve. In essence, practitioners will favour the construction of a yield curve which not only is nearest the theoretical paradigm but one which also facilitates trading and hedging activities.

Against this set of constraints, most practitioners utilise two separate yield curves for valuation purposes:

1. A risk free curve—usually constructed from the available series of interest rates on government securities of the relevant tenor.
2. A risk adjusted (the swap) curve—usually constructed from a mixture of instruments (including short term inter-bank rates, near term short term interest rate futures contracts or forward rate agreement (FRA) prices/rates, and interest rate swap rates. The swap curve is used to value credit risk affected (that is, non government risk financial instruments) where appropriate, incorporating adjustment spreads where the underlying

instrument is considered to have fundamentally different risk or other characteristics.

The derivation of the risk free curve follows the established procedures identified. Key considerations include, depending on the market, noisy data (reflecting a large number of government securities of identical or similar maturities with different coupons trading at different yields) and data sparseness (whereby there may be significant gaps in the yield curve). In practice, these are overcome by constructing a fitted curve (using one or other of the techniques identified) and generating the required zero rates from that curve.

The generation of the swap curve is more complex. As noted above, this curve is constructed by combining a series of interest rates from different instruments. The following example, which uses the US market, is indicative of both the approach and the key considerations which are relevant. It should be noted that the problem of yield curve construction in other markets is necessarily more complex and less readily soluble than those in the US example, reflecting the relative maturity, liquidity and efficiency of that market.

In practice, the swap curve in US$ is constructed as follows:

- The cash rate (usually based on an interbank rate such as LIBOR) to the first IMM eurodollar delivery date (the near month contract) is taken. Some interpolation is usually required, as the period may not coincide with the traded maturities for cash, which are usually overnight, one week, one, three, six et cetera months.
- The next series of rates taken are the eurodollar futures rates. The number of successive eurodollar futures contracts varies but in practice will be between 12 and 20 quarterly contracts (three to five years).[7] For each contract, the traded futures price is deducted from 100 to determine the forward rate which is then incorporated into the yield curve. This process is repeated for each contract.
- Beyond the futures contracts, available interest rate swap rates are utilised to complete the yield curve.
- Certain dominance rules are specified; for example, the eurodollar rates may or may not be overridden by the relevant swap rates.

The curve once derived is fitted and zero rates generated in the established manner. *Exhibit 3.21* sets out a simple example of the process described.

7. It is likely in currencies other than US$ the number of futures contracts used would be lower, say four to eight contracts (one to two years), reflecting the fact that the futures markets in the relevant currency do not allow trading beyond this maturity and/or the liquidity of the market. Technically, in the US$ market, it is theoretically possible to trade eurodollar futures out to 10 years (40 successive quarters), which would mean it would be possible to derive a 10 year yield curve from the futures rates.

Exhibit 3.21

Constructing a Zero Coupon Curve

This is an example of how to construct a zero coupon curve using bills, bill futures and swaps. We use the bootstrap method. Interpolation is based on keeping the forward rates constant. Internal calculations will be in terms of continuously compounding rates and discount factors.

The instruments

For the curve, we use instruments whose prices are available in the marketplace. These are bills, bill futures and swaps. We will assume that the bill futures take precedence where there is any overlap.[i]

To keep the procedure general, all times will be in days or years. This means that dates can be omitted. In a real application, dates would be converted to days. The spot date is assumed to be day zero. All days/years are relative to the spot date unless otherwise specified.

Bills

These are pure discount instruments whose maturity is a fixed number of days out from the spot date.

# Days	Rate
1	6
30	6
60	5.98
150	6

The price of a bill is just its face value discounted over the appropriate number of days. Thus we can directly obtain discount factors for bills. If the bill has d days to maturity then the discount factor is

$$df = \frac{1}{(1 + r\frac{d}{365})}$$

This gives a continuously compounded zero rate of

$$r(zero) = \frac{-365}{d}\ln(df)$$

i. This is a realistic assumption as they are the most liquid of the instruments used. Also, short dated swaps are usually hedged using bill futures.

Exhibit 3.21—*continued*

# Days	Rate	df	Cts Zero Rate
1	6	0.999836	5.999507
30	6	0.995093	5.985254
60	5.98	0.990266	5.950799
150	6	0.975936	5.927221

Bill futures

Bill futures are just like bills, except that they start out of a forward date. In this example, we have a strip of bill futures. Where one bill future matures, the next begins. From the table below, the first bill future starts at day 40 and matures on day 132. The next starts on day 132 and expires on day 223 and so forth. The yield of a bill future is just 100 minus its price.

Days to Start	Days to Expiry	Price	Yield
40	132	94.000	6
132	223	93.920	6.08
223	314	93.660	6.34
314	405	93.350	6.65
405	496	93.110	6.89
496	587	92.940	7.06
587	678	92.790	7.21
678	769	92.640	7.36
769	860	92.520	7.48
860	951	92.420	7.58
951	1042	92.330	7.67
1042	1133	92.220	7.78

It is a simple matter to get the forward discount factor and the forward rate continuously compounded for a bill future. These are given by the same formulae as for bills. If the bill future starts at d_s and expires at d_e, then

$$df_s^e = \frac{1}{(1 + r\frac{(d_e - d_s)}{365})}$$

$$r_s^e(zero) = \frac{-365}{(d_e - d_s)} \ln(df_s^e)$$

Exhibit 3.21—*continued*

These are forward discount factors and rates. They need to be linked to spot rates. This will be illustrated below when we combine all the components of the curve.

Days to Start	Days to Expiry	Price	Yield	Cts Zero Rate	df
40	132	94.000	6	5.955082	0.985102
132	223	93.920	6.08	6.034379	0.985068
223	314	93.660	6.34	6.290415	0.984439
314	405	93.350	6.65	6.595475	0.983691
405	496	93.110	6.89	6.831492	0.983112
496	587	92.940	7.06	6.998586	0.982703
587	678	92.790	7.21	7.145964	0.982342
678	769	92.640	7.36	7.293288	0.981981
769	860	92.520	7.48	7.411109	0.981693
860	951	92.420	7.58	7.509266	0.981452
951	1042	92.330	7.67	7.597587	0.981236
1042	1133	92.220	7.78	7.705509	0.980972

Swaps

Swaps are priced as fixed coupon instruments whose coupon is equal to the swap rate. In other words they are par bonds. Conventionally, swaps of three years and less pay quarterly coupons while swaps of longer maturity pay coupons semi annually. It is convention to quote rates for swaps of maturity one to five years, then for seven and ten year swaps. For maturities of six, eight and nine years, we shall interpolate the rates linearly.[ii]

# Years	Rate	Freq
1	6.5	4
2	7	4
3	7.2	4
4	7.25	2
5	7.35	2
6	7.415	2
7	7.48	2
8	7.52	2
9	7.56	2
10	7.6	2

The swaps will be priced when the curve is assembled.

ii. It may be more appropriate to use higher order interpolation. Alternatively, the curve can be built using only the quoted swap rates. The other rates obtained from the curve.

Exhibit 3.21—*continued*

Putting it together

We have chosen to construct the curve giving precedence to bill futures where there is overlap. Thus the bill of 150 days will be omitted. The 60 day bill will not be used explicitly, but will be used to obtain a bill rate to the beginning of the bill futures strip. Swaps of three years and less are not needed either.

Up to 30 days, the bills provide zero coupon instruments directly. Once a rate at 40 days is known, the bill futures can be used to compute the curve out to 1133 days. The swaps are then required for further dates.

As we know zero rates from the bills to 30 and 60 days, a 40 day bill rate can be implied by interpolation. In this example we assume that forward rates are constant. Using the 30 and 60 day bill rates, the discount factor and forward rate from 30 to 60 days is given by:

$$df_{30}^{60} = \frac{df_0^{60}}{df_0^{30}} = \frac{0.990266}{0.995093} = 0.995149$$

$$r_{30}^{60} = \frac{365}{(60-30)} \ln(df_{30}^{60}) = 5.916344\%$$

This forward rate is used from 30 to 40 days (as the interpolation keeps the forward rates flat). Thus the 40 day discount factor and zero rate can be obtained.

$$df_{30}^{40} = df_0^{30} e^{-0.05916344.10/365} = 0.993481$$

The process so far has used bill rates directly to obtain the zero curve to 30 days. The 60 day bill is used to obtain the curve to the beginning of the bill futures strip. The futures can be combined directly to generate the curve out to 1133 days.

	Days (Start)	Days (Expiry)	Zero Rate	Spot DF	Fwd DF	Fwd Rate
Bill	30	60	5.985254	0.995093	0.995149	
Bill	60		5.950799	0.990266		**5.916344**
Futures Splice	40	60	5.968027	**0.993481**	0.99838	**5.916344**
Futures	40	132	5.968027	**0.993481**	0.985102	5.955082
	132	223	5.959005	0.97868	0.985068	6.034379
	223	314	5.989763	0.964067	0.984439	6.290415
	314	405	6.076895	0.949065	0.983691	6.595475

Exhibit 3.21—*continued*

The forward discount factors for the bill futures have already been obtained above. These can be combined with the discount factor to the beginning of the futures strip to build the curve. For the first future

$$df_0^{132} = df_0^{40} df_{40}^{132} = 0.993481 \times 0.985102 = 0.97868$$

and the continuous zero rate is

$$r_0^{132} = \frac{-365}{132} \ln(0.97868) = 5.959005\%$$

This can be continued for all the bill futures.

# Days	df	Cts Zero Rate
1	0.999836	5.999507
30	0.995093	5.985254
60	0.990266	5.950799
150	0.975936	5.927221
132	0.985102	5.955082
223	0.985068	6.034379
314	0.984439	6.290415
405	0.983691	6.595475
496	0.983112	6.831492
587	0.982703	6.998586
678	0.982342	7.145964
769	0.981981	7.293288
860	0.981693	7.411109
951	0.981452	7.509266
1042	0.981236	7.597587
1133	0.980972	7.705509

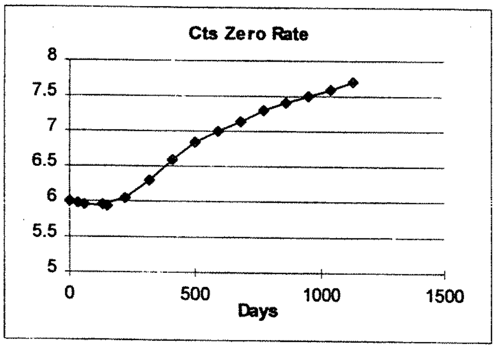


Exhibit 3.21—*continued*

The Zero Curve to the End of the Bill Futures Strip

The swaps are now required. The first swap to be used is the four year swap. The swap rate of 7.25% implies the following cashflows

Years	Cashflow
0	−100.000
0.5	3.625
1	3.625
1.5	3.625
2	3.625
2.5	3.625
3	3.625
3.5	3.625
4	103.625

As the curve has already been generated out to 1133 days (or 3/104 years), this can be used to value the cashflows to three years. The remaining two cashflows are required to generate the curve.

Years	Days	Cashflow	DF	NPV
0	0	−100.000	1	−100
0.5	182	3.625	0.970623	3.51851
1	366	3.625	0.940189	3.408186
1.5	547	3.625	0.908889	3.294723
2	731	3.625	0.876684	3.177981
2.5	912	3.625	0.845036	3.063255
3	1096	3.625	0.81323	2.947957
3.5	1278	3.625		
4	1462	103.625		

The NPV of the swap must equal exactly zero if the cashflow on the spot date is included. Of the cashflows where the curve is available, the NPV is −80.5894. The complete equation is

$$df_{1278} \times 3.625 + df_{1462} \times 103.625 = 80.5894$$

This cannot be solved directly as there are two unknown discount factors. If we use our interpolation rule keeping the forward rate constant, then an iterative solution can be generated.[iii]

This gives a solution for the forward rate of $r^{1462}_{1096} = 8.022778$ and discount factors $df_{1278} = 0.781339328$ and $df_{1462} = 0.750369643$. This solves the equation for the swap price above. This procedure is then repeated for all the other swaps.

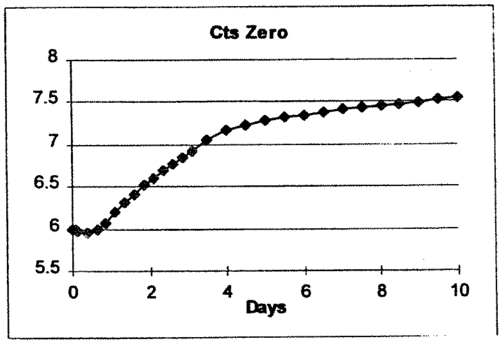
iii. Any common numerical method such as Newton-Raphson can be used.

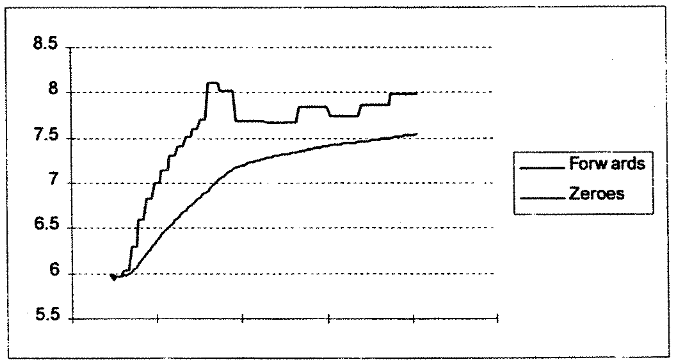
Exhibit 3.21—*continued*

Instrument	Years[iv]	Discount Factor	Continuous Forward Rate	Continuous Zero Rate
Spot		1		
1 Day Bill	0.003	0.9998	5.9995	5.9995
30 Day Bill	0.082	0.9951	5.9848	5.9853
Futures	0.110	0.9935	5.9163	5.9680
	0.362	0.9787	5.9551	5.9590
	0.611	0.9641	6.0344	5.9898
	0.860	0.9491	6.2904	6.0769
	1.110	0.9336	6.5955	6.1934
	1.359	0.9178	6.8315	6.3105
	1.608	0.9019	6.9986	6.4172
	1.858	0.8860	7.1460	6.5150
	2.107	0.8701	7.2933	6.6071
	2.356	0.8541	7.4111	6.6922
	2.605	0.8383	7.5093	6.7703
	2.855	0.8226	7.5976	6.8426
	3.104	0.8069	7.7055	6.9119
	3.501	0.7813	8.1037	7.0471
4y Swap	4.005	0.7504	8.0228	7.1699
	4.501	0.7223	7.6876	7.2269
5y Swap	5.005	0.6948	7.6876	7.2733
	5.501	0.6689	7.6782	7.3098
6y Swap	6.005	0.6435	7.6782	7.3407
	6.501	0.6190	7.8404	7.3789
7y Swap	7.005	0.5950	7.8404	7.4121
	7.504	0.5725	7.7330	7.4334
8y Swap	8.008	0.5506	7.7330	7.4523
	8.504	0.5295	7.8649	7.4763
9y Swap	9.008	0.5089	7.8649	7.4981
	9.504	0.4892	7.9816	7.5233
10y Swap	10.008	0.4699	7.9816	7.5464

iv. Note that the number of years is not exactly integral for the swaps. This is because when this example was generated, it was assumed that there were 365 days per year. Exact calendar dates were used.

Exhibit 3.21—*continued*



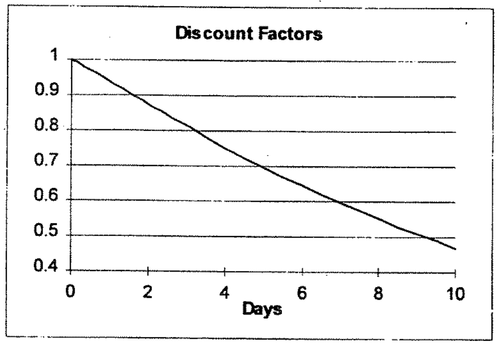


The graph showing the forward rates illustrates the interpolation method. The forward rates are stepped. They change discontinuously. Where there are frequent and liquid instruments (the bills and bill futures), the change is regular. Once the instruments become less liquid, there is some irregularity in these rates. The zero coupon rates are much smoother. It is rare that a long dated forward instrument will require pricing. If it does, the spread will have to be large to account for the irregularity in the forward rates. Alternatively an interpolation method where the forwards are kept smooth will need to be employed.

Obtaining discount factors

Once the curve has been generated, we have discount factors (and rates) at the points where cashflows from the underlying instruments occur. The curve will price these instruments. This in itself is not useful as the prices are already used in the curve construction. Indeed this is a circular situation. The value of the zero curve is in having a model which gives discount factors at any time in the future. To obtain these, we can either graphically retrieve them, or use the interpolation method implicit in the curve.

Exhibit 3.21—*continued*



The Discount Factors

Using the interpolation method to obtain discount factors requires minor calculation. As an example, we will obtain the discount factor at 7.4 years.

Instrument	Years[v]	Discount Factor	Continuous Forward Rate	Continuous Zero Rate
7y Swap	7.005	0.5950	7.8404	7.4121
	7.504	0.5725	7.7330	7.4334
8y Swap	8.008	0.5506	7.330	7.4523

We know the discount factor to 7.005 years. We also know that the forward rate for any period between 7.005 and 8.008 years is 7.7330% (as our interpolation method keeps the forward constant). We can obtain the discount factor from 7.005 to 7.4 years.

$$df_{7.005}^{7.4} = e^{-0.07733t} = 0.969952$$

This then allows the discount factor and zero rate to be calculated.[vi]

Instrument	Years[vii]	Discount Factor	Continuous Forward Rate	Continuous Zero Rate
7y Swap	7.005	0.5950	7.8404	7.4121
7.4y	**7.4**	**0.577089**	**7.7330**	**7.429182**
	7.504	0.5725	7.7330	7.4334
8y Swap	8.008	0.5506	7.7330	7.4523

Discount factors can be obtained to any dates by this process.

v. Note that the number of years is not exactly integral for the swaps. This is because when this example was generated, it was assumed that there were 365 days per year. Exact calendar dates were used.
vi. In this equation, $t = 0.394520$.
vii. Note that the number of years is not exactly integral for the swaps. This is because when this example was generated, it was assumed that there were 365 days per year. Exact calendar dates were used.

Exhibit 3.21—*continued*

Pricing cashflows

Arbitrary cashflows can now be valued by obtaining the discount factors to the dates where the cashflows occur. To price any instrument, decompose it into cashflows, then use this method. If there are liquidity or other conditions, the zero curve can be modified to account for these. Also the difference between market prices and the price on the curve can be calculated. This is useful to assess the premium or discount the market is building in to non standard instruments. As an example, corporate bonds can be valued on a zero coupon curve built using government bonds. The discount for credit liquidity and other factors can be quantified.

The approach described requires careful consideration of the following factors:

- The nature of the instruments is different, including differences in credit risk and instrument features. For example, the use of futures contracts introduces the following factors: the payment of deposits and margins; the differential credit risk of the clearing house; and, in the case of eurodollar futures, the problems of the fixed tick point value (0.01% is equal to US$25) or the negative convexity.
- The problem of overlapping dates. This can be illustrated by comparing and contrasting the use of FRAs versus futures. The FRAs out of spot will trade at regular runs (3 × 6; 6 × 9 et cetera) which allows the calculation of the relevant forward interest rates and discount factors which are incorporated in the yield curve. In contrast, the eurodollar futures contracts are standardised. They are traded to pre-specified dates and are cash settled against three month LIBOR. As these dates may not be precisely three months apart, the prospect exists for gaps or overlaps between the end of the reference period and the maturity date of the next eurodollar contract. The practical import of this is that the discount factors cannot be multiplied or forward rates compounded as the interest periods are not exactly linked. This requires adjustments to the futures rates.
- The forward rates embodied in eurodollar futures prices reflect some of the inherent biases in futures prices, including the impact of margin payments and the negative convexity. Where interest rates are expected to increase, the holder of a short (long) futures position will receive (be required to pay) margin payments which can be invested (must be funded) at higher interest rates. This uncertainty forces the futures rate to trade above the theoretical arbitrage free forward rate. In practice, the bias in futures prices and the negative convexity necessitate additional adjustments based on the expected volatility of interest rates.
- The selection of the transition point between the futures curves and the swap curve is relatively arbitrary reflecting institutional and market considerations as well the margining and convexity issues identified.

The presence of these factors means that the generation of the relevant yield curve is unlikely to be an objectively verifiable activity.

The major practical problem to arise relates to the fact that more than one equally tractable yield curve can be created from the same set of market data reflecting differences in interest rate selection and mode of adjustment for the problems identified. In reality, the problems are generally likely to be confined to the shorter end, particularly the transition points from one set of interest rates to the next source. A major area is the transition from the futures or FRA rates to the swap rates.

Given that the zero rates are likely to be different, either a selection must be made between the yield curves or adjustment made to the set of rates to be used. One possible basis for selection between the curves is by application. Where an adjustment is to be made it is necessary to either segment the curve to ensure the absence of overlap or create blended curves. There are problems with each approach:

- the use of different curves differentiated by application will create different values and prices for different transactions opening up the possibility of value loss; and
- the adjustments will either create severe discontinuity in the curve and irregularities in the rates or require complex and highly subjective adjustments.

In practice, there is little uniformity in approach to these issues. Each institution generally employs its own set of techniques to deal with the problem, reflecting the nature of the market and the purpose for which the generated zero rates are to be utilised.

5. SUMMARY

The pricing, valuation and trading of financial instruments, irrespective of whether it is a simple fixed income security or a derivative transaction, requires and assumes the availability of interest rate or discount factors extending across the maturity spectrum. In practice, the practitioner is required to choose between different interest rates to value these transactions. The best practice method now applied universally in capital markets is to discount the transaction cash flows using zero rates derived from the relevant yield curve. The zero rate itself requires the construction of the yield curve using sophisticated mathematical techniques within a framework of economic theory which is consistent with observed interest rate behaviour. While the requirement for accurate zero rates is now recognised, deficiencies in market structure and data availability present significant challenges to the derivation of accurate, consistent and computationally efficient yield curves.

Part 3

Derivative Pricing

Chapter 4

Pricing Forwards and Futures Contracts[1]

by John Martin

1. INTRODUCTION

A forward is any contract which obliges you to buy or sell a financial instrument or physical commodity at some date in the future at an agreed price. For our purposes, forwards includes OTC forward contracts and exchange-traded futures contracts, and includes instruments such as:

- foreign exchange forward contracts;
- forward rate agreements;
- forward bonds;
- short-term interest rate futures;
- bond futures;
- stock index futures; and
- commodity futures contracts.

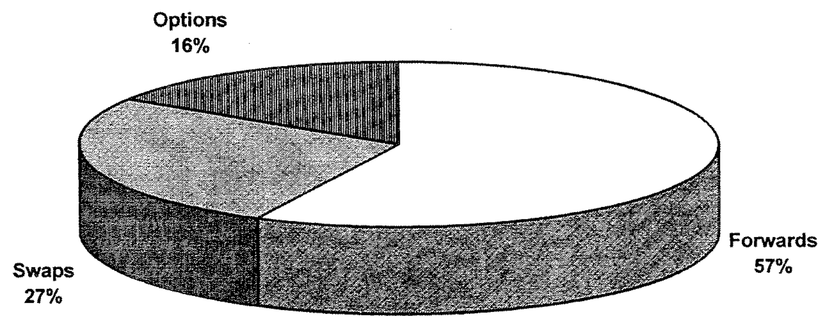
Forward contracts represent an extremely useful starting point for all derivative valuation. As we will see in other parts of this book, more complex derivatives such as swaps and options can be decomposed into portfolios of forward contracts. As a consequence, the valuation of these more complex instruments will be based partly on the forward valuation techniques developed in the following chapters.

While forward contracts are the simplest form of derivative, they represent the largest derivative type by outstanding face value and volume. *Exhibit 4.1* highlights the relative importance of forwards versus swaps and options.

1. Parts of this chapter are based on the discussion on forward contracts in J S Martin, *Derivative Maths* (IFR, 1996).

Exhibit 4.1

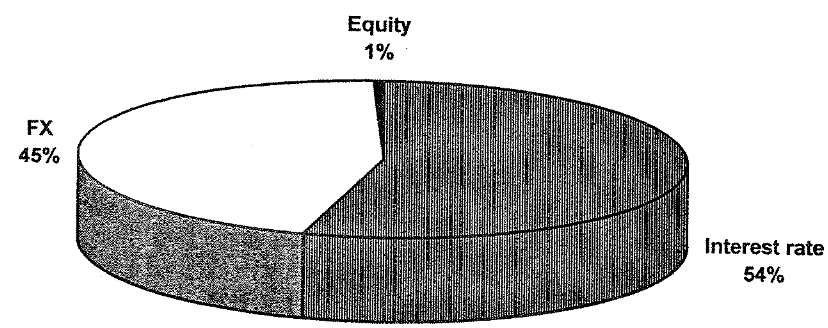
Derivatives Outstanding by Type, 1993



Source: GAO and BIS

Exhibit 4.2

Forwards Outstanding, by Underlying, 1993



Source: GAO and BIS

As *Exhibit 4.2* demonstrates, foreign exchange and interest rate products dominate forward transactions. It is interesting to note that most of this volume is comprised of OTC foreign exchange forwards and exchange traded interest rate futures. The volume of equity forwards is relatively low and is mainly stock index futures. Like all derivatives markets, forward markets have seen very rapid growth since the mid 1980s.

2. CHARACTERISTICS OF FORWARDS

2.1 Forwards versus cash transactions

The definition of when a transaction is a cash transaction and when it is a forward is important. We might expect any transaction which settles today to be a cash transaction and a forward is anything settling from tomorrow onward. Unfortunately this is not always the case and, depending on the underlying financial asset, a "cash" transaction can range from today for a money market transaction to several weeks, or longer, in some securities markets. Some examples of cash instrument settlement days are listed below:

Cash Market	Settlement Date
Money market transactions	Today
Euromarkets	Today + 2 business days
Foreign exchange	Today + 2 business days
Stock exchanges	Today + 5 business days
Crude oil	Today + 1 month
Some property markets	Today + 6 weeks

All of these represent "cash" transactions, however, in each market the convention applying to the settlement of these transactions changes according to market convention. The market convention usually reflects the ease with which settlement can be arranged and/or the relative complexity of changing ownership of a financial asset. For example, money market instruments in most currencies are lodged on electronic networks, where both payment and transfer of ownership can occur in a matter of minutes—same day settlement is easily possible. However, where change of ownership involves different time zones and bank accounts such as in foreign exchange, or the shipment of commodities in the oil market, or the completion of legal documentation as in property, the time to settlement becomes longer.

When considering a forward, the market convention on the time to settlement underlying a cash or spot price has to be known. A forward transaction does not commence till the settlement day passes the cash settlement date. So, in the foreign exchange market, a forward is a transaction which settles after two business days.

2.2 Forward price and value

A forward contract gives us the right to buy or sell a financial asset at some date after the normal cash settlement date at an agreed price. The attraction of forward contracts is that they provide a method of obtaining a fixed price on an asset regardless of movements in the cash price between the trade date and the settlement date.

As in the cash market, a forward *price* is agreed between the buyer and seller which reflects the relative cost or benefits to both parties of delaying settlement of that transaction. Once this price is agreed then the market replacement cost, or the *value* of reversing, will also change as market conditions change. An important feature of forwards, and derivative valuation in general, is that there are always two components to valuation:

1. What is the forward price?
2. What is the value of an open transaction based on this price?

The forward price and cash price are usually different. When valuing a forward transaction we first need to determine the prevailing market price for a forward, and then determine the present value based on that forward price. This is simplified in futures markets, as a transparent forward price is the subject of trading and, in liquid markets, is a fair reflection of the price at which open contracts can be reversed.

Determining the forward price is not only a requirement of valuation. It is also an essential tool in comparing different forward instruments such as forwards and futures, whether you are a market maker, arbitrager or hedger. For example, suppose you know that the true ''fair'' forward price of a security is $100 and the futures market is trading at $99. This suggests that an arbitrage opportunity of $1 exists if you buy the futures contract and enter into another contract to sell it forward. Similarly, if you are a market maker in forward instruments you need to be able to calculate a constantly updated forward price to quote to your clients.

In this chapter we review the following:

- general forward pricing and valuation (Section 3);
- interest rate forwards and futures (Section 4);
- foreign exchange forwards (Section 5); and
- equity forwards (Section 6).

2.3 Terminology

In these chapters on forwards and futures we will use the following standardised terminology:

Term	Description
Valuation Date	The date on which a valuation is being performed (usually today)
Forward Expiry Date	The date on which the forward contract expires and the obligation to buy or sell forward falls due
Forward Settlement Date	The date on which the forward obligations arising from the forward contract must be settled in cash (usually the forward expiry date or soon after)
Forward Period	The number of days between the valuation date and the forward expiry date
Cash or Spot Price	The price paid on the valuation date for a "cash" purchase of the underlying asset
Forward Price	The price agreed to be paid for the underlying asset on the forward expiry date
Asset Income	The income paid to the owner of a financial asset—usually during the forward period. Examples of asset income include coupons and dividends.

3. GENERAL FORWARD PRICING AND VALUATION

3.1 Deriving the forward price

The easiest way to understand forward pricing is to break it down into its underlying components. Like most derivatives, forward transactions can be reproduced by a series of physical positions. This can probably be best explained by some simple examples.

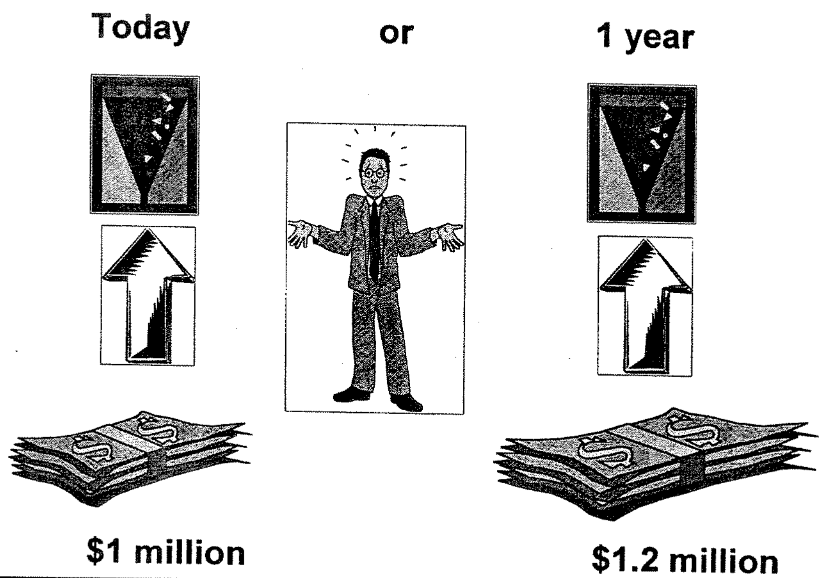
3.1.1 Example: buying a painting

Calculating the forward pricing is the same question as how much should I pay to buy something in the futures. To remove the complications sometimes presented by financial instruments it is often easier to understand derivative pricing using a tangible good such as a piece of artwork. In both of these examples we ignore any compounding effects and assume that there are no other benefits from the asset apart from any described.

Suppose an art dealer offers to sell you a painting today for US$1 million today or US$1.2 million in one year's time. Would you buy it today or in one year? (As a hint the current one year interest rate is 10% pa.)

Exhibit 4.3

Buying a Painting Forward



Your relative position can be summarised by looking at your asset and cash balances

	Balance	Cashflows	Balance
Spot Purchase			
Assets	Painting		Painting
Cash	−$1m	−$0.1m	−$1.1m
Forward Purchase			
Assets	0		Painting
Cash	0		−$1.2m
Net Cash difference of today versus forward			+$0.1m

In *Exhibit 4.3* the problem is represented diagrammatically. The art dealer is offering to buy the painting at a cash price of $1 million or a forward price of $1.2 million. We wish to take the deal which is financially beneficial.

A useful way of looking at the transaction is in terms of the cash and asset balances. If we buy the painting today we give up $1 million. This is either financed by borrowing the funds or reducing our existing cash balances, which at the end of the year incurs an interest cost or reduces interest income by $0.1 million. While the forward purchase avoids spending cash today, it requires paying $1.2 million in one year. At the end of one year both deals give us the same asset, however the cash cost of the forward purchase is $0.1 million higher than the spot purchase—so we would buy the painting today.

While a simple example, it displays an important idea: a forward transaction can be replicated by purchasing the asset today and borrowing the money to finance it. The ''fair'' forward price is then the cash price plus the interest cost over the life of the forward transaction. In this example, the fair one year forward price is $1.1 million.

This example shows the forward price on an asset that provides no cash return in the form of coupons or dividends. Most financial assets provide an income so we need to incorporate that into our pricing framework. In the following example we use another simple tangible asset.

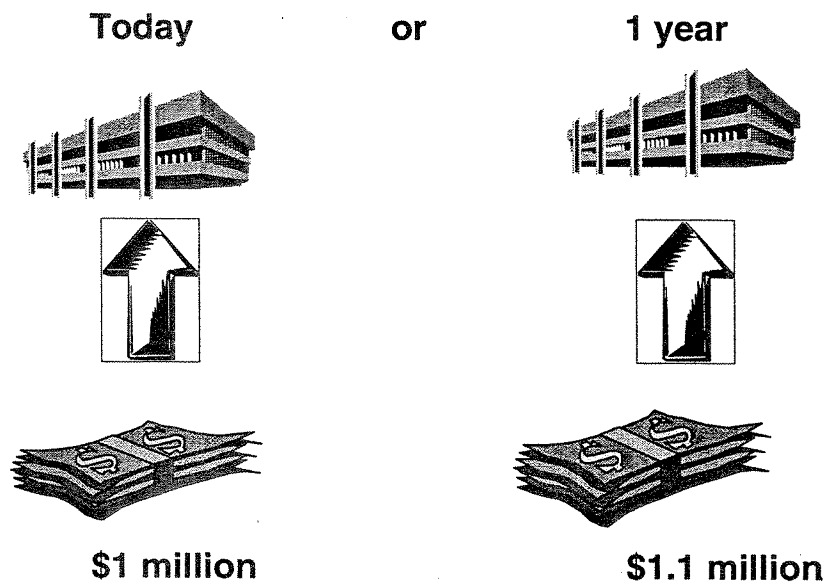
3.1.2 Example: buying a warehouse

You are given the opportunity to buy a warehouse as an investment for DEM 1 million today or DEM 1.1 million in one year's time. Would you buy it today or in one year? (As a hint, the warehouse is currently earning rental income of 2% pa and you can borrow DEM for 1 year at 14% pa.)

This time we need to take account of the cash income. In this case, if we buy the property today we receive DEM 0.02 million in income, which we do not receive if we purchase the warehouse forward. The rent has the effect of reducing the net cost of ''carrying'' that property.

Exhibit 4.4

Buying a Warehouse Forward



Spot Purchase	**Balance**	**Net Cashflows**	**Balance**
Assets	Warehouse		Warehouse
Cash	−$1m	−$0.12m	−$1.1m
Forward Purchase			
Assets	0		Warehouse
Cash	0		−$1.1m
Net Cash difference of today versus forward			−$0.02m

Using the same cash balance approach as in the above example we can see from *Exhibit 4.4* that, while the rental income reduces the net cost of buying the property the net cash cost, and the ''fair'' forward price, at the end of the year is DEM 1.12 million. So, in this example we would buy the property forward.

3.2 Some conclusions regarding the forward price

While simple, these two examples display the fundamentals of forward pricing. Essentially the ''fair value'' forward price makes buyers and sellers indifferent between buying and selling the underlying asset today or in the future based on the current market cash price, cost of financing the asset and the expected return on the asset. In other words the forward price is essentially a summary of the net financial obligations of owning an asset.

The examples also highlight four of the key features of the forward prices:

1. *Replication*: A forward purchase of an asset can be replicated by buying the asset in the cash market, financing the purchase by borrowing the cash required and then receiving any asset income.
2. *Fair price*: The "fair" forward price is given by the cash price plus the net cost of financing the asset over the term of the forward contract.
3. *Interest effect*: The interest cost tends to *increase* the forward price versus the cash price.
4. *"Dividend" or "coupon" effect*: Any cash return on the asset over the term of the forward contract tends to *decrease* the forward price versus the cash price.

These four general rules should apply to all forward prices on financial assets regardless of whether it is an interest rate, foreign exchange or equity product provided they operate in freely operating markets. It is worth noting that these relationships start to break down when you move away from financial assets, particularly to consumable commodities such as agricultural and energy products. This is because the decision to have the physical commodity today or in the future is not just a financial or investment decision, the decision to buy a commodity today or in the future also has to take into consideration when the commodity is required for consumption.

While these examples were from the point of view of a forward purchase, the same logic applies to a forward sale except it works in reverse. Looking at the painting example, if we agree to sell the painting forward we forgo receiving the cash today and any interest earned over the year. Correspondingly, if we sell the painting forward we want to ensure that we will receive a cash amount, which is, at least, equivalent to the cash price plus interest—giving us the same price as suggested by the "buy" example.

The forward price can be "synthetically replicated" using physical transactions. A forward purchase can be replicated by buying today and financing the purchase over the forward term by borrowing. While economically, the price achieved by the replication is equivalent to a forward which usually makes the derivative more attractive. First, the transaction costs of the physical replication are often greater due to the number of extra "legs" and also physical transactions often have higher execution costs than derivatives. Secondly, forward transactions are a future commitment and are "off-balance" sheet. The transactions in the physical replication are included in the balance sheet which has the effect of increasing assets and liabilities and possibly increasing capital costs.

3.3 Cashflow timing and the cost of carry

A cash and forward purchase provide ownership of the asset at different times. However, providing the only benefit offered by these assets is their income stream, and the repayment of principal in debt instruments, then this benefit of ownership now or in the future is only notional.[2] As we have seen the forward price incorporates both the net interest cost of holding the asset

2. This is an important condition of forward pricing. If there are other tangible or intangible benefits of owning an asset such as the appreciation of artwork, or the ability to use it for consumption, then, unless these factors can be quantified, the forward price is unknown.

as well as the asset return. As a result the difference between cash and forward transactions is only a difference of cashflow profiles:

1. *Cash purchase cashflow profile*: Requires cash today, however, it will provide income between today and the forward date.
2. *Cash sale cashflow profile*: Receives cash today but will miss out on any income between today and the forward date.
3. *Forward purchase cashflow profile*: Does not require cash till the forward settlement date and as such misses out on any asset income.
4. *Forward sale cashflow profile*: Forgoes receiving cash till the forward settlement date, however it will provide income between today and the forward date.

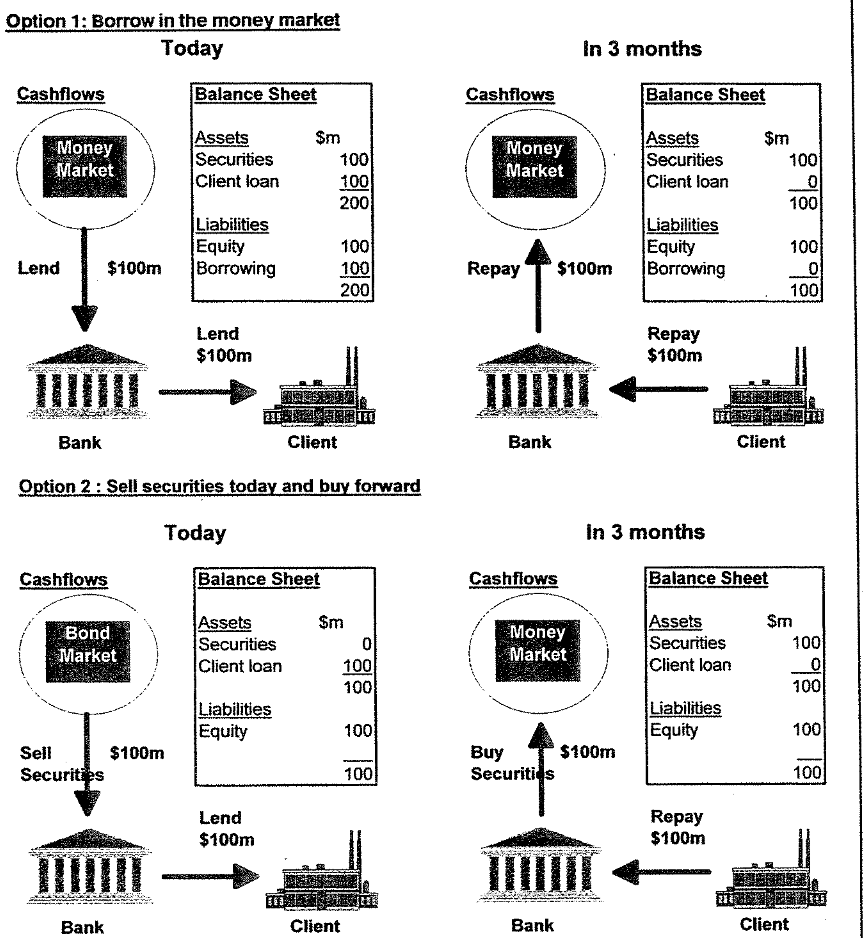
These cashflow profiles are interesting and explain an important part of the use of forward transactions. Any of the four alternatives represent the exchange of cash and an asset. However, if opposite spot and forward transactions are combined (for example, a spot sale and a forward purchase) then two offsetting cashflows at different points in time are created. In effect, we are creating transactions, using an underlying financial asset, which have the cashflow profile of a borrowing or lending. This is a fundamental driving force in forward markets globally and is a key reason for transactions such as "repos" (see section 4) and FX swaps (see section 5).

Suppose you are a small merchant bank which owns $100 million of liquid government securities. Your organisation has a requirement for $100 million in short-term funding needs over the next three months. The traditional way to finance this would be to borrow in the short-term money market. This creates three problems: your ability to borrow is dependant on your credit standing, as is the cost of borrowing, and the transaction will increase gearing. If, however, you entered into an agreement to sell the securities today and then buy them back in three months, you have created the underlying borrowing required. Further, the ability to raise the funds, and the cost of those funds, is primarily determined by the government securities, not your own credit rating. This type of transaction is often described as "security lending", "sell and buy", "repurchase or reciprocal purchase agreement (repo)", or as a "liquidity swap"—we will refer to it as a "repurchase agreement". Both the money market and "sell and buy" transactions are summarised in *Exhibit 4.5*. The cashflows are identical, however, the balance sheet and gearing consequences are very different—with the money market doubling total assets and liabilities while the other security transaction has no effect on the totals, just the composition.

Exhibit 4.5

Using Forwards to Raise Finance

Your bank's only asset is $100m worth of securities and all its liabilities are shareholders funds. It requires $100m in funding to make a loan to a client. You can borrow the funds or enter into an agreement to simultaneously sell and buy the securities.



Whereas the cost of borrowing in the money market is given by its interest rate, in the security transaction it will be given by the difference between the spot and forward price. As we know the forward price of the securities is determined by the current cash price plus the net cost of financing those positions. Given the forward purchase can offer the government securities as collateral the implied interest cost should be lower than the direct money market borrowing.

From the point of view of large holders of financial assets repurchase agreements are a balance sheet efficient and, potentially, low-cost form of financing. For example, an investment bank which is a market maker in securities and holds large bond portfolios will use repurchase agreements to finance its bond holdings in a similar way to the example in *Exhibit 4.5*. We will discuss the intricacies of these agreements in the interest rate and foreign exchange forward chapters.

3.3.1 Cost of carry

The key pricing concept in forward transactions is the net financing cost of creating a synthetic replica using cash instruments. The usual terminology for this net financing cost is the "cost of carry".

$$\text{Forward Price} = \text{Cash Price} + \text{Cost of Carry}$$

$$\text{Cost of Carry} = \text{Interest Cost} - \text{Asset Income}$$

3.4 Forward pricing formulae

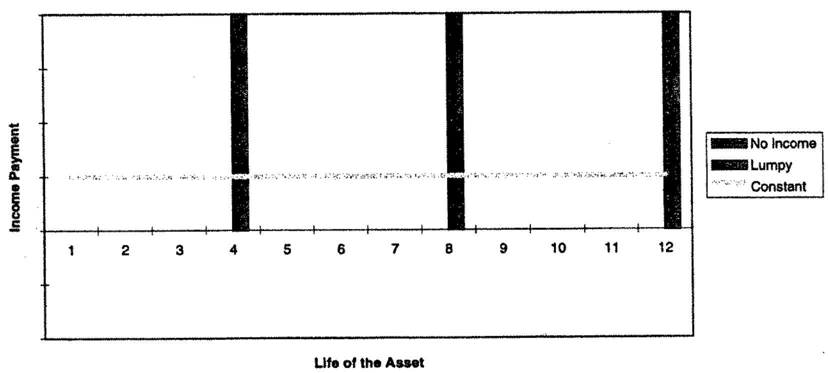
We now know the general relationship between cash and forward prices. In this section we will develop three simple formulae for pricing forward transactions depending on the nature of the income stream generated by the underlying financial asset during the period of time to the forward expiry date. The three forms of asset considered are:

- an asset which pays *no* income;
- an asset which pays *constant* income; and
- an asset which pays "*lumpy*" income.

The difference in these three income streams is summarised in *Exhibit 4.6*. Distinguishing by asset income allows us to develop three models which can price most financial assets. As a result, we find that we can apply the same "lumpy" pricing models to bonds and shares—even though, apart from the income streams, the underlying instruments have very different characteristics.

Exhibit 4.6

Different Asset Income Assumptions



3.4.1 Forward pricing on an asset which pays no income

If the asset pays no income between the day we calculate the forward price and the expiry of that forward contract, then the forward price is the cash price adjusted for the interest cost only. Examples of these types of financial assets includes precious metals, some commodities, and artwork.

It is worth noting that this formula also applies to financial assets which pay an uneven or "lumpy" income (see section 3.4.3 below), but will not pay any income between the pricing date and the forward expiry date.

Forward Price: Asset Pays No Income

Using simple interest the calculation is as follows

$$F = S \times (1 + r \times f / D)$$

Where

F = Forward price

S = Cash or spot price of the underlying instrument

r = Interest rate to forward date (preferably zero-coupon rate)

D = Day count basis (365 or 360)

f = Number of days to the forward expiry date

Examples of underlying assets: precious metals; artwork; and some commodities.

An important assumption in all of the forward formulae is that the interest rate, r, is a simple interest rate over the term to expiry of the forward. Formally this means the interest rate in the model is an effective zero-coupon rate. While zero-coupon rates will be discussed in detail in later chapters the important point about zero-coupon rates is that they pay no interest between today and the maturity date, and there is no risk associated with the re-investment of coupons.

In practice, most money market instruments are zero-coupon. So pricing forward transactions with up to six months to expiry is accurate. For longer terms, pricing using a coupon paying interest rate should be viewed as an estimate—accurate pricing requires zero-coupon yields.

3.4.2 Forward pricing on an asset which pays constant income

The assumption in this formula is that the underlying financial asset pays income at an even, constant rate over the life of the forward contract. In practice that means the asset will pay income for every day that it is held. Examples of this type of instrument include discount money market instruments, foreign exchange and broad-based equity indexes.

Forward Price : Asset Pays Constant Income

Using simple interest, the calculation is as follows

$$F = S \times (1 + (r - q) \times f / D)$$

Where

F = Forward price

S = Cash or spot price of the underlying instrument

r = Interest rate to the forward expiry date

D = Day count basis (365 or 360)

f = Number of days to the forward expiry date

q = Asset income expressed as a % per annum

Examples of underlying assets: money market instruments; foreign exchange; and broad-based equity indexes.

3.4.3 Forward pricing on an asset which pays "lumpy" income

In this formula it is assumed that the underlying financial asset pays income only at certain points over its life. Typically, the asset income is accrued over a period and then paid at the end of the period—this gives a "lumpy" appearance to the income cashflows. From the point of view of forward pricing the important consideration is how many income payments will occur during the life forward term.

Some examples of this type of instrument include bonds and shares.

Forward Price: Asset Pays "Lumpy" Income

Using simple interest and one income payment, the calculation is as follows

$$F = S \times (1 + r_1 \times f_1 / D) - c \times (1 + r_2 \times f_2 / D)$$

Where

F = Forward price
S = Cash or spot price of the underlying instrument
r_1 = Interest rate to the forward expiry date
r_2 = Interest rate between the income payment and forward expiry dates
D = Day count basis (365 or 360)
f_1 = Number of days to the forward expiry date
f_2 = Number of days between the income payment and forward expiry dates
c = Asset income expressed in the same units as the cash price

Examples of underlying assets: bonds and shares.

Sample calculations are provided in *Exhibit 4.7*, where the forward price is calculated on a security under each of the three asset income assumptions.

Exhibit 4.7

Forward Price Example

You intend to buy a security 180 days forward. The current spot price is $90 and the six month interest rate is 6.7% pa (A/360). Calculate the forward price under the following three asset income scenarios:

- *no income;*
- *income paid at rate of 8% pa on a constant basis; and*
- *a lump sum of $4.50 will be paid in 91 days—assume the three month interest rate in three months is also 6.7% pa.*

i) No income

$S = \$90 \quad r = .067 \quad f = 180 \quad D = 360$

$$\begin{aligned} F &= S \times (1 + r \times f / D) \\ &= 90 \times (1 + .067 \times 180 / 360) \\ &= 93.015 \end{aligned}$$

ii) Income = 8% pa constant

$S = \$90 \quad r = .067 \quad f = 180 \quad D = 360 \quad q = .08$

$$\begin{aligned} F &= S \times (1 + (r - q) \times f / D) \\ &= 90 \times (1 + (.067 - .08) \times 180 / 360) \\ &= 89.415 \end{aligned}$$

iii) Income = lump payment of $4.50

$S = \$90 \quad r1 = .067 \quad r2 = .067 \quad f1 = 180 \quad f2 = 89 \quad D = 360 \quad c = 4.50$

$$\begin{aligned} F &= S \times (1 + r1 \times f1 / D) - c \times (1 + r2 \times f2 / D) \\ &= 90 \times (1 + .067 \times 180 / 360) - 4.5 \times (1 + .067 \times 89 / 360) \\ &= 88.44046 \end{aligned}$$

As the results demonstrate, the nature of the income payment has a considerable impact on the forward price. And as the cost of carry concept tells us, where the asset income exceeds the cost of financing the security (scenarios i and ii) the forward price is lower than the cash price.

3.5 Valuation of forward contracts

In the previous sections we have examined how to determine a forward price based on cash market information. Now that we can generate a forward price we can determine the present value of open forward contracts. We divide this calculation into two steps: determining the value on the forward expiry date; and then determining the present value.

3.5.1 Valuation on the forward expiry date—the forward value

In the financial mathematics of financial assets, value is given by the present value of all future coupon and principal cashflows. As a result, the present value generally represents a premium or discount to the face value of the asset. This is not the case with forward contracts—when a forward contract is initially executed its value is zero, as the forward price this value can change to be positive or negative.

A forward contract represents a commitment to purchase or sell a financial asset, they are not financial assets in their own right. Unlike a financial asset which has value arising from future cashflows, the value of a forward contract only arises from the benefit or loss arising from the obligation to buy or sell the underlying asset.

If we think of the cashflows of a forward contract on an interest bearing security, then as well as the future coupon and principal repayments there is the initial cashflow associated with the purchase or sale of the security on the forward settlement date. So, if a bond is purchased forward, the cashflows consist of a cash payment on the forward settlement date and then cash receipts in the form of coupons and principal repayments over the remaining life of the bond.

At the time a forward contract is executed the forward price and the value of all of the future cashflows created by the bond after the forward expiry date are equal. However, as interest rates change the relative values of the forward price and all of the future cashflows are different. This relationship for a forward purchase of a bond contract can be summarised as follows:

Forward Value = Forward Bond Value − Forward Contract Price

where forward bond value is the value of all of the cashflows created by the bond after the forward expiry date and the forward contract price is the price agreed under the forward contract. As the forward price is fixed, the contract value will change as the forward bond value changes: if the yield to maturity on the bond falls (increasing the forward bond value) then the forward contract value will rise above zero and if bond yields rise the forward contract value will fall below zero.

Over the life of the contract the forward contract price is fixed. The forward bond value is simply calculated by determining the current forward price using the appropriate formula from Section 3.4 above. So, if the forward contract price was \$110 and the current forward price is \$120, the value of a forward purchase on the forward expiry date would be positive \$10.

This relationship holds for all forward contracts. The value as at the forward expiry date is the difference between the forward contract price and the current forward price. This relationship also explains the risk profile, or potential for profit or loss, of forward contract. This relationship is described as the "pay-off" of the forward contract and the graphical representation as a "pay-off diagram". We will utilise this concept regularly in our investigation of derivative value.

Exhibit 4.8 provides an example of a full forward pricing and valuation exercise. It also illustrates the sensitivity of the contract value to changes in the current forward price using a pay-off graph.

3.5.2 Forward contract valuation today—present value

A common mistake in using forward contracts is forgetting that the forward valuation occurs on the forward expiry date. As we know from the time value of money, cash today is worth more than in the future. The implication is that, depending on the time period involved, the forward valuation overstates the true present value. We need to be very careful when dealing with forward contracts to ensure we know whether we are calculating forward or present values. This is an important consideration when comparing futures and forwards and is discussed in Section 3.6 below.

Calculating the present value of a forward contract can be performed using one of the present value formulae from Chapter 3. If we can apply a simple interest rate, then the present value of the forward contract value is:

$$\text{Present Value} = \text{Forward value} / (1 + r \times f / D)$$

where r is a simple interest zero-coupon rate between today and the forward expiry date. As already noted, if the interest rate that is being used is a money market interest rate it already is zero-coupon interest rate and can be entered directly into this calculation.

The distinction between forward and present values is demonstrated in *Exhibit 4.8*.

Exhibit 4.8
Forward Price and Valuation

You have entered into the forward contract discussed in Exhibit 4.7 where the asset pays no income at a price of $93.015. You decide to calculate the value of this contract after 30 days have passed. In that time interest rates have risen to 8% pa and the cash price of the security has declined to $84.2. Calculate the current forward price, the forward valuation and then the present value of this contract.

Current forward price

S = $84.2 r = .08 f = 150 D = 360

F	=	S × (1 + r × f / D)
	=	84.2 × (1 + .08 × 150 / 360)
	=	<u>87.0067</u>

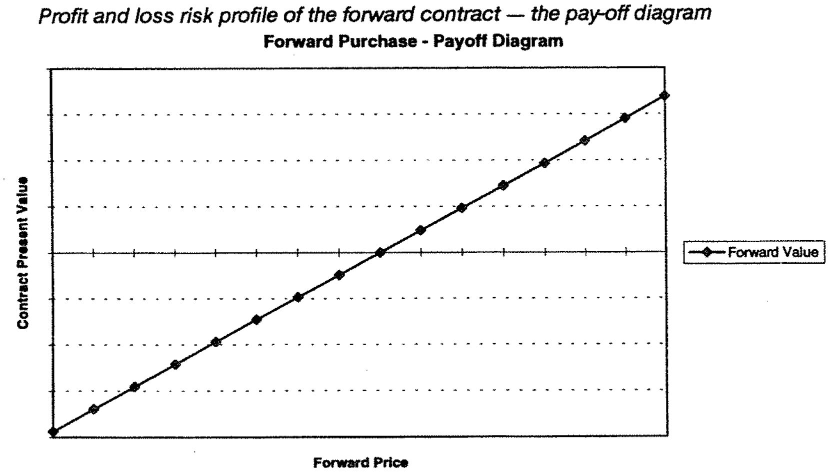
Value at forward expiry date

Forward value	=	Current forward price − forward contract price
	=	87.0067 − 93.0150
	=	<u>−6.0083</u>

Present value of forward contract

Present value	=	forward value / (1 + r × f / D)
	=	−6.0083 / (1 + .08 × 150 / 360)
	=	<u>−5.81448</u>

Profit and loss risk profile of the forward contract — the pay-off diagram



3.6 Valuation differences between forwards and futures

Futures contracts are a standardised form of forward contracts: a futures contract represents a commitment to buy or sell a fixed amount of the underlying financial asset, at a single date in the future. An OTC and futures contract with the same forward expiry date should have the same forward price.

While the pricing and valuation methodologies for the forwards and futures are similar, there are some key differences that need to be considered. In the following two chapters we will deal with specific differences between comparable OTC and exchange traded contracts.

The differences arise from the fact that futures contracts are subject to daily mark-to-markets and upfront initial margins (or performance bonds). These margining requirements have been created to minimise the risk that the clearing house of futures exchanges take to position takers. The effect of these margins, however, is to alter the valuation of futures versus forwards as they cause cashflows prior to the forward expiry date. In terms of the forward valuation/present valuation distinction in Section 3.5, a futures contract generates present values rather than forward values.

3.6.1 The impact of daily mark-to-markets on valuation

While equivalent forward and futures contracts will have the same value at the forward expiry date, it is often not realised that prior to the expiry date they have different present values. It is still common practice in many organisations to directly compare the valuations of forward and futures contracts without recognising that there is a difference in timing.

The valuation effect of daily mark-to-markets is to create a cashflow timing difference: in essence the forward values are being paid early. On any given day, the present value of a futures contract (represented by the mark-to-market gain or loss) is the same as the forward value on an equivalent forward contract.

For example, suppose we buy identical futures and forward contracts for expiry in one year. Today the difference between the current forward price and contract price at which we bought both contracts gives a value equivalent to \$100 profit. The present value of these two instruments will reflect the relative timing of the values. If the one year rate is 10%, then the present values are:

$$\begin{aligned} \text{Futures Profit} &= \$100 \\ \text{Forward Profit} &= \$100 / (1 + 10\%) \\ &= \$90.91 \end{aligned}$$

The difference in these values is important, as the futures contract in fact demonstrates greater sensitivity to movements in the forward price. In this example we can say that for every \$1 movement in the forward value, the futures contract will generate a present value change of \$1, while the present value of the forward contract will only change by 0.9091.

3.6.2 *The impact of initial margins on valuation*

Initial margins are a security deposit that must be paid by both buyers and sellers of futures contracts to the clearing house. These initial margins are held to cover any losses incurred by a defaulting position holder. In general the initial margins are set to cover the losses generated by a very large movement in the futures price over a 24 hour period. As a result the more volatile the futures price, the higher the level of initial margins.[3]

The impact of initial margins on valuation is not as straightforward to quantify as the daily mark-to-market because it depends on the level of initial margins and the rules of the futures exchange clearing house with respect to the types of collateral (for example, cash, government securities, precious metals, shares) allowed and the interest payment policy. These costs can be divided into two categories: interest cost and capital cost.

3.6.2.1 Interest cost

If the clearing house only accepts cash then the cost of initial margins is equivalent to the interest spread between the cost of funding the initial margin deposits and the interest paid on that deposit by the clearing house:

Interest Cost = Initial Margin × (Funding Cost − Clearing House Rate)

Suppose you enter a futures contract with a face value of $100 million and an initial margin requirement of $3 million. Your company borrows at the overnight money market rate while the clearing house pays the overnight rate minus 0.50% pa on your initial margin deposit. The additional funding cost of these initial margins is equivalent to 0.75% pa or $15,000 annually. In terms of the total contract value this has increased the cost of financing the position by 0.015% pa or 1.5 basis points.

To incorporate the interest cost into the forward pricing formula we need to increase the cost of carry to reflect the additional financing cost of the initial margin. In this example we would add 1.5 basis points to the interest rate, which will have very little effect on the forward price, as is shown in the calculation below where the forward price from *Exhibit 4.8* is recalculated using an interest rate of 8.015% pa:

$$\begin{aligned} F &= S \times (1 + r \times f / D) \\ &= 84.2 \times (1 + .08015 \times 150 / 360) \\ &= 87.0119 \end{aligned}$$

This represents a change in the forward price of 0.0052—a very small impact. Often the interest cost is ignored by market participants because it is viewed as relatively unimportant.

If the clearing house accepts the lodgment of other forms of collateral such as bonds, shares and money market instruments without imposing any charges—a common practice in most large exchanges—then the interest cost will be the difference between the return on the asset and your organisation's cost of funds.

It is quite common for financial institutions to consider that providing securities as initial margin collateral incurs no cost. This is because they

3. For more detail on how initial margins are derived, see the Appendix to Chapter 7 in Martin, op cit n 1.

already hold the assets used for collateral for regulatory or investment reasons—they are simply re-using assets already held.

3.6.2.2 ***Capital cost***

Whether initial margin collateral is in the form of cash or some form of security, there is a capital cost. By providing this collateral to the clearing house, the position taker is potentially transferring ownership of the assets and there is the possibility that those assets will not be repaid—initial margins represent a credit risk to the clearing house and some capital has to be set aside for that possibility.

The capital cost will depend on the capital allocation policies of the organisation involved. However, for a bank the Bank for International Settlements (BIS) Capital Adequacy Standards would view the initial margins deposited with the clearing house as a deposit with a corporation—requiring an allocation of capital equivalent to 8% of the deposit. Given the assumed cost of capital of an organisation then we can calculate the capital cost as follows:

$$\text{Capital Cost} = \text{Initial Margin} \times 8\% \times \text{Cost of Capital}$$

So, on the example above, if the cost of capital is assumed to be 20% pa then the annual capital cost is as follows:

$$\begin{aligned} \text{Capital Cost} &= \$3{,}000{,}000 \times 8\% \times 20\% \\ &= \$48{,}000 \text{ pa} \end{aligned}$$

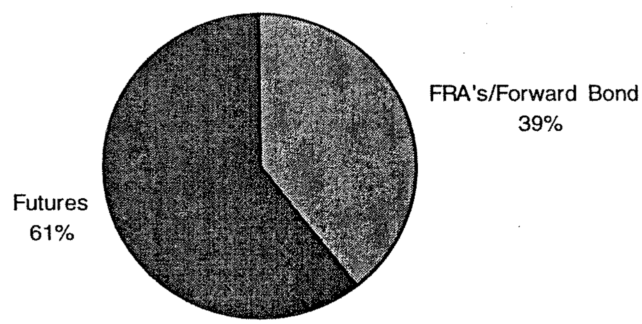
In the same manner as the interest cost, the capital cost is considered an additional cost in financing the futures position and increases the cost of carry. Once again the capital cost is often viewed as too small to worry about and is ignored by some market participants.

4. INTEREST RATE FORWARDS AND FUTURES

4.1 Introduction

As we saw in the previous section, interest rate forwards and futures represent the largest single category of volume in financial derivatives. This is a recognition of the importance of these instruments as a day-to-day hedging and trading tool for all participants of the financial market place. While the volume of OTC forward contracts in some currencies is substantial (for example, US$ Treasury Bonds), *Exhibit 4.9* shows that the global volume of interest rate futures is considerably higher than forwards. This is a reflection of the fact that short-term and long-term interest rate futures are the primary forward interest rate instruments in most currencies, whereas OTC forwards tend to be for more specialised use.

Exhibit 4.9
Composition of Interest Rate Forwards Outstanding Volume - 1993



Source: GAO

In this chapter we will develop pricing and valuation models for forward and futures contracts on interest bearing financial assets.[4] These models will be divided into categories reflecting the different characteristics of forwards on short-term and long-term debt securities. The categories used are as follows:

- forward rate agreements (section 4.2);
- short-term interest rate futures (section 4.3);
- bond forwards (section 4.4); and
- long-term interest rate futures (section 4.5).

For each model our starting point is the generalised model developed in Section 3. These models will then be adapted to the specific cashflow and convention characteristics of each instrument.

4.2 Forward rate agreements

4.2.1 General description

Forward rate agreements (FRAs) are the predominant form of OTC forward on short-term interest rate securities. They represent an agreement between two parties who wish to "fix" the interest rate on an underlying short-term security at a future date. FRAs do not have physical delivery, instead, any profits and losses are realised by way of a cash settlement at the end of the forward period. While the underlying instrument in an FRA is usually a short-term instrument with a term of three or six months, the forward period can range from one month to several years. An FRA is agreed

4. Another description of the underlying assets is "debt securities".

in terms of a forward interest rate as opposed to a forward price and the pricing formulae need to be adjusted to reflect this.

The liquidity of FRAs in most countries is very high, with most financial institutions providing market making services. Reflecting the level of activity, standard documentation and dealing terms and conditions have been developed in most countries.[5] A summary of the general terms and conditions is provided in *Exhibit 4.10*.

5. Examples of this documentation and terms and conditions can be obtained from most bankers' associations or the local branch of ISDA in the relevant country. Otherwise, see S Das, Swaps and Financial Derivatives (2nd ed, IFR, 1994), pp 89-96.

Exhibit 4.10

General FRA Terms and Conditions

Item	Description
Broken period	A settlement period which differs in length from that used in fixing the interest settlement rate.
Buyer/borrower	The party wishing to protect against a rise in interest rates.
Cash settlement	There is no delivery under an FRA, instead any profits or losses are realised as a cash settlement on the settlement date.
Contract amount	The notional sum on which the FRA is based (that is, the principal).
Contract/trade date	The date the FRA is entered into.
Contract rate	The rate of interest agreed between the parties on the contract date (that is, the forward rate).
Maturity date	The date that the settlement period ends (that is, the maturity date of the security which notionally underlies the FRA).
Run	The period or term of the notional underlying security, usually three or six months.
Seller/lender	The party wishing to protect against a fall in interest rates.
Settlement date	The expiry of the forward period, the start of the notional underlying security and the day the settlement sum is paid.
Settlement period	The term of the notional underlying security represented by the number of days between the settlement date and the maturity date.
Settlement rate	The mean rate quoted by the specified reference banks for the settlement period of the notional underlying security. In US$ based FRAs this is commonly given by the Reuters page "LIBO".
Settlement sum	The amount representing the difference between interest calculated at the contract rate and the settlement rate.

Source: BBA and AFMA

Under an FRA the two parties agree the interest rate (the "forward rate") applying to a notional principal amount of an underlying money market security at a forward settlement date. Depending on how the relevant interest rate moves between the trade date and the forward settlement date, one of the parties will owe the other party a net settlement amount equivalent to the difference between the forward rate and the actual rate for the forward settlement date. The party which benefits from a fall in interest rates is defined as the "lender" or "seller". The other party which benefits from a rise in interest rates is the "borrower" or "buyer".

FRAs are normally quoted in terms of monthly combinations of the time to the forward settlement date and the time to maturity of the notional underlying security. For example, an FRA with one month to forward settlement on a three month security is referred to as a "1 × 4". On quote vendor services such as Reuters, FRA dealers generally provide indications of FRA rates in terms of the standard combinations set out below:

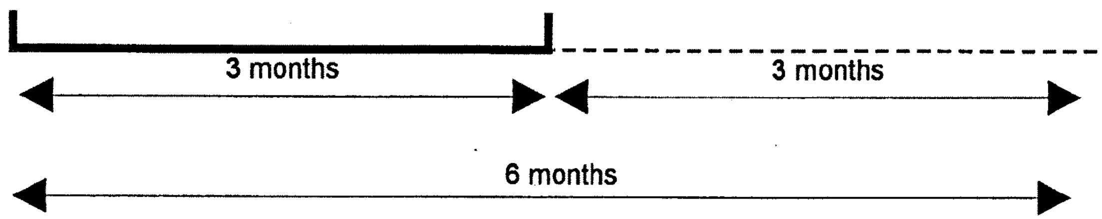
Tenor	**Rate**	**Description of Forward**
1 × 4	7.35	A three month security starting in one month
3 × 6	7.25	A three month security starting in three months
6 × 9	7.23	A three month security starting in six months
3 × 9	7.24	A six month security starting in three months
6 × 12	7.20	A six month security starting in six months

An example of an FRA transaction starting in three months on a three month security is summarised in *Exhibit 4.11*.

Exhibit 4.11

A 3 × 6 Forward Rate Agreement

Trade Date	Forward Settlement Date	Notional Maturity Date
Agree forward rate of 6% pa on US$10m.	Actual settlement rate is 8% pa.* Buyer pays seller $49,019.61.	Maturity is only notional.



$$\text{Settlement amount} = \frac{(.08 - .06) \times 90 / 360 \times 10\text{m}}{1 + .08 \times 90 / 360} = 49{,}019.61$$

Note: *The settlement rate is commonly set two business days prior to the settlement date.

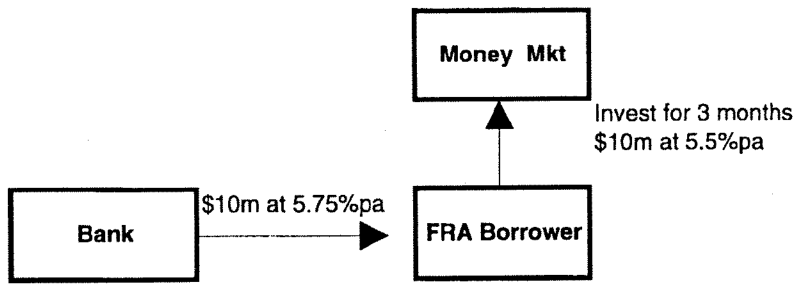
4.2.2. Synthetic replication of an FRA

To understand the pricing of an FRA we will look at how an FRA can be synthetically replicated using cash instruments. The example in *Exhibit 4.11* is from the point of view of the buyer/borrower, and it is agreeing on the interest rate of a three month borrowing commencing in three months time. This can be replicated by borrowing on the trade date for six months and investing the funds raised for three months till the forward settlement date as is shown in *Exhibit 4.12*.

Exhibit 4.12

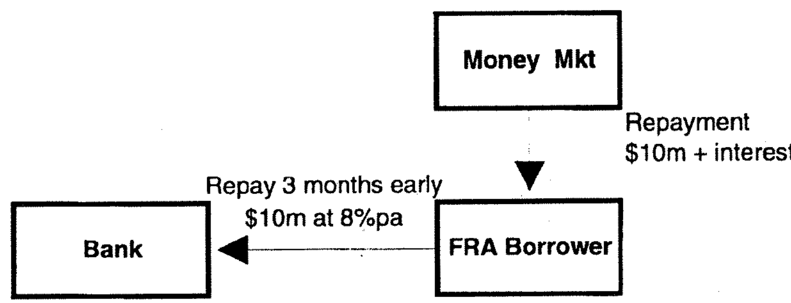
Synthetic Replication of an FRA Buyer/Borrower Position

Trade date



Borrow $10m for six months from a bank where interest is paid quarterly at 5.75% pa. Invest for three months at 5.50% pa. This creates a net interest cost, or cost of carry, over the first three months equivalent to approximately 0.25% pa.

In three months



The three month investment matures and is repaid using the funds received. The borrowing is repaid in three months. As interest rates have risen a profit is crystalised on the transaction.

Exhibit 4.12—*continued*

Forward rate calculations

Investment at three months	=	10m × (0.055 × 90 / 360)
	=	137,500.00
Borrowing interest at three months	=	(1 + 0.0575 × 90 / 360)
	=	143,750.00
Net cost at three months	=	6,250.00
Principal left at three months	=	9,993,750.00
Borrowing at six months	=	10,143,750.00
Forward rate	=	(10,143,750 / 9,993,750 − 1) × 360 / 90
	=	6.00%

Settlement sum calculation

In the synthetic replication the settlement sum will be equivalent to the net cashflows after the borrowing has been repaid early at three months.

Settlement amount at three months	=	Principal left at three months minus the present value of the remaining interest on the borrowing (at 8% pa)
	=	9,993,750 − 10,143,750 / (1 + .08 × 90 / 360)
	=	48,897.06 *

Note: * There is a small rounding difference of $122.55 between this calculation and Exhibit 4.11. This arises because the forward interest rate in this example is actually 6.00375% pa.

As with any forward transaction, a cost of carry will be created depending on the difference between the three and six month interest rates. The forward interest rate will be given by the six month interest rate adjusted for the cost of carry. So, if the six month interest rate is 5.75% pa[6] and the three month rate is 5.5% pa then the cost of carry is 0.25% pa. The forward interest rate is consequently 6.00% pa. At the end of three months there has been a net interest cost of $6,250 which effectively means the principal amount of the original borrowing remaining is $9,993,750.

To replicate the cash settlement of the FRA on the forward settlement date, the borrowing is repaid three months early using the principal amount available at three months. The cost of repaying this borrowing is the present value of the principal or interest accrued at the end of the borrowing. The net cashflow arising from repaying the borrowing is $48,897.06, or approximately the same as the same strategy using an FRA in *Exhibit 4.11*.

Using the terminology developed in Sections 2 and 3:

1. The forward rate in this example is the forward price and is derived using the now familiar cost of carry concept.
2. The settlement sum or amount is the same as the forward value.
3. The present value of an FRA is given by taking the present value of the settlement sum.

4.2.3 A model for FRA forward interest rates

FRAs and interest rate futures introduce a number of new characteristics for the general model developed in Section 3:

- an underlying instrument with a limited life, where
- the calculation is expressed in terms of interest rate not price.

In this section we will convert the generalised price formulae from Section 3 into a formula that generates a forward interest rate on a security which accrues interest for a limited period.

An FRA is an instrument in which the underlying asset is cash provides a constant income in the form of interest payments. We know that the underlying asset is a security which pays interest on a principal amount, S, from today until its maturity date. This future value of the cashflows on this security can be expressed as:

$$FV = S \times (1 + q \times d / D)$$

where the asset income, q, is the yield to maturity on the asset expressed as a per cent per annum and d is the number of days from today until maturity of the asset.

Using the forward pricing model with constant income in Section 3.4.2 we know this can be expressed as:

$$F = S \times (1 + (r - q) \times f / D)$$

where r, the financing cost, is the interest rate over the forward period. The cash price, S, is the principal value of the security and the forward price is

6. To avoid compounding differences this rate, while fixed for a term of six months, is compounded quarterly.

this amount adjusted for the cost of carry. Our aim is to express this same concept in terms of a forward interest rate calculation. Essentially we need to incorporate the cost of carry over the forward period into the interest calculation from the forward settlement date to the maturity date of the underlying security.

In simplistic terms, the interest on the forward security will be equivalent to the difference between the interest earned between today and the forward settlement date and between today and the maturity date of the underlying security. Given that we know the values of q and r, then the amount of interest earned by the forward security can be expressed as:

$$\text{Forward Interest} = S \times (q \times d / D - r \times f / D)$$

The forward interest rate can then be expressed as:

$$\text{Forward Rate} = \frac{\text{Forward Interest}}{\text{Forward Price}} \times \frac{D}{(d-f)}$$

If we insert the formulae above then we have:

$$\text{Forward Rate} = \frac{S \times (q \times d / D - r \times f / D)}{S \times (1 + (r-q) \times f / D)} \times \frac{D}{(d-f)}$$

which simplifies to:

$$\text{Forward Rate } (r_f) = \frac{(q \times d / D - r \times f / D)}{(1 + (r-q) \times f / D)} \times \frac{D}{(d-f)}$$

This formula approximates the calculation performed in *Exhibit 4.12*. However, it ignores the timing of cashflows and the compounding of interest. In most forward interest rate calculations interest rates r and q have different compounding frequencies, which means they cannot be directly compared.

A common method of avoiding the compounding problems is to convert the interest rates into continuously compound rates. This avoids any compounding differences and simplifies the forward rate calculation. We know that the future value of using a continuous rate is as follows:

$$FV = S \times \exp(q \times d / D)$$

Further, we know that the future value of an amount invested for the full term d and an amount invested for the combined term of f and (d−f) must be the same. Using the future value formula we can express this as follows:

$$S \times \exp(q \times d / D) = S \times \exp(r \times f / D + r_f \times (d-f) / D)$$

If we cancel S and take the natural logarithm of both sides of this equation, this simplifies to:

Forward Interest Rate: Continuous Compounding

$$r_f = \frac{q \times d / D - r \times f / D}{d / D - f / D}$$

Where

r_f = Forward interest rate % pa
r = Interest rate to the forward settlement date % pa
q = Interest rate to the maturity date % pa
D = Day count basis (365 or 360)
f = Number of days to the forward expiry date
d = Number of days to the maturity date of the underlying instrument

Examples of underlying assets: FRAs and short-term interest rate futures

Market interest rates are rarely quoted in continuously compounded form, to use this model we need to convert to and from continuously compounded rates. *Exhibit 4.13* provides an example of a spreadsheet which calculates forward interest rates using the continuous compounding method.

Exhibit 4.13
Forward Rate Agreement Calculator

Spreadsheet example

Field	Cell	Cell: Formula
Inputs		
Trade date	01-Nov-95	E8 :
Forward settlement date	30-Jan-96	E9 :
Underlying maturity date	29-Apr-96	E10 :
Spot rate to forward settlement date %	5.5000	E11 :
Frequency (1, 2 or 4)	4	E12 :
Interest rate for maturity %	5.8050	E13 :
Frequency (1, 2 or 4)	2	E14 :
Outputs		
Term to forward settlement in days — f	90	N16 : +E9-E8
Term to maturity in days — d	90	N17 : +E10-E8
Continuous rate to settlement date % — r	5.4625	N18 : =LN(E6/(E7*100)+1)*E7*100
Continuous rate to maturity % — q	5.7224	N19 : =LN(E8/(E9*100)+1)*E9*100
Forward rate % —continuous compounding	5.9822	N20 : =(E11*(E5-E3)-E10*(E4-E3))/(E5-E4)
—quarterly compounding	6.0271	N21 : =4*(EXP(E12/(4*100))-1)*100
—semi-annual compounding	6.0725	N22 : =2*(EXP(E12/(2*100))-1)*100
—annual compounding	6.1647	N23 : =(EXP(E12/100)-1)*100

The approach in this model is to take the interest rates based on market rates and adjust them to continuous rates to calculate the forward rate. Once the continuously compounded rate has been calculated this can be re-converted to any compounding basis required.

4.2.4 A model for FRA valuation

The value of an FRA on the forward settlement date is the difference between the agreed contract rate in the FRA and the prevailing reference interest rate for the remaining term to maturity of the underlying security (the "settlement rate").

There are two general methods of calculating the forward amount, depending on the conventions in the money market. In most markets where money market securities are traded in terms of face value or discounted using the discount method, such as in the United States and United Kingdom, the settlement formula is as follows:

FRA Settlement Calculation: Full Face Value

If $r_s > r_c$ then the settlement sum is

seller pays buyer $= \dfrac{(r_s - r_c) \times d / D \times S}{1 + r_c \times d / D}$

If $r_s < r_c$ then the settlement sum is

buyer pays seller $= \dfrac{(r_c - r_s) \times d / D \times S}{1 + r_c \times d / D}$

Where

r_c = Contract rate % pa

r_s = Settlement rate % pa

d = Settlement period (days till maturity of underlying security)

D = Day count basis (365 or 360)

S = The contract amount

FRA markets commonly using this method: US$ and most European currencies

This calculation is derived so that all obligations of the FRA can be terminated on the forward settlement date rather than the maturity date of the notional underlying security. That explains why the difference between the contract and settlement interest amounts is calculated at the maturity of the notional underlying security and then present values this difference to the forward settlement date.

In markets where money market securities are traded at a discount to face value using the yield method, such as in Australia and New Zealand, the forward value is based on a discounted proceeds calculation as follows:

FRA Settlement Calculation: Discounted Face Value

If $r_s > r_c$ then the settlement sum is

$$\text{seller pays buyer} = \frac{S}{1 + r_c \times d / D} - \frac{S}{1 + r_s \times d / D}$$

If $r_s < r_c$ then the settlement sum is

$$\text{buyer pays seller} = \frac{S}{1 + r_c \times d / D} - \frac{S}{1 + r_s \times d / D}$$

Where

r_c = Contract rate % pa

r_s = Settlement rate % pa

d = Settlement period (days till maturity of underlying security)

D = Day count basis (365 or 360)

S = The contract amount

FRA markets commonly using this method: A$ and NZ$

4.2.4.1 Forward value

Prior to the forward settlement date, the forward value is given by the relevant settlement calculation above. However, instead of the settlement rate, r_s, the prevailing forward rate, r_f, is used. As we noted in Section 3, at initial execution the forward value of an FRA will be zero as the contract rate and the prevailing forward rate are the same. As time passes and the forward rate changes so will the forward value of the FRA. To illustrate this point, *Exhibit 4.14* extends the previous example and examines the change in value of the FRA contract over the forward period using both settlement calculations.

Exhibit 4.14

Forward and Present Value of an FRA

You have entered into the FRA from Exhibit 4.11 as the buyer/borrower. Two months have passed. Calculate the forward rate, the forward value and the present value using the market date provided. Calculate values using both the full face value and discounted face value settlement methods.

Forward Period Remaining

3 months — Enter into 3 × 6 buyer/borrower FRA at 6% pa

↓

1 month — Market data

1 month rate	=	6.00%
4 month rate	=	6.50%

Forward rate calculation

f = 30

d = 120

r = 5.99% pa (continuous compounded rate)

q = 6.43% pa (continuous compounded rate)

$$rf = \frac{.0643 \times 120/360 - .0599 \times 30/360}{120/360 - 30/360}$$

= 6.58% pa (continuous compounded rate)

Then converted to a quarterly rate

rf = 6.63%

Exhibit 4.14—*continued*

Forward values

rc = 6.00%

dc = 90

Full face value $= \frac{(0.0663-0.06) \times 90 / 360 \times 10m}{1 + 0.663 \times 90 / 360}$

$= 15{,}493.20$

Discounted face value $= \frac{10m}{1 + .06 \times 90 / 360}$ $\frac{10m}{1 + .06 \times 90 / 360}$

$= 15{,}264.24$

Present values

Discount the forward values to today using the prevailing one month interest rate

Full face value $= \frac{15{,}493.20}{1 + .06 \times 30 / 360}$

$= 15{,}416.12$

Discounted face value $= \frac{15{,}262.24}{1 + .06 \times 30 / 360}$

$= 15{,}188.30$

4.2.4.2 Present value

The present value of an FRA is easily calculated once the forward value has been generated using the standard present value formulae. An example of this calculation is provided in *Exhibit 4.14*. The present value should be calculated using a zero-coupon interest rate.

4.2.5 FRA risk characteristics

A forward rate agreement is the right to purchase or sell a short-term money market instrument at some date in the future. Correspondingly, the sensitivity to movements in interest rates of an FRA is very similar to the money market instruments.

In the case of an FRA the duration and convexity is equivalent to the underlying instrument. The point value of a basis point (PVBP), however, is less than that of the underlying instrument. An FRA generates gains and losses on the forward settlement date equivalent to the underlying security, however, these amounts are present valued in the PVBP and so are consequently smaller.

An example of the PVBP is provided later in the chapter in *Exhibit 4.18*, where an FRA PVBP is calculated and compared to short-term interest rate futures contracts.

4.3 Short-term interest rate futures

4.3.1 General description

Short-term interest rate futures represent standardised, exchange traded forward contracts on money market instruments. In general, most major currencies have one futures contract on a tradeable short-term money market instrument such as a bank deposit or bank bill. The pricing and valuation of these instruments is very similar to FRAs and the two markets can often be viewed as direct substitutes. The global volume in these instruments is enormous, representing the largest single category of futures contract. A list of the major short-term interest rate futures contracts are listed in *Exhibit 4.15* along with total volume for 1994.

Exhibit 4.15

List of Short-term Interest Rate Contracts and Volumes

Contract	Currency	Exchange(s)	1994 Futures Volumes	
			No of contracts	Face Value (Bn)
Bank accepted bills*	A$	SFE	9,369,008	6,933
3 month Euro-Swiss franc	CHF	LIFFE	1,698,736	1,493
FIBOR futures	DEM	DTB	428,516	305
3 month Euro-deutschmark	DEM	LIFFE	29,312,222	62,575
3 month ECU interest rate	ECU	LIFFE	622,457	814
PIBOR 3 month	FFR	MATIF	13,176,354	2,717
3 month Sterling interest rate	GBP	LIFFE	16,603,152	25,821
3 month Euro-yen	JPY	TIFFE	37,425,846	367
3 month Euro-yen	JPY	SIMEX	6,820,673	67
3 month Euro-lira	ITL	LIFFE	3,456,437	2
Bank accepted bills	NZ$	NZFOE	608,460	393,963
MIBOR 90	ESP	MEFF	3,730,008	155
3 month Eurodollar	US$	CME	104,823,245	524,116
3 month Eurodollar (fungible with CME)	US$	SIMEX	8,687,969	4,344
3 month Eurodollar	US$	LIFFE	91,738	9,174
1 month Eurodollar	US$	CME	1,911,184	191,118
90 day T-bills	US$	CME	1,020,491	510
		Total	239,786,496	1,224,476

Note: *SFE contract upsized from 500,000 to 1,000,000 in April 1995

The benchmark contract for short-term contracts is the Eurodollar contract traded on the Chicago Mercantile Exchange (CME). As the table in *Exhibit 4.15* shows the Eurodollar contract is the most heavily traded reflecting its status as the primary hedging vehicle for short to medium term exposures. It is also traded on the London International Financial Futures and Options Exchange (LIFFE) and the Singapore International Money Exchange (SIMEX). The SIMEX contract is ''fungible'' with the CME contract, which means contracts traded on the two exchanges can be offset.

The Eurodollar contract was the first of the short-term futures contracts when it was listed in 1981. Most other short-term interest rate futures contracts have been a copy of the Eurodollar contract with only the currency, settlement interest rate and face value changed. The A$ and NZ$ bank bill contracts traded on the Sydney Futures Exchange (SFE) and the New Zealand Futures and Options Exchange (NZFOE) respectively, are the only contracts which have different valuation formulae.

In order to familiarise ourselves with short-term interest rate contracts we will firstly review the features of the Eurodollar contract and then examine the differences with other contracts.

4.3.1.1 Eurodollars

The Eurodollar is a cash settled contract on a three month Eurodollar time deposit. The name ''Eurodollar'' derives from the fact that it is a forward contract on a US dollar money market instrument traded in Europe (or London to be more specific). The importance of the contract reflects the importance of the US$ in global financing and the willingness of US-based market participants to use futures.

The CME lists contracts to expire in quarterly rests in March, June, September and December. Currently, there are 40 consecutive quarters listed, that is, expiries out to 10 years. The Eurodollar has obvious appeal to corporations, banks and fund managers with short-term interest rate exposures. However, a substantial driving force behind Eurodollar volumes is from organisations with medium-term exposures such as interest rate swap market makers.

The contract expires on the third Monday of the delivery month and is cash settled against a three-month London Interbank Offered Rate (LIBOR) in a similar fashion to a US$ FRA. If the current month is not a quarterly delivery month then a single ''spot'' contract is also listed to ensure traders have a very short-term instrument. For example, after the March contract expires an April contract is listed. This ''spot'' contract concept is currently only offered on the Eurodollar.

The price of a contract is expressed as:

$$\text{Futures Price} = 100 - \text{Interest Rate} \times 100$$

So, if the current interest rate for a Eurodollar deposit starting on the futures expiry date is 5.00% pa, then the futures price is 95.00. The aim of quoting in terms of price rather than yield is primarily to keep interest rate contracts in line with other price-based contracts on bonds, shares and commodities.

A buyer of a Eurodollar contract gains if the futures price rises (interest rate falls) above the price at which they purchase and the seller gains if the price falls (interest rate rises). Be careful when comparing futures to FRAs as the terminology is opposite; a Eurodollar futures buyer is equivalent to an FRA seller/lender as they both benefit from a fall in interest rates.

While the detailed contract specifications of all short-term interest rate futures contracts are provided in the Appendix to this chapter the major features of the Eurodollar contract are summarised below:

Summary of Eurodollar Futures Specifications

Feature	Description
Underlying	90 day Eurodollar time deposit
Face Value	US$1,000,000
Delivery Months	March, June, September, December and spot month
Delivery Method	Cash settled
Settlement Rate	LIBOR rate for three month Eurodollar deposits on the last trading day
Last Trading Day	Third Wednesday of the delivery month
Quotation Method	100 minus the rate of interest
Valuation Formula	Term deposit
Tick Size	The value of each price point is $25
Margining	Initial margin (currently $500/contract) and daily mark-to-market

The details of most of the other short-term interest rate contracts are similar except for differing face values as is summarised below:

Contract	Exchange	Face Value
90 Day T-Bills	CME	1,000,000
Bank Accepted Bills	SFE	1,000,000
3 Month Euro-Swiss Franc	LIFFE	1,000,000
FIBOR Futures	DTB	1,000,000
3 Month Euro-deutschmark	LIFFE	1,000,000
3 Month ECU Interest Rate	LIFFE	1,000,000
MIBOR 90	MEFF	10,000,000
PIBOR 3 Month	MATIF	5,000,000
3 Month Sterling Interest Rate	LIFFE	500,000
3 Month Euro-Yen	TIFFE	100,000,000
Bank Accepted Bills	NZFOE	500,000

For most contracts the underlying instrument is the same as the Eurodollar, that is, a three month deposit on a discount security where interest is calculated using the discount method. However, in the case of the SFE and NZFOE contracts the underlying instrument is a bank bill, which is a discount security valued using the yield formula.[7] This has an impact on valuation, as discussed in section 4.3.3.

4.3.2 A model for futures prices

The price of a futures contract is equal to 100 minus the forward interest rate. So, the futures pricing model will be based on the forward pricing models from section 3 and the FRA model from section 4.2.3.

The method of synthetically replicating a futures contract is exactly the same as an FRA (see section 4.2.2). However, as we have already noted, futures contracts have a different cashflow profile to similar forward contracts because of initial margins and the daily mark-to-market of gains and losses.[8] As a result of initial margins, futures may need to incorporate a small funding cost, while the impact of the mark-to-market is unknown, as it will depend on the level of interest rates over the life of the futures contract.

In summary, the short-term futures contract price is primarily determined by the prevailing forward rate using the formula in section 4.2.3 above. There is, however, an element of the interest rate which will not be known till expiry of the contract due to the unknown funding requirements during the life of the contract. This can be summarised as follows:

$$\text{Futures Price} = 100 - (\text{Forward Rate} + \text{Funding Adjustment})$$

For contracts with a forward period of up to six months the differences between short-term interest rate futures and FRAs are very small, as well as unknown, and can often be ignored.[9] However, for longer term futures consideration should be given to incorporating the possible funding consequences of a futures contract. It has to be remembered that this is just an estimate; it is common for market users to estimate the "worst case" funding cost requirement and incorporate that into their estimate of the effective forward rate given by short-term interest rates.

7. A discount formula calculates interest based on the future face value and then deducts this from the face value of instrument. A yield formula is a present value of the future face value of the contract. For more on this distinction see Martin, op cit nl, Ch 3.
8. See section 3.6 for a more detailed discussion on this point.
9. In this case the spreadsheet model provided in Exhibit 4.13 is appropriate.

Exhibit 4.16

Synthetic Replication of a Sold Eurodollar Futures Position

In the following example the forward interest rate provided by a sold Eurodollars position is compared against the forward interest rate provided by a forward rate agreement. The Eurodollar contract expires in 1 year's time for a term of 90 days and it is sold today at a price of 92.74. Over the year interest rates fall and the futures price converges to a three month rate of 5% pa. The table summarises the resulting cashflows by quarter.

To simplify the analysis all interest rates are continuously compounded and converted to a 365 day basis. Also, the futures price is assumed to remain steady until the end of the quarter, at which time it falls to the price shown in the table.

Dates	
Trade date	24-Oct-96
Forward settle	24-Oct-97
Maturity date	22-Jan-98

Market Rates	
Overnight/3 month/1 year =	6.00%
1.25 year rate =	6.25%

Implied FRA Rate Calculations			
f	365		
d	455	Forward rate =	7.2639%
r	6.00%		
q	6.25%		

Exhibit 4.16—*continued*

		Interest Rates			Synthetic Replication		Futures Replication				
	Qtr	Overnight rate and 3 mth rate	Futures Price	Current fwd rate	Borrow for 1.25 years @ 6.25%	Invest for 1 year	Mark-to-Market	Funding requirement	Interest on initial margins (o/n - 1% pa)	Quarterly interest	Total interest
Today	0	6.00%	92.74	7.26%	1,000,000.00	(1,000,000.00)		(500.00)			
	1	6.75%	92.30	7.70%			1,100.00	600.00	8.51	18.72001	
	2	7.50%	91.86	8.14%			1,100.00	1,718.72	9.46	41.99346	
	3	8.25%	91.42	8.58%			1,100.00	2,860.71	10.42	70.03446	
Forward	4	9.00%	91.00	9.00%		1,061,836.55	1,050.00	3,980.75	11.38	101.9596	
Maturity	5	9.00%			(1,081,026.40)			4,082.71	11.38	104.2797	336.9872

Effective Forward Rate from Futures Contract	
Investment return after 1 year =	1,061,836.55
Initial borrowings after 1.25 years =	1,081,026.40
Extra funding cost of futures =	(336.99)
Total borrowings after 1.25 years =	1,080,689.41
Effective interest cost =	7.1374%
Difference between FRA and futures =	−0.1264%

Exhibit 4.16 gives an example of the impact that the funding requirements of a futures contract can have on the effective forward rate. In this case we examine the synthetic replication of a single sold Eurodollar contract. As with the FRA synthetic replication we borrow till the maturity date of the underlying (1.25 years) and invest for the forward period (1 year). The additional complication of the future contract is the upfront initial margin of $500 and the mark-to-markets based on the prevailing forward price. In the example the Eurodollar future is sold at a price of 92.74 with one year till expiry. It is assumed that the forward price rises over the life of the futures contract to settle at 95.00 (a three month interest rate of 5.00% pa). As well as the initial margin this generates substantial funding requirements for a sold position. The interest costs of funding these cashflows is incorporated into the effective futures forward rate calculation. Under this scenario, the effective interest rate by nearly 10 bp over the equivalent FRA forward rate.

The difference between the FRA and the effective forward rate in the futures contract is dependant on the actual path taken by interest rates over the forward period. The effective forward rate on the example above is calculated for a range of outcomes in *Exhibit 4.17*.

Exhibit 4.17

Effective Futures Rate Over a Range of Outcomes

Using the example from Exhibit 4.16, the effective forward rate given by the futures contract is calculated for a range of final three month interest rates.

3 Month Rate After 1 Year	Effective Futures Rate	FRA/Futures Difference
5%	7.3620%	0.0981%
6%	7.3273%	0.0634%
7%	7.2784%	0.0146%
8%	7.2152%	−0.0487%
9%	7.1374%	−0.1264%

It is important to note that these funding issues also face the buyer of a futures contract. However, in the case of the buyer a fall in rates generates positive cashflows and improves the effective forward interest rate.

It is important to realise that these funding problems effect both buyers and sellers of futures contracts. The effective forward rates achieved are the same but they have a different effect. For a seller, if interest rates fall the higher effective forward rate increases its cost of borrowing. However, for a buyer, if interest rates fall, the higher effective forward rate represents an improvement in their investment yield.

This difference in cashflows also gives rise to so-called "convexity adjustments" when hedging OTC products such as FRAs and swaps. This issue will be discussed in the following section on valuation.

4.3.3 Valuation of short-term interest rate futures contracts

As we saw above, the valuation of futures contracts can be a source of considerable confusion. It is important to remember that futures contracts have the peculiar property of constant PVBP—there is no distinction between present and future values. Whereas in cash financial assets and OTC derivatives there is a difference equivalent to the time value of money.

In this section we will consider the formulae for determining the contract value of futures contracts and then examine the impact of constant PVBP when using FRAs.

4.3.3.1 Contract values

All open futures contracts are subject to at least a daily mark-to-market, sometimes more.[10] Whenever a mark-to-market is made the valuation method is unchanged and is based on the same formula used to determine the final settlement value of the futures contract. In the case of a Eurodollars contract, the underlying security is the interest on a 90 day deposit—each point change in the futures price is equivalent to a 0.01% pa change in the interest rate in the underlying security. We can then express the value of this type of contract as:

$$\text{Contract Value}_{ED} = \text{Face Value} \times (100 - \text{Price}) \times d / D / 100$$

where price is the prevailing futures price. So if the futures price is 96.24 then the contract value of a Eurodollar contract is:

$$= 1{,}000{,}000 \times (100 - 96.24) \times 90 / 360 / 100$$
$$= 9{,}400.$$

On any given day the mark-to-market gain or loss will be equivalent to the difference in the contract value at the previous mark-to-market price and today's market price.

Most market participants recognise that the contract value formula always implies a constant PVBP, commonly known as the "tick value" in futures markets, equivalent to $25, regardless of the time to expiry of the futures contract. As we saw in section 3.5.3, this is quite an unusual property, and is commonly referred to as "non-convexity". In fact it represents a form of slightly negative convexity as convexity relates to the percentage change in price and in these contracts the PVBP is declining as a percentage of the present value as interest rates fall. It is more appropriate to describe this property as "non-dollar convexity".

This formula can be applied to all short-term interest rate contracts except the two SFE and NZFOE bank bill contracts. In the case of these contracts the contract value is given by a discount security formula using the yield method:

$$\text{Contract Value}_{BB} = FV / (1 + (100 - \text{Price}) \times d / D / 100)$$

10. Some exchanges such as the CME and CBOT mark-to-market twice a day, once at the close of business and once at the end of morning trading. Further, most exchange clearing houses require "intra-day" margins when market conditions are volatile—effectively marking-to-market during the day.

So in the case of the SFE contract a price of 96.24 is:

$$= 1{,}000{,}000 / (1 + (100 - 96.24) \times 90 / 365 / 100)$$
$$= 990{,}813.93.$$

In these instruments the PVBP is not constant as it changes slightly according to the level of interest rates. At the price in the example above the tick value is $24.21, however, at a price of 86.24 the tick value is $23.06. In practice most market participants assume the tick value is approximately $24. Because of the underlying yield discount method, bank bill future contracts are displaying a "positive" PVBP relationship as PVBP rises with a fall in yields. It is important to realise that this convexity is arising just from the valuation of the underlying three month bank bill contract—it is not related to the term to forward expiry at all and is just a reflection of the bank bills convexity on the futures expiry date. As such, it shares the same property as the Eurodollar contract: at a given yield the PVBP of the futures contract will be the same regardless of the number of days to the future's expiry date. In the following section we will refer to this property as "constant PVBP".

4.3.3.2 Time value of money, hedge ratios and convexity adjustments

Futures contracts do not appear to obey the rules of the time value of money (TVM). At a given yield the value of a contract is the same today as in the future—there seems to be no compensation for the passing of time. Of course, this is not the case. We need to remember that futures contracts represent a highly structured form of financial derivative. The constant PVBP is a result of the risk management practices of exchange clearing houses; it has not been specifically designed to work this way.

One of the "cornerstones" of valuation is that all derivatives must obey the TVM. As a result, when using futures contracts they must be used to conform with the TVM characteristics for the purpose they are being used. So, when using futures contracts to hedge an instrument such as physical financial assets or OTC derivatives, the number of futures contracts needs to be adjusted in accordance with the TVM characteristics of the instrument being hedged. This is an extremely important rule, and if it is not followed then it will lead to over or under-hedging.

In this section we will consider using short-term interest rate futures to hedge FRAs. For a future and FRA with the same forward settlement date and notional maturity date, the forward interest rate represents is very similar—with differences arising due to the funding consequences of the futures contract. So, as market interest rates change the current forward interest rate used to determine both FRA rates and futures prices can be viewed as identical. Any differences in the two contracts will result from the different treatment of present values.

This is highlighted in *Exhibit 4.18* where the PVBP of an A$ FRA and equivalent bank bill future are compared. While the future value is the same, the FRA PVBP is lower, reflecting the TVM. If we wish to hedge the FRA with futures contracts the aim is to ensure that any profits and losses today on the FRA are offset by the futures contracts. The appropriate amount of futures to hedge the FRA is that amount which equates the PVBPs of the two instruments—in this case 91 contracts. This equating of PVBPs is often referred to as determining the "hedge ratio". (Note that as time passes the

PVBP of the FRA will rise toward the futures PVBP and the number of futures contracts will need to rise correspondingly.)

Exhibit 4.18

Futures and FRAs: Dealing with Different PVBPs

It is 13 June 1995. Calculate today's PVBP on bank bill futures (BAB) contract and FRA is listed below for face values of A$1 million. Using this information if you had bought $100 million face value of FRAs, how many futures contracts would you sell to hedge the price risk?

Current Market Data

Instrument	Tenor	Underlying Days	Expiry Date	Current Yield	PVBP Yield
1. FRA	15/18	90	13-Sep-96	7.58%	7.59%
2. BAB	Sep-96	90	13-Sep-96	7.58%	7.59%

Note: Zero coupon rate to 13 Sep 96 = 7.90%

Calculations

Instrument	Current Value	PVBP Value	Future Value	Present Value	PVBP
1. FRA	981,652.51	981,628.75	(23.76)	(21.61)	21.61
2. BAB	981,652.51	981,628.75	(23.76)	(23.76)	23.76
		Difference		(2.15)	

Exhibit 4.18—*continued*

Number of futures contract to hedge $100m FRAs.

We assume that the futures price and FRA are very closely correlated. And then apply the hedge ratio formula developed in section 3.5.3.

Hedge Ratio	=	PVBP(FRA) / PVBP
	=	21.61 / 23.76
	=	0.9093

So for every $1 face value of FRA we would sell 0.9099 BAB contracts.

Number of contracts	=	FRA face value × Hedge Ratio / BAB face value
	=	$100,000,000 × 0.9093 / $1,000,000
	=	90.93
	=	91 contracts (rounded to nearest whole contract)

This analysis suggests that FRAs and futures can be equated by adjusting for the differences in PVBP. This is generally true, however, there is another small effect that FRA and swap market makers call the "convexity adjustment".[11] As the name implies it is an adjustment to take account of the differences in the convexity characteristics of futures contracts and other closely related OTC derivatives. This adjustment starts with the recognition that the interest earned on futures mark-to-markets is negatively correlated with the futures price. That is, as interest rates fall futures prices rise. From the point of view of a short-seller of futures contracts this means if interest rates fall then they will pay mark-to-market losses. However, the interest rate to fund these losses is *lower* than the rate prevailing when they executed the transaction. On the other hand, if interest rates rise, the seller receives mark-to-market profits and earns a *higher* rate of investment interest. This creates a natural bias in futures contracts which favours short-sellers whether interest rates rise or fall. This bias works against a buyer of short-term futures contracts.

As with the funding adjustment, the exact effect of this convexity effect is not known till the expiry of the futures contracts. Estimating the convexity adjustment depends on determining an expected path for interest rates over the life of the futures contract—not a straightforward task. As with the funding adjustment the approach is to make an estimate of the net convexity approach and convert it into a forward interest rate equivalent. The implied futures yield should then be equivalent to the FRA rate plus the convexity adjustment.

A simple example of showing how the convexity adjustment can be calculated is provided in *Exhibit 4.19*. In this case we look at the outcome of the previous hedging example assuming interest rates either increase or decrease by 1% pa. The convexity effect can be seen at work in this example—regardless of whether interest rates rise or fall the futures generate a net benefit of around 1 bp. This is an extremely simple example as it assumes that the zero-coupon and forward rates rise or fall by 1% pa on the first day and stay there. It is a complex estimation problem for a small increase in accuracy and as such is generally of concern only to FRA and swap market makers.

11. Convexity is a measure of the sensitivity of duration to a movement in the yield on the underlying instrument. For more details see Martin, op cit n 1, Ch 3.

Exhibit 4.19

Futures and FRAs: Estimating Convexity Adjustments

To see the convexity adjustment at work let us look at the cashflows generated by the transactions in Exhibit 4.18. We will examine the cashflows impact if interest rates rose by 1% pa or fell by 1% pa on the trader date and then stayed there till the forward expiry and examine the relative costs of futures and FRAs.

Original Transactions — 23 June 95

Instrument	Amount		Forward Settlement	Maturity Date	Traded Rate/ Price
1. FRA	100,000,000	Seller/investor	13-Sep-96	12-Dec-96	7.58%
2. BAB	91 contracts	Seller	13-Sep-96	12-Dec-96	92.42

Interest Rates

	Original	After 1% rise	After 1% fall
FRA	7.58%	8.58%	6.58%
BAB	7.58%	8.58%	6.58%
1.25 year ra	7.90%	8.90%	6.90%

Calculations for 1% Rise in Rates

Instrument	Traded value Value (1)	Settlement Value	Settlement Amount	Present Value (2)	
1. FRA	98,165,251.11	97,928,214.59	(237,036.52)	(215,545.25)	
2. BAB	(89,330,378.51)	(89,114,675.28)	215,703.23	215,703.23	
		Difference		157.98	

Exhibit 4.19 — *continued*

Calculations for 1% Fall in Rates

Instrument	Traded value Value (1)	Settlement Value	Settlement Amount	Present Value (2)	
1. FRA	98,165,251.11	98,403,437.92	238,186.80	215,545.25	
2. BAB	(89,330,378.51)	(89,547,128.51)	(216,749.99)	(216,749.99)	
		Difference		(158.75)	

Notes: (1) A long position is shown as a positive, a short position is negative
(2) The futures present value is the mark-to-market, where a gain is positive and a loss is negative.

Transaction Cashflows and Net Benefit of Futures

Date/Cashflows		Interest Rates Up 1% pa		Interest Rates Down 1% pa	
		Futures	FRA	Futures	FRA
Trade date	13-Jun-95				
Traded Contract value		(89,330,379)		(89,330,379)	
Value after rate change		(89,114,675)		(89,547,129)	
Mark-to-Market		215,704		(216,750)	
Expiry date	13-Sep-96				
Settlement		0	(237,036.52)	0	238,186.80
Interest on mark-to-market		24,258.32		(18,853.23)	
Total Future value		239,962.32	(237,036.52)	(235,603.22)	238,186.80
Futures Benefit – $		2,925.80		2,583.58	
Futures Benefit – % pa		0.0120%		0.0105%	
Futures Benefit – bp		1.20		1.05	

This example displays the futures/FRA convexity effect. Given the simple scenario used, the convexity adjustment suggests that the futures yield should be a little more than 1 bp greater than the FRA rate.

In practice, the amount of the convexity adjustment is ignored for forward period of up to one year. For longer forward terms the adjustment is in the order of 1 or 2 basis points—gradually rising as the forward period increases.

4.3.3.3 A complete futures pricing model

If we incorporate all of the special features of a futures contract relative to a forward contract we can summarise the "complete" short-term interest rate futures pricing model as follows:

Futures Price = 100 − (Forward Rate + Funding Adjustment + Convexity Adjustment)

4.4 Forward bonds

4.4.1 General description

Forward bonds are an OTC forward contract on fixed interest bearing bonds and can be likened to an FRA on a long-term interest rate security. Under a forward bond agreement the two parties agree to deliver a specified bond series at a fixed price at a future date. While FRAs relate to a generic money market interest rate such as LIBOR, forward bond agreements relate to a specific bond issue. So, every forward bond agreement must reflect the characteristics (such as issuer, maturity date, coupon and yield) of the underlying bond. This variety of issues combined with the more complex valuation formula, tend to make forward bonds a more complex and specialised transaction than FRAs.

Typically the forward bond market revolves around government and other high quality bond issues as there is already considerable activity in the underlying securities. As we will see, cash, forward and futures transactions in the bond market are closely related.

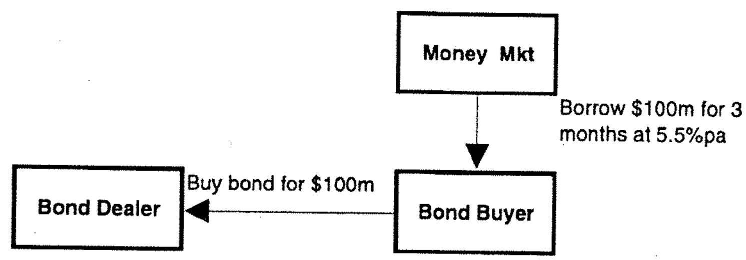
4.4.2 Synthetic replication

A forward bond purchase can be synthetically replicated in the same way as an investor FRA by purchasing the security today and financing the bond for the forward period. This is illustrated in *Exhibit 4.20*.

Exhibit 4.20

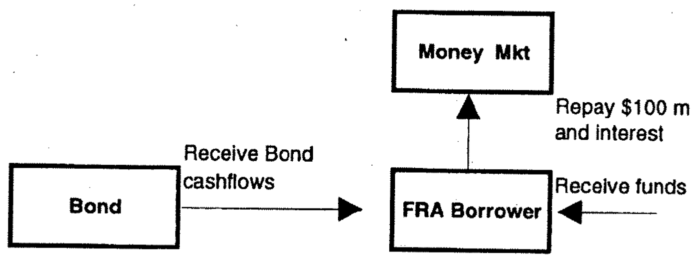
Synthetic Replication of a Forward Bond Purchase

Trade date



Suppose you are a fund manager and you wish to buy bonds but will not receive the funds to buy the bonds for three months. To replicate buying the bonds forward you can buy the bond today for $100 million and fund the purchase by borrowing for three months at a rate of 5.5% pa.

In three months



After three months the funds are received from fund members, these proceeds are used to repay the money market loan and interest. The cost of buying the bonds forward will reflect the borrowing cost and any cashflows received from the bond. Let us assume a coupon of $5 million is received at the end of the three months.

Exhibit 4.20—*continued*

Forward Rate calculations

Original cash cost	=	100,000,000.00	
Bond value at three months	=	95,000,000.00	The bond value is reduced by the coupon payment
Borrowing interest at three months	=	100m × (1 + 0.0575 × 90 / 360)	
	=	1,437,500.00	
Asset cashflows	=	5,000,000.00	
Net Cash value at three months	=	96,437,500.00	
Forward Price	=	96.4375	

As with all forward transactions the forward price represents the current cash price adjusted for the cost of carry. In this case the cost of carry is the difference between the coupons received on the bond and the cost of financing the bond:

$$\text{Cost of Carry} = \text{Financing Cost} - \text{Bond Coupons}$$

A common mistake in forward bond calculations is to use the yield to maturity as the asset return instead of the bond coupon. A forward calculation is concerned with actual cashflows that take place during the forward term—a yield to maturity reflects the asset return over the whole life of the underlying security.

While the example uses a money market interest rate to determine the financing cost this provides only an estimate. If a bond is a government bond then the credit quality of that instrument is likely to be better than most money market instruments. In a synthetic forward purchase the buyer could offer the bond as security and borrow at an interest rate appropriate to the credit quality of the bond. This will be discussed more in the section below on repurchase agreements.

4.4.3 A model for forward bond prices

The synthetic replication indicates that the forward bond price conforms with the "lumpy" income model developed in section 3.6.3, where the lumpy income is the coupon payment on the bond. If the bond does not pay any coupon during the forward period then we use the "no income" model from section 3.6.1.

Forward Bond Price Model—One Coupon Payment

The forward price per $100 can be expressed as:

$$F = S \times (1 + r_1 \times f_1 / D) - c \times (1 + r_2 \times f_2 / D)$$

Where

F = Forward price per $100 face value including accrued interest ("dirty price")
S = Cash bond price including accrued interest
r_1 = Interest rate to the forward expiry date
r_2 = Interest rate between the coupon payment and forward expiry dates
D = Day count basis (365 or 360)
f_1 = Number of days to the forward expiry date
f_2 = Number of days between the coupon payment and forward expiry dates
c = Periodic coupon payment per $100 of face value

It is interesting to note that the forward calculation is based purely on cashflows between today and the forward settlement date. Apart from calculating the initial cash price, S, there is no reference to the bond pricing formula. As with all forward calculations the aim of the model is reflect the cashflow consequence of entraining into a forward transaction.

This formula solves for the forward price. To determine the forward yield, enter the forward price into the bond price calculator and solve for the yield on the forward settlement date. This yield will reflect the cost of carry, however, it is not just a function of the difference between the financing cost and coupon rate, it also reflects the timing and payment of coupons.

This model only allows the incorporation of one coupon payment. Including other coupon payments is simply a matter calculating the future value of each extra coupon using the same methodology as the first coupon. In practice the bulk of forward bond transactions have a forward term of three months or less, so encountering more than one coupon is uncommon.

The best way of building forward bond pricing models is to combine them with a cash bond price calculator. This allows you to automatically generate the current cash price, as well as the next coupon dates and coupon amounts. An example of a forward bond pricing spreadsheet is provided in *Exhibit 4.21*. This spreadsheet assumes that the two short-term interest rates, r_1 and r_2, are the same.

Exhibit 4.21

Forward Bond Price and Yield Calculator

Spreadsheet Example

Field	Cell	Cell Address: Formula (blank for input cells)
Inputs		
Trade date	20-Dec-95	F11 :
Forward settlement date	15-Jun-96	F12 :
Maturity date	15-Jul-99	F13 :
Coupon rate %	8.0000	F14 :
Number of periods/year (1, 2 or 4)	2	F15 :
Current yield to maturity	6.0000	F16 :
Repo rate till forward settlement	4.8500	F17 :
Repo rate day count basis (360 or 365)	365	F18 :
30/360 days count (y or n)	n	F19 :
Underlying Bond Details		
Settlement date	20-Dec-95	Q22 : =F11
Forward date	15-Jun-96	Q23 : =F12
Maturity date	15-Jul-99	Q24 : =F13
Last coupon date	15-Jul-95	Q25 : =COUPPCD(F22,F24,F28,F30)
Next coupon date 1	15-Jan-96	Q26 : =COUPNCD(F22,F24,F28,F30)
Coupon rate %	8.0000	Q27 : =F14
Number of periods/year (1, 2 or 4)	2.0000	Q27 : =F15
Current yield to maturity % pa	6.0000	Q29 : =F16
MS excel day count method	1	Q30 : =IF(F19="n",1,0)
Clean price	106.3361	Q31 : =PRICE(F22,F24,F27/100,F29/100,100,F28,F30)
Accrued interest at trade rate	3.4348	Q32 : =IF(F22=F25,0,ACCRINT(F25,F26,F22,F27,1,F28,F30))

Exhibit 4.21 — *continued*

Field	Cell	Cell Address: Formula (blank for input cells)
Financing (or Repo) Details		
Current financing rate	4.85	Q35 : =F17
Repo rate day count	365	Q36 : =F18
Dirty bond price on trade date	109.7708	Q37 : =F32+F31
Number of coupons during repo	1	Q38 : =IF(F23>F26,1,0)
Number of repo days in forward period	178	Q39 : =F23-F22
Number of days from coupon date to fwd date	152	Q40 : =IF(F38=1,F23-F26,0)
Repo finance cost of bond	112.3671	Q41 : =F37*(1 + F35/(F36*100)*F39)
Forward Price Calculation		
Cumulative coupon 1 value at forward date	4.0808	Q44 : =F27/F28*(1 + F35/(100*F36)*F40)*F38
Dirty forward price	108.2864	Q45 : =F41-F44
Last coupon at forward date	15-Jan-96	Q46 : =COUPPCD(F23,F24,F28,F30)
Next coupon date at forward date	15-Jul-96	Q47 : =COUPNCD(F23,F24,F28,F30)
Accrued interest at forward date	3.3407	Q48 : =ACCRINT(F46,F47,F23,F27,1,F28,F30)
Clean forward price	104.9457	Q49 : =F45-F48
Forward yield % pa	6.2093	Q50 : =YIELD(F12,F13,F14/100,F49,100,F15,F30)*100

To appreciate the impact of a different coupon, *Exhibit 4.22* takes the bond in the previous example and applies different coupon levels. While the yield to maturity and financing rate on each bond is the same, the forward yield at each coupon level is different. In general, as the coupon rate increases the absolute level of the cost of carry increases.

Exhibit 4.22

The Impact of Coupons on Forward Bond Yields

Using the example in Exhibit 4.21, determine the impact on the forward yield of different coupon levels leaving all other inputs unchanged.

Original bond

Trade date	20-Dec-95
Forward settlement date	15-Jun-96
Maturity date	15-Jul-99
Coupon rate %	8
Number of periods/year (1, 2 or 4)	2
Cash yield to maturity % pa	6
Financing/repo rate % pa	4.85
Forward price	104.9457
Forward yield — % pa	6.209315
Yield cost of carry	−0.20931

Impact of Coupons

Coupon % pa	Forward Yield % pa	Cost of Carry % pa
4	6.1973	(0.1973)
8	6.2093	(0.2093)
12	6.2199	(0.2199)
16	6.2292	(0.2292)

4.4.4 A model for forward bond valuation

The forward value of a forward bond is the difference between the contract price in the forward bond agreement and the prevailing forward bond price:

$$\text{Forward Value} = \text{Forward Bond Price} - \text{Contract Price}$$

When determining the present value of the forward bond we need to be wary of the discounting interest rate used. While it is common practice to use the prevailing short-term money market rate to the forward term to determine the present value, it is not strictly correct. Determining the present value involves converting a known future cashflow with specific characteristics into a known amount today. A forward bond forward value is obviously characterised by the difference between the current forward price and the contract price. However, these cashflows are also dependent on the characteristics of the underlying bond—most notably the credit quality of the bond. Consequently, the interest rate used to present value these cashflows should be a short-term interest rate on the bond. So, in simplistic terms the interest rate used for government bonds should be equivalent to a treasury bill rate, while the interest rate for bank bonds should be the same as bank-related money market instruments.

In summary, the present value interest rate should be the same as the financing interest rate, r_1, used in the forward bond price formula. So, on a simple interest basis the formula is as follows:

$$\text{Present Value} = \text{Forward Value} / (1 + r_1 \times f / D)$$

In *Exhibit 4.23* an existing forward bond position is marked-to-market by calculating the current forward price and then determining the forward and present value of the forward bond. The forward value can be determined with reference to the "dirty" or "clean" price of the bond. Either method is acceptable as the difference in the two is simply accrued interest. In this example we use the clean price, which gives just the capital gain or loss on the position.

Exhibit 4.23

Forward Bond Valuation

Suppose you have purchased forward $100 m face value of bonds on the following basis:

Original forward purchase

Trade date	17-Jun-96
Forward settlement date	02-Nov-96
Maturity date	15-Dec-05
Coupon rate %	6.5000
Number of periods/year (1, 2 or 4)	2
Current yield to maturity	7.2500
Repo rate till forward settlement	8.0000
Repo rate day count basis (360 or 365)	360
30/360 days count (y or n)	n
Dirty forward price	97.8629
Clean forward price	95.3765
Forward yield % pa	7.1988

Exhibit 4.23—*continued*

On 20 September interest rates have risen and you decide to mark the position to market (that is, calculate its present value). The forward price is now as follows:

Forward price on 20 September

Valuation date	20-Sep-96
Forward settlement date	02-Nov-96
Maturity date	15-Dec-05
Coupon rate %	6.5000
Number of periods/year (1, 2 or 4)	2
Current yield to maturity	7.7000
Repo rate till forward settlement	8.4000
Repo rate day count basis (360 or 365)	360
30/360 days count (y or n)	n
Dirty forward price	94.8213
Clean forward price	92.3350
Forward yield % pa	7.6829

In general the forward value is calculated excluding the effects of accrued interest, that is using the "clean" price.

Forward value = (forward price − contract price) / 100 × face value
= (92.335 − 95.3765)/100 × 100,000,000
= (3,041,500)

Present value = (forward value / 1 + repo rate × f / D)
= − 3,041,500 / (1 + .084 × 43 / 360)
= (3,011,287)

4.4.5 *Repurchase agreements*

In any forward bond market, a large proportion of forward transactions are linked to repurchase (repo) and reverse repurchase (reverse repo) agreements. In some markets it is estimated that the majority of bond dealer transactions is some form of repo or reverse repo. The bulk of bond repos are very short-term—in the order of one day to one week. Even in very liquid markets such as the US treasury bond market, repos for longer than six months are relatively rare.

A repo is the simultaneous execution, with one counterparty, of the sale of a cash bond and a forward bond purchase. That is, it is an agreement to sell a bond today and repurchase it at a date in the future at a fixed price.[12] A reverse repo is the simultaneous purchase of a cash bond and forward bond sale. A repo is arranged so that the sale of the bond today is at the prevailing cash price for the bond and the future repurchase of the bond is based on the forward bond price. Given that we know the cash price of the bond, the forward period and the coupon on the bond the only unknown in a repo is the financing interest rate. This makes the financing interest rate the key variable in any repurchase agreement and explains why this financing interest rate in the forward pricing formula is referred to as the ''repo rate''.[13]

Exhibit 4.24 diagrammatically illustrates the mechanics of a repo.

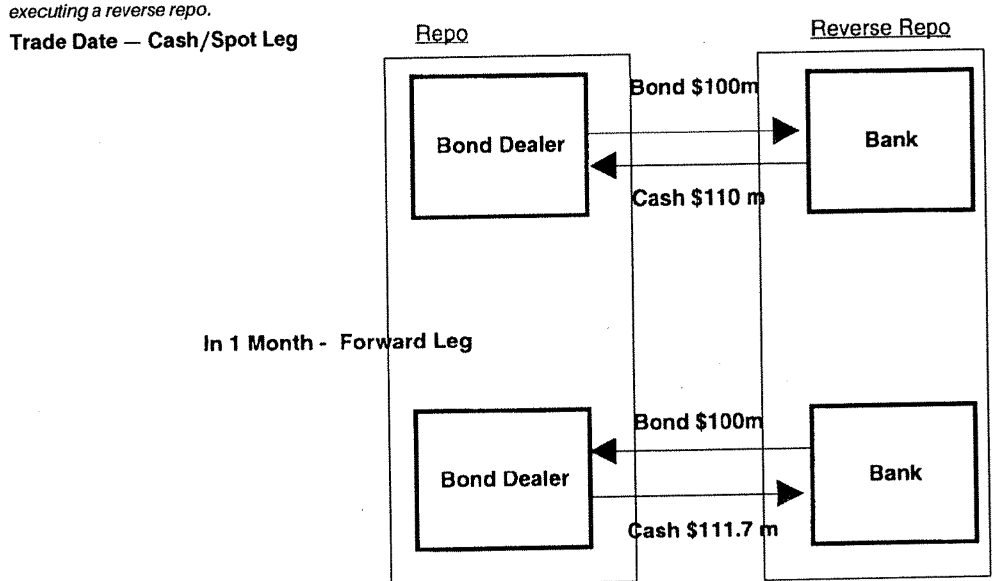
12. Other names for repo transactions include buy-backs, reciprocal purchase agreements and bond lending. All have similar economic results, however, the mechanics can be different. For more detail see T Shanahan, ''The Repo Market'' (1991) (Summer) *Journal of International Securities Markets*.

13. In some markets this is referred to as the ''cash'' or ''term'' rate. Both of these terms can be confused with other interest rates, so they will be avoided in the remainder of this book.

Exhibit 4.24

Repurchase and Reverse Repurchase Agreements

A bond dealer decides to execute a Repurchase Agreement on a $100 million bond holding with a bank. The current cash price of the bond is 110 per $100 face value. The first leg of the transaction consists of the bond dealer selling the $100 million face value of bonds and the bank paying $110 million in cash for the bond. At the same time the bond dealer agrees to buy the bonds back from the bank in 1 month's time. Note that the bank is executing a reverse repo.



At the end of 1 month the dealer buys the bond back. The price that they pay will be equivalent to the forward price — in this case a price of 111.7.

The effect of this transaction is to allow the bond dealer to borrow money but retain a forward ownership of the bonds. The bank has invested cash with the dealer and holds the bonds as a form of security.

As we discussed in Section 3,[14] the effect of a repo is to shift cashflows from one point of time to another—it does not actually change the participant's exposure to changes in the value of bonds. In *Exhibit 4.24*, the repo has provided a very efficient method of funding its bond positions, as it retains the same long exposure to bond price movements. However, it is able to finance this bond holding at a rate which is usually lower than its normal funding rate. Notice that by continually using repos, that is executing a new repo as soon as each repo matures, the dealer could fund its bond holding over long periods at an attractive rate. This explains why the judicious use of repos is a fundamental aspect of dealing in bonds.

Some of the reasons behind entering into repos and reverse repos are summarised as follows.

4.4.5.1 Repos

1. Obtaining funding at an attractive rate.
2. "Grossing up" investment returns—selling bonds already owned into a repo and then using the cash to invest in more securities.
3. Both repos and reverse repos offer a method of liquidity management to central banks. Repos are widely used to manage the cash position of the banking system. A repo allows the central bank to withdraw cash from the economy today and then re-inject that cash at a date in the future when it will be required.

4.4.5.2 Reverse repos

1. Investors wishing to invest cash and obtaining the underlying bond as security.
2. Often banks are required to hold government bonds for regulatory reasons. Under a reverse repo, the bank could obtain ownership of the bond. However, it is not exposed to the potential price volatility of the underlying bond.
3. An organisation which has created a short position in bonds (for example from the maturity of a forward sale or from an option transaction) could cover this position temporarily through a reverse repo.

A fascinating aspect of repos is that for every term to maturity there are multiple repo rates. Every bond series issued has its own characteristics such as the issuer, the maturity date, the coupon payment dates, and the coupon amounts. As we saw in *Exhibit 4.22*, just by varying the size of the coupon changes the forward yield to maturity on otherwise identical bonds. Given these different characteristics, the market can have greater or less interest in owning bonds. If a particular bond series is in demand, then the cost of financing it will be lower than less in demand bonds. This is because reverse repo counterparties will be willing to invest cash at a lower rate to obtain temporary ownership of highly favoured bonds.

The US treasury bond market is the most liquid cash and forward bond market in the world. While the issuer is constant and the terms similar the repo rate on different bond issues can vary substantially depending on the level of demand for specific issues. As an example, in late October 1995

14. See section 3.3.

while the overnight US$ interest rate was 5.75% pa, the repo rate for treasury bonds varies from 5.50% pa down to 1.75% pa for the series, which are in heavy demand. Reasons for these very low repo rates reflects the existence of large short positions in these securities relative to their supply. These short positions are covered with reverse repos, and as the availability of bonds declines the short-position holders are willing to accept a lower return on the cash invested in a reverse repo.

We can solve for the repo rate implied in a forward bond transaction by re-arranging the formula from section 4.4.3 as follows (if we assume r_1 and r_2 are the same):

Calculating the Implied Repo Rate

The repo rate in the forward leg of a repo can be solved as follows:

$$r_1 = \frac{F - S + c}{S \times f_1 / D - c \times f_2 / D}$$

Where

F	=	Forward price per $100 face value including accrued interest at futures date (dirty price)
S	=	Cash bond price including accrued interest
r_1	=	Repo rate to the forward expiry date
D	=	Day count basis (365 or 360)
f_1	=	Number of days to the forward expiry date
f_2	=	Number of days between the coupon payment and forward expiry dates
c	=	Periodic coupon payment per $100 of face value

Exhibit 4.25 calculates the implied repo rate on a forward bond transaction using this formula.

Exhibit 4.25

Repo Rate Calculation

Calculate the implied repurchase rate in the following forward bond transaction.

Cash Bond Details

Trade date	20-Dec-95
Forward settlement date	15-Jun-96
Maturity date	15-Jul-99
Coupon rate %	8.0000
Number of periods/year (1, 2 or 4)	2
Current yield to maturity	6.0000
Dirty cash price	109.7708
Day count basis	Act/Act

Forward Details

Dirty forward price	108.2864
Repo rate day count basis (360 or 365)	365
Number of days in forward period	178
Coupon payment on 15-Jan-95	4.0000
Number of days from coupon date to fwd date	152

Repo calculation

Inputs:

$S =$	109.7708	$f_1 =$	178
$F =$	108.2864	$f_2 =$	152
$c =$	4.0000	$D =$	365

Formula:

$$r = \frac{F - S + c}{(S \times f_1 / D - c \times f_2 / D)}$$

$$= 4.8500\%$$

4.5 Bond futures

4.5.1 General description

Bond futures represent a standardised, exchange-traded forward bond contract. Like short-term interest rate futures contracts they have become an integral part of most financial markets, and typically represent a benchmark for long-term interest rate transactions.

The pricing and valuation of these instruments is derived from the forward bond calculations in section 4.4 above. As with all futures contracts, adjustments may need to be made for margin funding costs. The users of both bond futures and forward bonds are usually very similar and the reasons these products are used are closely related. In those countries where the underlying cash market is liquid, the volume in bond futures is substantial.

A list of the major bond futures contracts are listed in *Exhibit 4.26* along with the total volume for 1994.

Exhibit 4.26

List of Bond Futures Contracts and Volumes

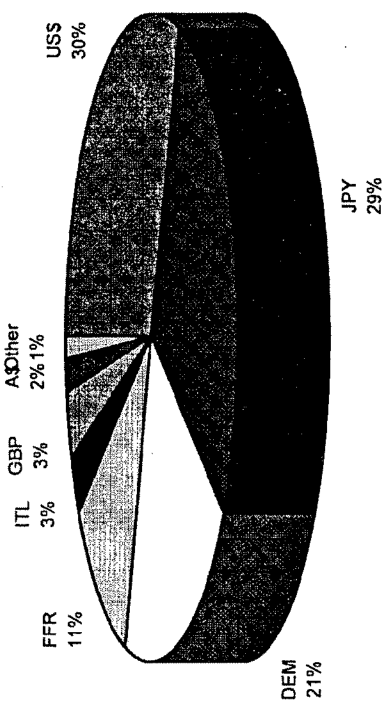
Contract	Currency	Quote Method	Delivery Method	Exchange(s)	1994 Futures Volumes	
					No of contracts	Face Value (Bn)
A$ 3 year bond	A$	Yield	Cash settled	SFE	9,709,791	719
A$ 10 year bond	A$	Yield	Cash settled	SFE	800,263	59
Medium term notional bond (BOBL)	DEM	PPH	Physical delivery	DTB	5,647,859	402
German government bond (Bund)	DEM	PPH	Physical delivery	LIFFE	37,335,437	6,642
German government bond (Bund)	DEM	PPH	Physical delivery	DTB	14,160,460	2,519
Spanish government bond	ESP	PPH	Physical delivery	MEFF	13,191,835	548
10 year government French bond	FFR	PPH	Physical delivery	MATIF	50,153,150	5,171
Long gilt future	GBP	PPH	Physical delivery	LIFFE	19,048,097	1,481
Italian government bond (BTP)	ITL	PPH	Physical delivery	LIFFE	11,823,741	1,492
Japanese government bond (JGB)	JPY	PPH	Cash settled	LIFFE	610,925	599
10 year Japanese government bond (JGB)	JPY	PPH	Cash settled	TSE	12,999,698	12,754
10 year Japanese government bond (JGB)	JPY	PPH	Cash settled	SIMEX	443,564	218
NZ$ 3 year bond	NZ$	Yield	Cash settled	NZFOE	101,229	7
NZ$ 10 year bond	NZ$	Yield	Cash settled	NZFOE	42,541	3
Swiss government bond	SFR	PPH	Physical delivery	SOFFEX	949,657	14
2 Year treasury notes	US$	PPH	Physical delivery	CBOT	939,043	188
5 Year treasury notes	US$	PPH	Physical delivery	CBOT	12,462,838	1,246
10 Year treasury notes	US$	PPH	Physical delivery	CBOT	24,077,828	2,408
US treasury bonds (T-bonds)	US$	PPH	Physical delivery	CBOT	99,959,881	9,996
US treasury bonds	US$	PPH	Physical delivery	MIDAM	1,385,904	69
					315,843,741	46,535

Notes: 1. "PPH" means the bond futures contracts are quoted in price terms as price per hundred units of face value.

2. "Yield" quotes mean the futures are quoted in terms of yield to maturity where the futures price is equivalent to 100 minus the yield.

Exhibit 4.26—*continued*

Composition of Bond Future Volume by Currency



Two important differences between these futures contracts is highlighted in the table in *Exhibit 4.26*:

1. *Quote method*: The price of most bond futures contracts is quoted as the current price per 100 units of face value (shown in the table as "PPH"). For these contracts, the futures price is essentially the same as the forward price previously calculated in section 4.4.3. The other alternative is the "yield" method. Futures prices are quoted as 100 minus the yield to maturity of the underlying forward bond. The futures quotation method is usually a reflection of the local bond market convention for quoting cash bond prices.
2. *Delivery method*: There are two alternative methods with which bond contracts are terminated: physical delivery and cash settlement. As its name implies, physical delivery requires that all open contracts at expiry must deliver (the futures contract seller), or take delivery of (the buyer), a defined amount and type of bonds. In the case of cash settlement, at expiry, all open contracts are reversed at the final settlement price of the contract (usually on the last day of trading). That is, all obligations under the futures contract are cancelled upon payment or receipt of the cash difference between the original traded price and the final settlement price of the futures contract.

In the case of some contracts, notably the A$ and NZ$ bond futures, they are quoted using the yield method and are cash settled against a basket of underlying bonds. This creates some additional pricing complexities for the forward bond formula, which are highlighted below.

4.5.2 Pricing and valuing bond futures

Conceptually, the pricing and valuation tools developed in section 4.4 can be applied directly to bond futures with the same sort of adjustments as were applied to short-term interest rate futures:

$$\text{Futures Price} = \text{Forward Price} + \text{Funding Adjustment} + \text{Convexity Adjustment}$$

As with all futures, there is no distinction between forward and present values and this should be incorporated into any hedging transaction using the PVBP in the same manner as the short-term futures contract.[15]

While the funding and convexity adjustments discussed in relation to short-term futures should strictly be applied, they are often ignored by market participants. The reason for this is twofold:

1. *Short forward period*: Most of the traded volume in bond futures across all contracts have a relatively short forward period (up to six months). As we have seen in earlier sections the funding and convexity adjustment calculations are usually extremely small for forward periods of under one year.

15. See section 4.3.3 for an example of dealing with the constant PVBP characteristics of futures contracts.

2. *Long-term instrument*: If the funding or convexity adjustment is calculated and spread over the fairly long life of the underlying bond, the impact of the adjustment tends to be fairly small.

Unfortunately, while it describes the conceptual relationship, deriving the final quoted futures price is not quite as simple as the formula above implies. In all bond futures additional adjustments are required depending on the nature of the delivery process. We can divide the futures price calculations into two general groups:[16]

1. *Delivery and conversion factors*—At the expiry of most bond futures contracts (for example, CBOT treasury notes and bonds) a physical delivery of bonds takes place. There is a specified list of approved bonds which can satisfy delivery and a ''conversion factor'' which is indented to convert each bond to the equivalent of the notional bond underlying the contract. This conversion process is not perfect and usually one of the approved bonds becomes the ''cheapest'' to deliver. Effectively, the price of the futures contract is based off the forward price per hundred (multiplied by the conversion factor) of the cheapest to deliver bond.
2. *Yield quotes and basket bonds*—These bonds are cash settled against the average yield on a basket of bonds on the last day of trading of the futures contract. The futures price is given by 100 minus the forward yield of the basket of bonds underlying the futures contract. Because each futures price point is one basis point in yield, and given bonds exhibit convexity, as the futures price changes so does the dollar, or ''tick'' value of each futures price point—the higher the futures price the higher the tick value. Both the SFE and NZFOE's bond contracts trade on this basis.

5. FOREIGN EXCHANGE FORWARDS

5.1 Introduction

The interesting feature about foreign exchange transactions is that they are purely about cashflows. In interest rate markets there is typically an underlying security which has specific interest paying and maturity characteristics and it may be in limited supply. With a foreign exchange transaction there is no specific underlying instrument; any organisation can create its own tailor-made foreign exchange transaction.

Additionally, foreign exchange transactions are a homogenous product and completely fungible.[17] That is, if a company enters into a foreign exchange transaction with a bank which settles in two days time, this can be offset by entering into an opposite transaction with another bank—the only difference will be the profit or loss due to differences in the exchange rate on the original and offsetting deals.

16. For more on this see Martin, op cit n 1, Ch 5.
17. If two financial instruments are fungible then they are perfect substitutes and can be used to replace one another.

Combining the underlying demand to execute foreign exchange deals for trade and capital transactions with this flexibility to create and manage foreign exchange positions, the enormous size and success of the OTC foreign exchange market can be understood. While currency futures exist, their volume is small relative to the OTC market.

The global foreign exchange market is the epitome of the "global financial village"—it is a huge marketplace spread across numerous financial centres and time zones. No matter what the time of day, it is possible to execute foreign exchange transactions involving major currencies. The BIS conducts a survey of global turnover every three years and the results from April 1995 are set out in *Exhibit 4.27*.

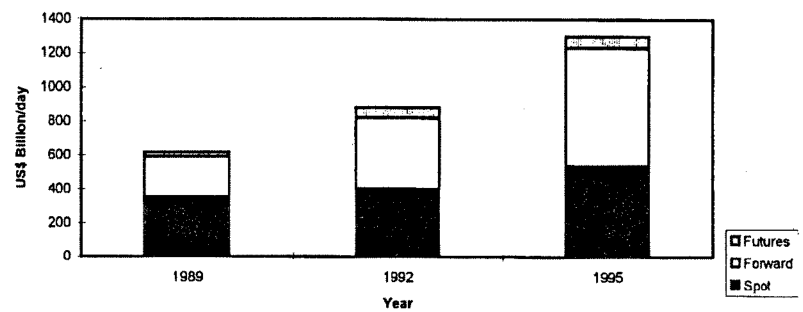
Exhibit 4.27

Global Foreign Exchange Turnover

Daily averages in billions of US Dollars (1)

Transaction	Apr-89	% share	Apr-92	% share	% change 1989-92	Apr-95	% share	% change 1992-95
Spot	350	56%	400	45%	14%	535	41%	34%
Forward (2)	240	39%	420	48%	75%	695	53%	65%
OTC Sub-Total	590		820		39%	1230		50%
Futures (3)	30	5%	60	7%	100%	72	6%	20%
Total	**620**		**880**		**42%**	**1,302**		**48%**

Foreign Exchange Volume



Notes:

(1) Figures have been adjusted for double counting.

(2) Includes outright forwards and forwards which are one leg of an FX swap.

(3) The Apr 95 volume was not provided with the BIS survey. Growth has been estimated from futures turnover growth of 20% from 1992 to 1995 (**Source:** FIA).

Source: BIS

In 1995 total daily volume for spot and forward FX instruments was US$1.3 trillion. This represents an increase in volume of 48% from 1992, an increase on the 1989-1992 change of 42%. The proportion of transactions executed as forwards has steadily increased since the survey was commenced. In 1989 forwards and futures represented 44% of total volume, but by 1995 this has risen to 59%.

An important feature of the market is that the bulk of volume is made up of FX swap transactions—a simultaneous execution of spot and forward or forward and forward transactions. The BIS do not include hard data in the survey but they do note that only 14% of forwards are outright, the remaining 86% are part of an FX swap transaction. This implies FX swap volume of US$ 598 billion per day in April 1995.

Reflecting the global nature of the FX market the survey also points to expanding turnover in most countries, as can be seen from *Exhibit 4.28*. Though, interestingly, the largest market, London, has grown at a more rapid rate since the survey started than its nearest rivals, the United States and Japan. In terms of geographical regions, the importance of Europe has grown in each survey, and it now represents more than half the total global volume.

Exhibit 4.28

Geographic Composition of Global FX Volume

Total OTC turnover by country

	Transaction	Apr-89 % share		Apr-92 % share		% change 1989-92	Apr-95 % share		% change 1992-95
1	United Kingdom	184	26%	290.5	27%	58%	464.5	30%	60%
2	United States	115.2	16%	166.9	16%	45%	244.4	16%	46%
3	Japan	110.8	15%	120.2	11%	8%	161.3	10%	34%
4	Singapore	55	8%	73.6	7%	34%	105.4	7%	43%
5	Hong Kong	48.8	7%	60.3	6%	24%	90.2	6%	50%
6	Switzerland	56	8%	65.5	6%	17%	86.5	6%	32%
7	Germany		0%	55	5%	—	76.2	5%	39%
8	France	23.2	3%	33.3	3%	44%	58	4%	74%
9	Australia	28.9	4%	29	3%	0%	39.5	3%	36%
10	Denmark	12.8	2%	26.6	2%	108%	30.5	2%	15%
11	Canada	15	2%	21.9	2%	46%	29.8	2%	36%
12	Belgium	10.4	1%	15.7	1%	51%	28.1	2%	79%
13	Netherlands	12.9	2%	19.6	2%	52%	25.5	2%	30%
14	Italy	10.3	1%	15.5	1%	50%	23.2	1%	50%
15	Sweden	13	2%	21.3	2%	64%	19.9	1%	−7%
	Other	21.6	3%	61.3	6%	184%	89.2	6%	46%
	Total (1)	717.9		1076.2		50%	1572.2		46%

Note: (1) The total in this table is higher than the actual turnover in Exhibit 4.27 because this table has cross-border double counting. For example, a transaction executed by a bank in the UK and a bank in the USA would be recorded in both countries.

Exhibit 4.28—*continued*

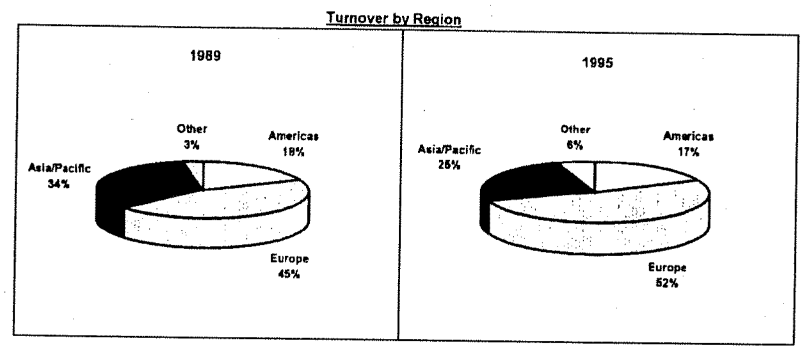


Exhibit 4.29 demonstrates the declining importance of the US$. In 1989 the US$ was one of the legs in 90% of FX transactions but by 1995 this had fallen to 83%. Most of the US$ share has been lost to non-DEM European Economic Union currencies.

Exhibit 4.29

Currency Composition of Global OTC FX Volume

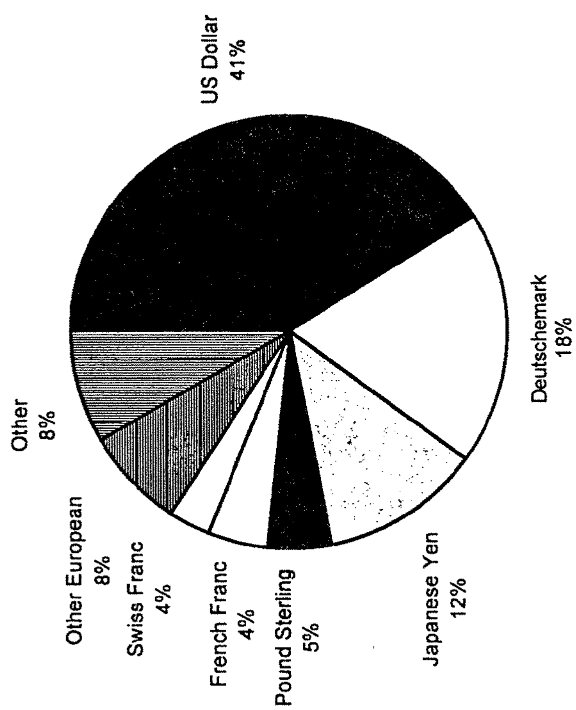
Total OTC turnover by currency as a percentage

	Currency	Apr-89 % share		Apr-92 % share		% change 1989-92	Apr-95 % share		% change 1992-95
1	US Dollar	531	90%	672.4	82%	27%	1020.9	83%	52%
2	Deutschemark	159.3	27%	328	40%	106%	455.1	37%	39%
3	Japanese Yen	159.3	27%	188.6	23%	18%	295.2	24%	57%
4	Pound Sterling	88.5	15%	114.8	14%	30%	123	10%	7%
5	French Franc	11.8	2%	32.8	4%	178%	98.4	8%	200%
6	Swiss Franc	59	10%	73.8	9%	25%	86.1	7%	17%
7	Canadian Dollar	5.9	1%	24.6	3%	—	36.9	3%	50%
8	ECU	5.9	1%	24.6	3%	317%	24.6	2%	0%
9	Australian Dollar	11.8	2%	16.4	2%	39%	36.9	3%	125%
10	Other EMS currencies	17.7	3%	73.8	9%	317%	159.9	13%	117%
11	Other	129.8	22%	90.2	11%	−31%	123	10%	36%
	Total (2)	590	200%	820	200%		1230	200%	

Note:
(1) The % share is expressed as a proportion of the global volume.
(2) As each FX deal involves two currencies the sum of the total volume in each currency is double the estimated global turnover.

Exhibit 4.29—*continued*

Turnover by currency — April 1995



5.2 A model for the forward foreign exchange price

In this section we will convert the generalised price formulae from Section 3 into a formula that generates a forward foreign exchange rate.

A forward FX transaction is an instrument in which a cash in one currency is exchanged for cash in another currency. Either currency can be thought of as the underlying asset. These assets provide constant and known incomes in the form of interest payments. We know that these assets can be exchanged on a spot basis at a rate of 1 unit of the base currency for S units of the terms currency.

The spot exchange rate is the current rate of exchange of two currencies. The synthetic replication above indicates that the forward exchange rate of two currencies is dependant on their respective interest rates. The forward FX pricing formula has a lot in common with the general forward models developed in Section 3, however, the formula needs to be adjusted for the fact that the interest amounts, r and q, are calculated on different currency amounts—1 unit of the base currency for every S units of the terms currency. We also need to incorporate the possibility that the calculation basis underlying each interest rate may differ. Generally for short-term forwards the main differences between money market interest rates are:

- the assumed number of days per annum (either 360 or 365); and
- whether the rate is based on a discount or yield calculation.[18]

The simplest approach is to look at the forward value of cash in each currency. As we have already discussed, cash by itself earns no income so we can apply the general "no-income" pricing model:

$$F = S \times (1 + r) \times f / D$$

This can be rewritten as follows for the forward cash value for the base (F_B) and the terms currency (F_T) as follows:

$$\text{Base Currency: } F_B = 1 \times (1 + r_B) \times f / D_B$$

$$\text{Terms Currency: } F_T = S \times (1 + r_T) \times f / D_T$$

where S is the exchange rate. The forward rate of exchange between these two currencies will be given by the ratio of these two forward amounts. This is summarised below:

18. See Martin, op cit n 1, Ch 3.

Short-term Forward Foreign Exchange Price

Using simple interest, the calculation is as follows:

$$F = \frac{S \times (1 + r_T) \times f / D_T}{(1 + r_B) \times f / D_B}$$

Where

- F = Forward exchange rate
- S = Spot exchange rate
- r_T = Terms currency interest rate to forward date
- r_B = Base currency interest rate to forward date
- D_T = Terms currency day count basis (365 or 360)
- D_B = Base currency day count basis (365 or 360)
- f = Number of days to the forward expiry date from the spot settlement date.

An example of this calculation is set out in *Exhibit 4.30*.

Exhibit 4.30

Forward Pricing Example

The current spot rate for US$/CAD is 1.3513. Calculate the rate of a forward FX deal settling in 30 days from the spot date. The 1 month US$ interest rate is 6.25% pa and the CAD rate is 8.2% pa. Calculate the implied forward FX rate.

S = 1.3513		f = 30	
rT = 8.20%		DT = 365	
rB = 6.25%		DB = 360	

$$F = \frac{S \times (1 + rT) \times f / DT}{(1 + rB) \times f / DB}$$

$$= \frac{1.3513 \times (1 + .082) \times 30 / 365}{(1 + .0625) \times 30 / 360}$$

$$= \mathbf{1.3534}$$

Forward Points = 0.0021 premium

There are a number of assumptions underlying this calculation, which, if they do not hold, may require an adjustment to the formula:

1. *Simple interest*: There is assumed to be no compounding in the interest calculation. This can be adjusted for by applying the compound interest calculation.
2. *Zero-coupon*: The interest rates are assumed to be zero-coupon rates. This is generally a satisfactory assumption for forward FX deals of up to six

months, most interest rates longer than that contain re-investment risk. This assumption is relaxed in the section on long-term foreign exchange below.

3. *Interest rates are yields*: The formula assumes that interest rates are yields; that is, the interest amount is given by multiplying the principal value today. If an interest rate is from a discount security which calculates interest on a discount basis, then this will need to be converted to a yield basis.[19]
4. *Calculations from spot date*: The forward period is from the spot settlement date to the forward settlement date. Strictly speaking, the interest rates that should be used are the two day forward interest rates, however, this is usually ignored as the impact is minimal.

Bids and Offers

A forward FX rate is derived from three market rates: the spot FX rate and the interest rate in both currencies. In each case the appropriate bid and offer rate has to be identified. In FX markets the "bid" is the rate at which a dealer is willing to buy the base currency; and the "offer" is where the dealer will sell the base currency—the bid rate is lower than the offer rate. As its counterparty we do the opposite of the dealer so we will sell the base currency at the bid and buy it at the offer. As we noted earlier, from an end-user's point of view a rule of thumb is that we will always lose money from the bid-offer spread, that is you "buy high and sell low".

Unfortunately, in money markets bids and offers can have two meanings, depending on the underlying money market instrument. If the underlying instrument is a direct term deposit with a bank, then the bid will be where you can invest funds with the bank, and the offer is where you can borrow from the bank—the bid interest rate is lower than the offer interest rate. However, if the instrument is a tradeable security, such as a bank bill, where the underlying security is bought and sold according to its present value, the bids and offers are expressed in terms of interest rates. As a lower price implies a higher yield, the "bid" interest rate is higher that the "offer" interest rate. One way of avoiding confusion with these conventions is to apply the rule of thumb to interest rates: an end-user will invest at the lower interest rate and borrow at the higher interest rate.

Assuming that we are end-users rather than FX dealers, the simplest method of identifying the appropriate rates is to make use of the synthetic replication concept to identify whether to use the bid or offer. If we wish to buy the base currency forward in three months, the synthetic replication is to buy the base currency in the spot FX market, borrow in the terms currency and invest in the base currency. Using this, then, we use the appropriate rates as follows for a forward purchase or sale:

19. Ibid.

Calculating Forward Rates (End-User Perspective): Bids and Offers

Leg	Buy Base Currency Forward	Sell Base Currency Forward
Spot Foreign Exchange	offer	bid
Base Money Market[1]	bid (low rate)	offer
Terms Money Market[1]	offer (high rate)	bid

Note: The quote convention in this table assumes underlying instruments are bank deposits

An example of using the correct bids and offers is set out in the first part of *Exhibit 4.31*.

5.3 A model for forward FX valuation

The forward value of a forward FX transaction is the difference between the original contract price and the prevailing market forward price. A forward FX transaction is usually expressed in terms of a constant amount of the base currency, while the terms currency is left to vary and all gains and losses are generated in the commodity currency. The forward value in the terms currency of a forward FX where the base currency is purchased/terms currency sold, the deal can be expressed as follows:[20]

$$\text{Forward Value}_{TERMS} = \text{Base Amount} \times F_M - \text{Base Amount} \times F_C$$

where F_M is the prevailing forward market rate and F_C is the contract rate.

If, instead, the terms currency is held constant then the situation is inversed as follows for a bought base currency position:

$$\text{Forward Value}_{BASE} = \text{Terms Amount} / F_C - \text{Terms Amount} / F_M$$

The mark-to-market on a forward FX position will be given by present valuing these forward value calculations. It is essential that we correctly identify the currency in which the forward value is generated. As it is usually the case that the interest rate differs between the two currencies, the present value of a forward cashflow in either currency will be different. So, if the forward value is calculated in the terms currency it should be present valued using the terms currency interest rate:

$$\text{Present Value}_{TERMS} = \text{Forward Value}_{TERMS} / (1 + r_T \times f / D)$$

otherwise if the forward value is in the base currency:

$$\text{Present Value}_{BASE} = \text{Forward Value}_{BASE} / (1 + r_B \times f / D)$$

Exhibit 4.31 provides a complete worked example for determining the current mark-to-market revaluation of a forward FX position.

20. A sold base currency/purchased terms currency has the same formula with the forward rates F_M and F_C reversed.

Exhibit 4.31

Forward FX Transaction

You currently hold a forward FX position where you buy GBP/sell USD 10 million at 1.5340. This contract will settle in 92 days time. Given the market rates below calculate the current revaluation of this position.

Market rates

	bid	offer
Spot GBP/USD:	1.5629	1.5634

Three Month Money Market rates (Deposit rates)

	Bid	Offer
GBP	6.55	6.65
USD	5.53	5.75

1. Calculate current forward FX price

We wish to revalue a bought GBP forward FX position. The current market value will be given by the forward FX price at which an offsetting position can be put in place. Consequently we need to generate a three month forward price to sell GBP/buy USD. The inputs to the forward price model will be the FX spot rate bid, the GBP money market offer and the USD money market bid.

S	=	1.5629	f	=	90
rT	=	5.53%	DT	=	360
rB	=	6.65%	DB	=	365

$$F = \frac{S \times (1 + rT) \times f / DT}{(1 + rB) \times f / DB}$$

$$= \frac{1.5629 \times (1 + .0553) \times 90 / 360}{(1 + .0665) \times 90 / 365}$$

$$= \mathbf{1.5589}$$

2. Calculate the forward and present values

The principal value of the transaction is expressed in the terms currency, that is, USD 10 million so we use the formula:

Forward Value = Terms Amount / Fc − Terms Amount / Fm

Where

Terms Amount	=	10,000,000 USD
Fc	=	1.5340
Fm	=	1.5589

Forward Value	=	10,000,000 / 1.5340 − 10,000,000 / 1.5589
	=	104,125 USD
	=	66,794 GBP
Present Value	=	Forward Value / (1 + rT × f / DT)
	=	102,705 USD
	=	65,715 GBP

The mark-to-market revaluation on this position is a profit of USD 102,705

5.4 Foreign exchange swaps

The BIS survey in section 5.1 identified that a very large percentage of global foreign exchange volume is in the form of FX swaps—the simultaneous execution of offsetting spot and forward foreign exchange transactions.[21] These transactions need to be distinguished from cross-currency interest rate swaps ("currency swaps"). FX swaps are generally short-term in nature and consist of just a spot and forward leg. Currency swaps are longer-term instruments, typically in the range of 1 to 10 years, and involve a series of periodic interest exchanges. While they are different instruments, the overall economics of the two transactions have the same effect of switching one currency exposure to another for the life of the instrument.

The popularity of FX swaps is a reflection of the important tasks for which it can be used:

- the temporary conversion of one currency to another without creating an exposure to foreign exchange movements;
- extending existing forward FX positions;
- a vehicle for trading the pure interest differential of two currencies without an exposure to exchange rates; and
- arbitrage related activities such as covered interest arbitrage.

Like a bond repurchase agreement, an FX swap allows the two counterparties to temporarily exchange one asset for another—in this case one currency for another. By itself, the transaction does not create an outright

21. An FX swap can also be the simultaneous execution of offsetting forward transactions with different terms.

foreign exchange exposure, as any gains or losses on the initial spot currency exchange are offset by gains and losses on the final forward exchange.

A useful example of the impact of an FX swap is the extension of an existing forward position which expires on the current spot settlement date. The mechanics of an extension or rollover of an existing FX position consist of two legs:

1. *The spot leg*: a spot FX deal is executed which offsets the cashflows from the original forward deal. This will typically crystallise a gain or loss equivalent to the difference between the original forward rate and the current spot rate.[22]
2. *The forward leg*: a forward FX deal is executed at the prevailing forward rate. The net cost of this transaction will be the spot and forward spreads and the forward points.

If these two legs are executed separately then there is a risk that the spot foreign exchange rate moves between executing the spot and forward legs. An FX swap reduces a rollover to one deal and removes the foreign exchange risk by simultaneously executing both legs. In an FX swap the transaction becomes insensitive to the spot exchange rate used, and the important variable in the transaction is the forward points—it is for this reason that forward points are also referred to as swap points or the swap rate.

Exhibit 4.32 demonstrates how a forward FX position expiring in two days can be extended for another three months with one FX swap transaction.

22. Some countries still allow "historic rate rollovers". In these transactions the gain or loss is not crystallised on the rollover date and is carried till the forward transaction finally matures. This introduces another component to calculating the forward FX price—an interest adjustment for any gains or losses funded by the FX dealer. In many countries these transactions have been banned due to the possibility of concealing trading losses for long periods of time.

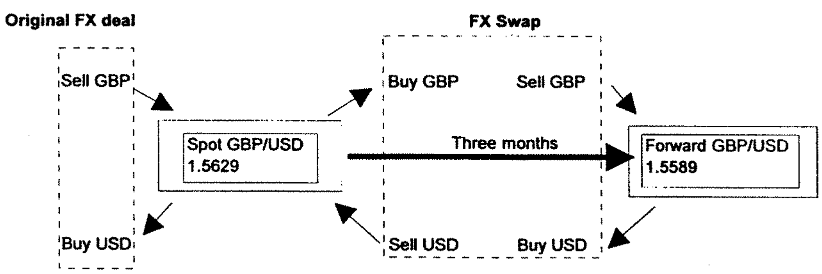
Exhibit 4.32

Extending a Forward FX Deal With an FX Swap

A company has a sold GBP 20 million/bought USD transaction expiring in two business days time. It wishes to extend this position for three months and decides to use an FX swap to do so where it buys GBP spot and sells GBP forward. The effect of the FX swap is to fund the settlement of the original transaction in two days time and create a new sold GBP/buy USD forward FX transaction in three months time. The net cost of this transaction is the forward or swap points over the next three months. Note that to simplify cashflows the original sold GBP position and the current spot FX rate are assumed to be same—this happy state of affairs rarely occurs in reality.

Current market rates:	Spot GBP/USD:	1.5629
	3 month Forward GBP/USD:	1.5589
	Forward or Swap Points:	(40)

Diagramatic representation of FX swap



Cashflows of original deal and FX swap

Cashflows		GBP	USD
Day 0	Execute FX Swap		
Day 2	Settle Original FX deal	(20,000,000)	(31,258,000)
	Spot Leg of FX swap	20,000,000	31,258,000
	Net cashflows	—	—
Day 92	Forward leg of FX swap	(20,000,000)	31,178,000

Net Cost of FX swap =	(80,000)

5.5 Long-term forward foreign exchange transactions

5.5.1 General description

As the name implies long-term forward foreign exchange (LTFX) transactions are a longer-term version of the forward FX transaction. It is an agreement between two parties who wish to agree on an exchange of currency cashflows at some date, possibly years in the future. For our purposes, we will consider an LTFX as any forward contract longer than six months.

LTFX contracts are relatively small proportion of total FX market volume. The BIS foreign exchange turnover statistics indicate that only 1% of FX volume is for longer than one year. While most FX dealers will quote LTFX transactions out for five years, given the lower liquidity of these instruments the bid-offer spread is wider than short-term forwards and reversing the position will not be as straightforward.

Typically, LTFX contracts are associated with hedging the FX exposures created by long-term borrowings or income streams created by assets in foreign currencies. Often LTFX transactions and currency swaps can be used interchangeably, the advantage of LTFX is that they can be more easily tailored to meet uneven future cashflows.

5.5.2 Pricing and valuing LTFX transactions

The synthetic replication of an LTFX is the same as a forward FX transaction—the borrowing and lending legs are just for a longer term. Similarly, the valuation procedure is identical. However, the valuation of LTFX is complicated by the following two effects:

1. *Zero-coupon yield*: The forward pricing and valuation models assume that there are not interest cashflows during the forward period—that is, the interest rates are zero coupon rates. This is a reasonable assumption when using money market interest rates, however, the quoted yields in most currencies which have a term to maturity of more than one year, usually are coupon-paying interest rates. The difficulty with coupon-paying yields is that there is a re-investment risk associated with each coupon payment. To price LTFX this risk has to be removed by deriving zero-coupon interest rates.
2. *Compounding*: Longer-term interest rates typically are expressed as compound interest rates—accordingly, compounding also needs to be incorporated into the model.

If we ignore either effect the LTFX price will be wrong, particularly if the yield curves in each currency have opposite shapes as this will exacerbate the difference between the zero-coupon and coupon interest differentials. *Exhibit 4.33* highlights the difference in forward price of a one year A$/US$ LTFX example when correctly calculated and when the short-term model is used.

Exhibit 4.33

LTFX Pricing: Using Zero Coupon Yields

To demonstrate the impact of using zero-coupon versus coupon yield curves, calculate the LTFX using the two sets of interest rates provided below.

Current Market Rates

FX Rate	0.7202
1 year, US$ coupon (sa)	5.50%
1 year, US$ zero coupon (sa)	5.56%
1 year, A$ coupon (ann)	7.60%
1 year, US$ zero coupon (ann)	7.79%

Correct Forward Price

Using zero-coupon rate & compounding sa rate	**0.7063**	
Incorrect Forward Prices (Short-term model)		Difference
Using coupon rate ignoring compounding	0.7067	−0.0003
Using coupon rate & compounding sa rate	0.7072	−0.0008

The three components of a short-term forward FX model are a spot exchange rate and two money market instruments; whereas the LTFX model is comprised of a spot FX deal and two zero-coupon bond transactions. That is, a LTFX purchase of the base currency can be synthetically replicated by buying the base currency in the spot market, funding the spot settlement of the terms currency by borrowing using a zero-coupon bond and investing the base currency proceeds in a zero-coupon bond which expires on the forward settlement date.

To incorporate this into our forward pricing model, the interest rate legs will calculate the forward value of a single amount (compounded using zero coupon rates).[23] If we incorporate these concepts into the forward pricing model we can express the LTFX model as follows:

23. Zero coupon rates are often not observable as a market quote, so the rates will need to be generated from coupon paying, or par, yields such as prevailing swap rates. See Martin, op cit n 1, Ch 8 for more detail on calculating zero coupon interest rates from par rates.

Long-term Forward Foreign Exchange (LTFX) Price

Using simple interest, the calculation is as follows:

$$F = \frac{S \times (1 + r_T / m_T)^{n_T}}{(1 + r_B) / m_B)^{n_B}}$$

Where

F	=	Forward exchange rate
S	=	Spot exchange rate
r_T	=	Terms currency zero coupon interest rate to forward date
r_B	=	Base currency zero coupon interest rate to forward date
m_T	=	Terms currency payment frequency (that is, 1, 2, 4, 12)
m_B	=	Base currency payment frequency
n_T	=	Terms currency # of payment periods to the forward date
n_B	=	Base currency # of payment periods to the forward date

An important characteristic of LTFX contracts is the impact of the interest differential on the forward price. Compared to a short-term forward FX contract, the interest rate legs of LTFX have considerably more impact on the forward price.

Exhibit 4.34

LTFX Pricing and Sensitivities

The graph below shows the sensitivity of a 5 year JPY/USD LTFX deal to changes in both the interest differential and the spot exchange rate. A feature of LTFX transactions is the increasing importance of the interest differential the longer the term to expiry. In this 5 year deal the impact of a move in the exchange rate of 1% is approximately equal to a change in the interest differential of 0.20% pa

Market Data	
Spot FX rate	101.00
USD 5 Year rate % pa (sa)	6.20
JPY 5 Year rate % pa (sa)	2.50
Interest Differential % pa	3.70

$$\text{LTFX Price} = \frac{S \times (1 + rT / mT)^{\wedge}nT}{(1 + rB / mB)^{\wedge}nB}$$

$$= \frac{101 \times (1 + 0.025/2)^{\wedge}10}{(1 + 0.062/2)^{\wedge}10}$$

$$= \underline{84.27}$$

The sensitivities of this position in foreign exchange points are as follows

PVBP = −0.0368

That is, a 1 bp rise in the interest differential will decrease the present value of the position by 0.0368 fx points.

PVFP = 0.0074

That is, a 0.01 change in the spot FX rate will alter the present value by 0.0074

PVD = 0.0075

Each day that passes increases the present value by 0.0075 fx points.

LTFX Price Sensitivity

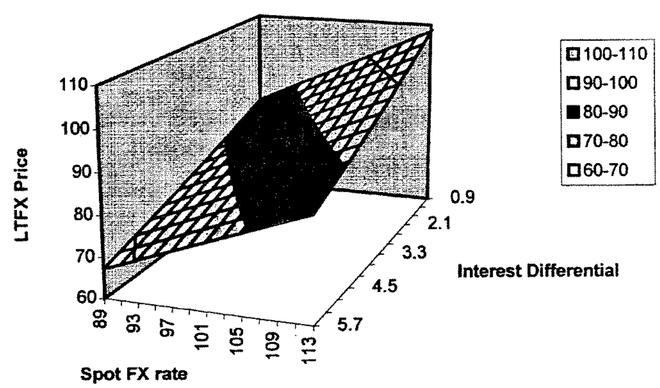


Exhibit 4.34 calculates the LTFX price of a five year JPY/US$ transaction and then graphs the sensitivity of the forward price to movements in the interest differential and the spot price. In this example a movement of 1% in the exchange rate has the same impact as a 0.20% pa change in the interest differential.

The forward value of a LTFX contract is the same as for a short-term forward FX contract: the difference between the contract value and the current market value. However, when calculating the present value we need to take account of the same yield considerations as the LTFX price. As a result we calculate the present value of the LTFX contract by using a zero coupon yield and applying the following compound interest, present value formula:[24]

$$\text{Present Value}_{BASE} = \text{Forward Value}_{BASE} / (1 + r_B / m_B)^{n_B}$$

5.6 Par forwards

5.6.1 Instrument description

Another form of LTFX is the par forward. It is a series of LTFX contracts with regularly spaced settlement dates (for example, monthly or quarterly) at a constant exchange rate. *Exhibit 4.35* compares the exchange rate of a five year CHF/US$ par forward transaction with a series of LTFX contracts which would achieve the same effect. The benefit from the point of view of an end user is that it allows the benefit or cost of a forward discount or premium to be spread out over the life of the transaction. In the example provided, from the point of view of a buyer of US$, in the first two years of the transaction you buy US$ at a substantially lower exchange rate than that implied by traditional forward instruments. The downside of course is that after two years the exchange rate is higher.

24. This is the base currency present value, to generate the terms currency PV just substitute terms currency variables for the base currency variables used.

Exhibit 4.35

Par Forward Rate Versus Series of LTFX Rates

In terms of the present value of these transactions the economics of a par forward and a series of LTFX are the same (as we will see below the pricing of par forwards ensures that this is the case). While in terms of the FX transaction alone there seems to be little added value in a par forward, the attraction of these instruments is that they can be very useful for cashflow management and also tax planning.

For example, suppose a Swiss-based distribution company is about to commence importing equipment from the United States. It has signed a five year contract which will require you to buy US$10 million of equipment every quarter. The initial set-up costs associated with selling this equipment will be substantial and at current exchange rates you are likely to have negative CHF cashflows for the first two years, after which time cashflows will turn positive. You talk to your banker and he or she is concerned about financing the new project given its long lead time and currency exposures and suggests that you consider covering the exposures using the forward FX rates outlined in *Exhibit 4.35*. From the Swiss manufacturer's point of view a par forward would be more attractive than a series of LTFX contracts. Par forwards provide and immediately lower CHF cost for the equipment imports and possibly create a positive cashflow in the first two years. The downside is that the CHF cost of the equipment is relatively higher than a series of LTFX contracts. The par forward has allowed the Swiss company to obtain forward FX cover and also smooth out its cashflows.

A par forward is also interesting as it also shares some similarities with fixed-to-fixed cross currency interest rate swaps as it involves a constant exchange of currency cashflows as opposed to the variable cashflows of LTFX contracts. We will see that in our discussion of currency swaps that par forwards can be a useful tool in managing swap exposures.

5.6.2 Pricing and valuing par forwards

A par forward is a "smoothed" series of LTFX contracts. Pricing a par forward involves determining the cashflows from a similar series of LTFX transactions and then making an adjustment for the funding cost or benefit of evenly spreading out the currency cashflows. As is the case for any forward transaction the present value of executing a par forward should be zero.

The spreadsheet set out in *Exhibit 4.36* shows how to solve for the par forward rate for the CHF/US$ example from the previous section. The first step is to calculate the LTFX rates for each periodic par forward date. We then need to determine what constant CHF delivery amount has an equivalent net present value to the CHF delivery amounts from the series of LTFX amounts. As can be seen the CHF par forward amounts will save the end user substantial CHF amounts—in effect the FX dealer is lending the difference between the LTFX rate and the par forward rate and having it repaid in the later delivery amounts. The dealer deserves a return on the money it has loaned to the end-user and so the par forward rate will include a funding cost. The solution of the par forward rate is to solve for a par forward rate which ensures that the total net present value of all of the funding differences is equal to zero. It is at this point that we know the money lent in the early par forward deliveries is re-couped, with interest, in the later deliveries.[25]

25. In practice FX dealers will want to ensure that the interest rates used reflect market bids and offers and that they earn a positive funding NPV on the transaction. This can be achieved in this model by setting the target NPV at a rate which provides the dealers required return. For example, in Exhibit 4.36 if the required NPV was CHF100,000 then the par forward rate would be 1.0476.

Exhibit 4.36

Pricing a Par Forward Transaction

Transaction Details

Spot FX Rate	1.13
Term (Yrs)	5
Delivery Frequency	Quarterly
Quarterly Amount	10,000,000

Solution

Unrounded Par Forward	1.04701178	Change

A	B	C	D	E	F	G	H	I
	Market Parameters			LTFX Cashflows		Par Forward	Net CHF	Net CHF
	Zero USD Rate	Zero CHF Rate	Forward FX rate	USD Amount	CHF Amount	CHF Amount	Funding Difference	Funding NPV
1	5.7500	2.0000	1.1196	10,000,000	11,195,564	10,470,118	−725,446	(721,837)
2	5.7500	2.0000	1.1092	10,000,000	11,092,093	10,470,118	−621,975	(615,802)
3	5.7817	2.0628	1.0992	10,000,000	10,992,155	10,470,118	−522,037	(514,043)
4	5.8134	2.1257	1.0895	10,000,000	10,894,820	10,470,118	−424,702	(415,793)
5	5.8358	2.2508	1.0810	10,000,000	10,809,644	10,470,118	−339,526	(330,133)
6	5.8582	2.3759	1.0731	10,000,000	10,730,621	10,470,118	−260,504	(251,409)
7	6.6166	2.8575	1.0589	10,000,000	10,588,703	10,470,118	−118,585	(112,821)
8	7.3751	3.3391	1.0435	10,000,000	10,434,820	10,470,118	35,297	33,026
9	7.2463	3.4347	1.0383	10,000,000	10,383,014	10,470,118	87,104	80,653
10	7.1175	3.5304	1.0343	10,000,000	10,342,908	10,470,118	127,210	116,508
11	7.0405	3.6254	1.0300	10,000,000	10,299,796	10,470,118	170,322	154,230
12	6.9634	3.7205	1.0266	10,000,000	10,265,579	10,470,118	204,539	183,032
13	6.9159	3.8062	1.0227	10,000,000	10,227,428	10,470,118	242,690	214,577
14	6.8685	3.8920	1.0196	10,000,000	10,196,121	10,470,118	273,997	239,261

Exhibit 4.36—*continued*

A	B	C	D	E	F	G	H	I
	Market Parameters			LTFX Cashflows				
	Zero USD Rate	Zero CHF Rate	Forward FX rate	USD Amount	CHF Amount	Par Forward CHF Amount	Net CHF Funding Difference	Net CHF Funding NPV
15	6.8380	3.9668	1.0161	10,000,000	10,161,093	10,470,118	309,025	266,506
16	6.8076	4.0416	1.0131	10,000,000	10,131,450	10,470,118	338,668	288,347
17	6.7881	4.1057	1.0098	10,000,000	10,097,918	10,470,118	372,200	312,883
18	6.7686	4.1697	1.0069	10,000,000	10,068,646	10,470,118	401,472	333,110
19	6.7534	4.2142	1.0032	10,000,000	10,032,394	10,470,118	437,724	358,691
20	6.7382	4.2587	0.9999	10,000,000	9,999,221	10,470,118	470,897	381,012
						Net CHF NPV (Target)		0

Summary of Results

Average LTFX rate =	1.0477
Funding Cost =	0.0023
Par Forward rate =	1.0470

How this Spreadsheet works

1. Generate the Zero Coupon interest rates (on a quarterly basis for columns B and C).
2. Calculate the LTFX rates in Column C.
3. Calculate the USD and CHF cashflows for each quarterly roll for column E and F.
4. Enter a "guess" of the Par Forward Rate and enter into the cell labelled "Unrounded Par Forward".
5. Calculate the Par Forward CHF amount in column H by multiplying the USD amount by the Unrounded Par Forward Amount.
6. The Net CHF amount is simply the difference between columns F and G.
7. Calculate the NPV in column I by taking the present value of column H using the CHF zero interest rates and the compound interest present value formula.
8. From the Tools menu invoke the "Solver" function.
9. Make the Net NPV cell (column I) the target by changing the cell with the Unrounded Par Forward rate so that the target becomes Zero and then press solve, this will iteratively solve for the Unrounded Par Forward Rate which makes the NPV zero.

Once a par forward price can be generated then the valuation procedures are identical to an LTFX contract. However, while a series of LTFX positions can be valued individually, all of the par forward legs should be valued together, because of the interdependence between the series of deliveries.

6. EQUITY FORWARDS

6.1 Introduction

In their relatively short lifetime equity forwards have gained a reputation as a highly risky instrument. The October 1987 stock market crash, and 1989 "mini-crash", prompted considerable conjecture that stock index futures exacerbated the market fall and, given the losses sustained by long position holders, were too risky an instrument to be used by the general public.[26] Then, in 1995, the collapse of British merchant bank Baring Brothers primarily due to unauthorised trading in share price index futures on the Japanese Nikkei index, prompted more regulatory "navel gazing" with respect to these instruments.

Despite the bad press, share price index futures have been an outstanding success if measured by volume growth since they were introduced in the United States in 1982. Stock index futures are a classic example of a derivative which "adds-value" to organisations and individuals with an exposure to share markets, as they provide a method of gaining an exposure to share market or hedging an existing exposure at considerably lower cost than transacting in the physical market. However, like their underlying market, the price volatility in share index futures is generally higher than most interest rate and currency markets—and as a result are a risky instrument in the hands of a novice user or uncontrolled trader.

An interesting feature of derivatives on equity index futures is the relatively high usage of options relative to both cash market volume and forward volume. Whereas in developed interest rate markets, option volume might be 10% of forward volume, in equity index markets the option percentage might be 20% or higher. This is partly explained by the high level of volatility, which encourages market participants to take insurance in the form of options.

Exhibit 4.37 summarises global exchange traded volume in equity forwards. Share price index futures dominate turnover, with very small volume in individual share futures. In the United States this is partly the result of regulatory restrictions, however, even where contracts have been listed on individual shares volume has been low.[27] The bulk of the volume is in US$ denominated indexes, followed by Japan and then Germany.

26. Possibly the most spectacular case of the damage done by stock index futures to over-leveraged users was the Hang Seng Stock Index Futures contract traded on the Hong Kong Futures Exchange. The majority of long position holders at the time of the crash were individual speculators. After the crash, most of these "longs" defaulted on payment of their losses and the market was effectively closed.
27. Apart from Sweden, individual share futures contracts are relatively new, and this may also contribute to the low volume.